1. In this question you will study how to change basis using polynomials as an example. Let $\mathbb{P}^3$ be the set of all the polynomials of degrees less or equal than 3.

   You can write any polynomial $p(x) \in \mathbb{P}^3$ in the form

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \tag{1}$$

   In that case you can easily write the following transformation $E : \mathbb{P}^3 \to \mathbb{R}^4$, which is given by:

$$E(p(x)) = E(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \tag{2}$$

   In other words you are writting the polynomail p using the coefficients from the monomial basis, which is given by $\mathcal{E} = \{1, x, x^2, x^3\}$.

   In order to alleviate the notation we will denote the $\mathcal{E}$-coordinates of a polynomial p as $[p]_{\mathcal{E}} = E(p(x))$

   (a) Compute $E(1 + x + x^2 + x^3)$.

   (b) Show that the transformation is linear

   (c) Show that the transformation E is one-to-one

   (d) Show that E transformations $\mathbb{P}^3$ onto $\mathbb{R}^4$.

   (e) Conclude that the linear transformation E is invertible, and explain why the inverse $E^{-1} : \mathbb{R}^4 \to \mathbb{P}^3$ is given by

$$E^{-1}\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3. \tag{3}$$

2. Now let us suppose that we have the transformation $T : \mathbb{P}^3 \to \mathbb{P}^3$ which is given by differentiating the polynomial, i.e.,

$$T(p(x)) = \frac{d}{dx} p(x), \tag{4}$$

   which is clearly linear.

   (a) Show (by computing) that the standard matrix of T with respect to the basis $\mathcal{E}$ is

$$[T]_{\mathcal{E}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

   (b) Check that your result is correct by computing the derivative of $r(x) = 1 + x + x^2 + x^3$ in a standard manner (i.e. how you were taught in 2B).

   (c) Now you will compute the same derivative but by using the following composition $T(r(x)) = E^{-1}([T]_{\mathcal{E}} \cdot E(r(x)))$. In other words, compute the coordinates of $r(x)$ in the $\mathcal{E}$ basis, then multiply the coordinates by $[T]_{\mathcal{E}}$ and transform back the new coordinates to a polynomial.

(d) Using $[T]_{\mathcal{E}}$, check that T is not one-to-one nor it maps $\mathbb{P}^3$ onto $\mathbb{P}^3$.

3. Now that you have understood how to represent the differentiation of polynomials in the monomial basis, we will explore how to change to different basis. Let consider the following set

$$\mathcal{B} = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}. \tag{6}$$

(a) Convince yourself that you can write any polynomial $p \in \mathbb{P}^3$ in the form

$$p(x) = \beta_0 + \beta_1(1+x) + \beta_2(1+x+x^2) + \beta_3(1+x+x^2+x^3). \tag{7}$$

This means that $\mathcal{B}$ is a basis of $\mathbb{P}^3$. In this case we will define the $\mathcal{B}$ coordinates of p as

$$[p]_{\mathcal{B}} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}. \tag{8}$$

(b) Use the fact that

$$\begin{aligned} p(x) &= \beta_0 + \beta_1(1+x) + \beta_2(1+x+x^2) + \beta_3(1+x+x^2+x^3) \\ &= (\beta_0+\beta_1+\beta_2+\beta_3) + (\beta_1+\beta_2+\beta_3)x + (\beta_2+\beta_3)x^2 + \beta_3 x^3. \end{aligned}$$

to show that the matrix of change of basis B is given by

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{9}$$

Argue that $[p]_{\mathcal{E}} = B \cdot [p]_{\mathcal{B}}$.

(c) Compute B^{-1}.

(d) Using the inverse of B, check that

$$[r]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \tag{10}$$

where $r(x) = 1+x+x^2+x^3$. Convince yourself that $[r]_{\mathcal{B}}$ and $[r]_{\mathcal{E}}$ are the SAME polynomial but written in different basis.

(e) Now that you all the necessary transformations, using the composition of transformations (and changes of coordinates) show that

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{11}$$

(f) Redo your computations by using the formula for standard matrices applied to T and $\mathcal{B}$.

(g) Check that $[T]_{\mathcal{B}}$ is correct by computing the derivative of $r(x)$ as defined before.

4. Can you generalize the same procedure. for higher order polynomials? How would the matrices $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{E}}$ look like then? Which one would to prefer and why?