1. For $c \in \mathbb{R}$, we define the matrix $\mathbf{A_c} \in \mathbb{R}^{3\times 3}$ by

$$\mathbf{A_c} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}. \tag{1}$$

   (a) Compute $det(\mathbf{A_c})$. Does it depend of c?
   (b) For which c is the matrix $\mathbf{A_c}$ invertible?
   (c) Compute $\mathbf{A_0}^{-1}$ (i.e. when $c = 0$).
   (d) Let $\mathbf{b} = (1, -4, 2)^t$, find the solution of $\mathbf{A_0 x} = \mathbf{b}$
   (e) Compute $det(\mathbf{A_c}^2)$.
   (f) Compute $det(5\mathbf{A_c})$.
   (g) Compute $det(\mathbf{E_k A_c})$, where

$$\mathbf{E}_k = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

   (h) Compute $det(\mathbf{D_k A_c})$, where

$$\mathbf{D}_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

   (i) Compute $det(\mathbf{A_0}^{-1})$.
   (j) Compute the eigenvalues of $\mathbf{A_0}$.
   (k) Compute the eigenvalues of $\mathbf{A_0}^{-1}$.

2. Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(\mathbf{x}) = \begin{pmatrix} 2x_1 + x_2 + \alpha x_3^2 \\ x_1 + 2x_2 \\ hx_3 + q \end{pmatrix}, \text{ where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \tag{4}$$

   in which, h and q are real numbers.

   (a) What are the conditions on q and α such that the transformation is linear?
   (b) **From now we suppose that** $q = 0$ **and** $\alpha = 0$. Write the associated matrix $\mathbf{A}$ of the transformation T. (**Hint:** remember that $\mathbf{A}(:, i) = T(\mathbf{e}_i)$.)
   (c) What is the condition on h such that the transformation T is **NOT** one-to-one? Explain briefly.
   (d) **Suppose that** $h = 0$, then

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

   Compute the eigenvalues of $\mathbf{A}$.

3. Let $\mathbf{A}$ be

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 3 & -4 \\ 4 & -4 & -2 \end{bmatrix}. \tag{6}$$

(a) Find the characteristic equation and eigenvalues of $\mathbf{A}$.

(b) Diagonalize $\mathbf{A}$, why can you do it?

(c) Compute $\mathbf{A}^5$.

(d) Is $\mathbf{A}$ invertible? Explain briefly.

(e) Determine if

$$\mathbf{C} = \begin{bmatrix} 4 & 1 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 6 \end{bmatrix}. \tag{7}$$

is similar to $\mathbf{A}$.

(f) Determine if

$$\mathbf{B} = \begin{bmatrix} 6 & 0 & 1 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}. \tag{8}$$

is similar to $\mathbf{A}$.

4. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation such that $Nul(T) = Col(T)$ and

$$T\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \qquad T\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}. \tag{9}$$

Find the standart matrix of T.

5. Let $\alpha, \beta \in \mathbb{R}$, and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\beta \\ \alpha \\ 0 \end{pmatrix}, \qquad T\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix}, \qquad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha - 1 \\ 0 \end{pmatrix}. \tag{10}$$

(a) Find the values of α, β such that T is **NOT** one-to-one.

(b) Assume that $\alpha = 1$ and $\beta = 0$, find the rank of T and the dimension of its nullspace.

6. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

(a) Find the eigenvalues of $\mathbf{A}$.

(b) Diagonalize $\mathbf{A}$.

7. Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad \mathbf{P} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{12}$$

(a) Compute $det(\mathbf{P})$. Is $\mathbf{P}$ invertible?

(b) Compute $\mathbf{P}^{-1}$.

(c) Verify that $\mathbf{A} = \mathbf{PDP}^{-1}$.

(d) Compute $\mathbf{A}^{10}$.

8. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation given by

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ -4 \\ -2 \end{pmatrix}, T\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ -3 \\ -3 \end{pmatrix}, T\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 1 \end{pmatrix}, T\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{13}$$

(a) Find the standard matrix (in the canonical or standard basis)

(b) You know that one of the eigenvalues of the standard matrix is $\lambda = 1$, can you diagonalize it?

9. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & b \\ 0 & 1 & 1 \\ 1 & 3 & a \end{bmatrix}, \tag{14}$$

for $a, b \in \mathbb{R}$.

(a) Compute the determinant of $\mathbf{A}$.

(b) Find the values of a and b such that

  (a) The matrix $\mathbf{A}$ is invertible.

  (b) The vector $(1, 1, 3)^t \in \mathbb{R}^3$ is in the $Col\mathbf{A}$.

  (c) The dimension of the nullspace of $\mathbf{A}$ is 1.

10. Let

$$\mathbf{A} = \begin{bmatrix} 1 & a & 0 \\ a & 1 & a \\ 0 & a & 1 \end{bmatrix}, \tag{15}$$

for $a \in \mathbb{R}$.

(a) Compute the determinant of $\mathbf{A}$.

(b) For which values of a, $\mathbf{A}$ is invertible?

11. Consider

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}. \tag{16}$$

(a) Find the eigenvalues of $\mathbf{A}$

(b) Compute a basis of $\mathbb{R}^4$ given by eigenvector of $\mathbf{A}$.

12. Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{17}$$

with $a, b, d, c > 0$. Show that $\mathbf{A}$ is diagonalizable. (Compute the eigenvalues, and show that the two roots are always different).