1. Let $M \in \mathbb{R}^{m\times n}$ such that $M^tM \in \mathbb{R}^{n\times n}$ is invertible. Define the matrix $P \in \mathbb{R}^{m,m}$

$$P = I_m - M(M^tM)^{-1}M^t,$$

   Show that

   (a) $P^2 - P$ and $P \cdot M = 0$, where 0 is the nul matrix of dimension m.

   (b) The matrices M^tM and P are symmetric. A matrix A is symmetric if and only if $A^t = A$.

   (c) P is not invertible.

2. Consider the linear system $Ax = b$ given by

$$A = \begin{bmatrix} -2 & 1 & (1-2\alpha) & (\beta+1) \\ 0 & 1 & -1 & (\beta-\alpha) \\ 0 & -2 & 2 & (2-2\beta) \\ 2 & 0 & 2 & \alpha \\ 2 & 1 & 1 & (\alpha+\beta-1) \end{bmatrix}, \text{ and let } \mathbf{b} = \begin{pmatrix} \beta-3 \\ -1 \\ -2 \\ 4\beta-3 \\ 0 \end{pmatrix}. \qquad (1)$$

   Find the conditions on α and β such that the system

   (a) has a unique solution

   (b) has infinity number of solutions

   (c) has no solution

3. Let $B \in R^{n\times n}$ such that $B^3 = 0$. For every $\lambda \in \mathbb{R}$ we define the matrix $M(\lambda) \in \mathbb{R}^n$ given by

$$M(\lambda) = I_n + \lambda B + \frac{\lambda^2}{2}B^2.$$

   (a) Show that

$$\forall \lambda, \beta \in \mathbb{R} \qquad M(\lambda+\beta) = M(\lambda)\cdot M(\beta),$$

   conclude that $M(\lambda)\cdot M(\beta) = M(\beta)\cdot M(\lambda)$

   (b) Show that $M(\lambda)$ is invertible and that $M(\lambda)^{-1} = M(-\lambda)$ (**Hint**: use $M(0)$)

4. Consider the following linear systems

$$L1 \begin{cases} x_1 + x_3 = 1 \\ \alpha x_1 + x_2 + x_3 = 0 \end{cases} \qquad L2 \begin{cases} 2\alpha x_1 + x_2 + x_3 = 1 \\ \alpha x_1 + x_2 + x_3 + 2 = 0 \end{cases} \qquad (2)$$

   (a) Solve the systems, and provide the conditions on α such that the solutions sets are lines,

   (b) Write the vectorial equations for $L1$ and $L2$

   (c) Suppose that th solution sets are straight lines, determine the value of α such that the solution sets of $L1$ and $L2$ do not intersect.

5. Let B be the set given by

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

and the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$T\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \qquad T\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \qquad \text{and } Nul(T) = Col(T) \tag{3}$$

(a) Show that the B is a basis of $\mathbb{R}^4$.

(b) Compute the standard (or associated) matrix of T using the canonical basis.

(c) Show that a basis of $Nul(T)$ is $\left\{\begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\}$.

(d) Compute the standard (or associated) matrix of T using B as basis in the domain and codomain.

6. Let S be the subspace given by

$$S = \text{span}\left\langle\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\3\\3\\3\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\}\right\rangle$$

(a) Find a basis of S

(b) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation such that $Nul(T) = S$ and

$$T\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \qquad T\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} \tag{4}$$

(a) Find the rank of the T and find a basis of $Col(T)$.

(b) Provide an explicit formula for T.

7. For $c \in \mathbb{R}$, we define the matrix $\mathbf{A_c} \in \mathbb{R}^{3\times 3}$ by

$$\mathbf{A_c} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \\ 3 & c & 2 \end{bmatrix}. \tag{5}$$

(a) Compute $det(\mathbf{A_c})$, for which c is $\mathbf{A_c}$ invertible

(b) Compute $\mathbf{A_0}^{-1}$

(c) Let $b = (2, -4, 1)^t$, find the solution $\mathbf{A_0}x = b$