Q4/

a) We know that

$$\sum_{n=0}^{+\infty} r^n = \frac{1}{1-r} \quad \text{when } r<1$$

in this case we have $xy=r$, then. ($\to 0\le x\le 1$, $0\le y\le 1$)

$$\frac{1}{1-xy} = \sum_{n=0}^{+\infty} (xy)^n$$

then

$$\int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy = \int_0^1\int_0^1 \sum_{n=0}^{+\infty} (xy)^n\,dx\,dy$$

in this case we formally swap the sum and the integral.

$$= \sum_{n=0}^{+\infty}\left(\int_0^1\int_0^1 (xy)^n\,dx\,dy\right) = \sum_{n=0}^{+\infty}\left(\int_0^1 x^n\,dx\right)\left(\int_0^1 y^n\,dy\right)$$

$$= \sum_{n=0}^{+\infty}\left[\frac{x^{n+1}}{n+1}\right]_0^1 \cdot \left[\frac{y^{n+1}}{n+1}\right]_0^1$$

$$= \sum_{n=0}^{+\infty} \frac{1}{(n+1)^2} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

$$= \sum_{n=1}^{+\infty} \frac{1}{n^2}$$

ζ(2)

Q6 |

we know that

$$\partial_{xx} f + \partial_{yy} f = 0,$$

and that $f(x,y) = g(u(x,y), v(x,y))$ where

$$u(x,y) = x \cdot y$$
$$v(x,y) = \frac{x}{y}$$

then we will compute $\partial_{xx} f$ using the chain rule.

$$\partial_x f(x,y) = \partial_x g(u,v)$$
$$= \partial_u g(u,v) \cdot \partial_x u + \partial_v g(u,v) \cdot \partial_x v$$

using the chain rule.

$$\Rightarrow \partial_{xx} f(x,y) = \partial_x (\partial_x f(x,y))$$
$$= \partial_x (\partial_x g(u,v))$$
$$= \partial_x \left[\partial_u g(u,v) \cdot \partial_x u + \partial_v g(u,v)\, \partial_x v \right]$$
$$= \underline{\partial_x (\partial_u g(u,v))} \partial_x u + \partial_u g(u,v) \cdot \partial_x \partial_x u$$
$$+ \underline{\partial_x (\partial_v g(u,v))} \partial_x v + \partial_v g(u,v) \cdot \partial_x \partial_x v$$

now we compute $\partial_x (\partial_u g(u,v))$ by chain rule.

$$\Rightarrow \partial_x (\partial_u g(u,v)) = \partial_u \partial_u g(u,v) \cdot \partial_x u + \partial_v \partial_u g(u,v) \cdot \partial_x v$$
$$\partial_x (\partial_v g(u,v)) = \partial_u \partial_v g(u,v) \cdot \partial_x u + \partial_v \partial_v g(u,v) \cdot \partial_x v$$

now we can add everything together.

$$\partial_{xx} f(x,y) = \partial_{uu} g(u,v)\,(\partial_x u)^2 + \partial_{vv} g(u,v)\cdot(\partial_x v)^2$$
$$+\ 2\,\partial_{uv} g(u,v)\cdot\partial_x u\cdot\partial_x v$$
$$+\ \partial_u g(u,v)\,\partial_{xx} u + \partial_v g(u,v)\cdot\partial_{xx} v.$$

in this case we used that g is twice differentiable. and Clairaut's theorem to change the order of the partial derivatives in g.

Obviously the same expression is valid for ∂_{yy}

$$\partial_{yy} f(x,y) = \partial_{uu} g(u,v)\,(\partial_y u)^2 + \partial_{vv} g(u,v)\cdot(\partial_y v)^2$$
$$+\ 2\,\partial_{uv} g(u,v)\,\partial_y u\cdot\partial_y v$$
$$+\ \partial_u g(u,v)\cdot\partial_{yy} u + \partial_v g(u,v)\cdot\partial_{yy} v$$

we can then add everything together and we obtain:

$$\partial_{xx} f + \partial_{yy} f = \partial_{uu} g(u,v)\cdot\left[(\partial_x u)^2 + (\partial_y u)^2\right]$$
$$+\ \partial_{vv} g(u,v)\left[(\partial_x v)^2 + (\partial_y v)^2\right]$$
$$+\ 2\,\partial_{uv} g(u,v)\cdot\left[\partial_x u\cdot\partial_x v + \partial_y u\cdot\partial_y v\right]$$
$$+\ \partial_u g(u,v)\,(\partial_{xx} u + \partial_{yy} u)$$
$$+\ \partial_v g(u,v)\,(\partial_{xx} v + \partial_{yy} v).$$

now we compute

$$\partial_x u(x,y) = \partial_x (x\cdot y) = y \qquad \partial_y u(x,y) = x$$

$$\partial_{xx} u = 0 \qquad \partial_{yy} u = 0.$$

$$\partial_x v(x,y) = \partial_x \frac{x}{y} = \frac{1}{y} \qquad \partial_y v(x,y) = \partial_y\left(\frac{x}{y}\right) = -\frac{x}{y^2}$$

$$\partial_{xx} v = 0 \qquad \partial_{yy} v(x,y) = \frac{2x}{y^3}$$

Then we have.

$$0 = \partial_{xx} f + \partial_{yy} f = \partial_{uu} g(u,v)\cdot\left(y^2 + x^2\right)$$
$$+ \partial_{vv} g(u,v)\cdot\left(\frac{1}{y^2} + \frac{x^2}{y^4}\right)$$
$$+ 2\,\partial_{uv} g(u,v)\left(1 + -\frac{x^2}{y^2}\right)$$
$$+ \partial_u g(u,v)\cdot 0$$
$$+ \partial_v g(u,v)\left(\frac{2x}{y^3}\right)$$

now $y^2 = \frac{u}{v}$ $\quad x^2 = u\cdot v$ $\quad \frac{x^2}{y^4} = \frac{v^3}{u}$ $\quad \frac{1}{y^2} = \frac{v}{u}$ $\quad \frac{x}{y^3} = \frac{v^2}{u}.$

then if we replace in the equation above we have.

$$0 = \partial_{xx} f + \partial_{yy} f = \partial_{uu} g(u,v)\cdot\left(u\cdot v + \frac{u}{v}\right) + \partial_{vv} g(u,v)\cdot\frac{u}{v}\left(1 + v^2\right)$$
$$+ \partial_{uv} g(u,v)\cdot 2\left(1 - v^2\right) + \partial_v g(u,v)\cdot 2\,\frac{v^2}{u} = 0$$

which is the expression we wanted to find.

Q12

We want to show that

$$\int_0^x\int_0^y\int_0^z f(t)\,dt\,dz\,dy = \frac{1}{2}\int_0^x (x-t)^2 f(t)\,dt.$$

to do so, we need to change the integration order; in particular, we want to integrate in t last.

for a fixed x, we have that.

$$\int_0^x\int_0^y\int_0^z f(t)\,dt\,dz\,dy = \iiint_{E_x} f(t)\,dA.$$

where $E_x = \{(t,y,z)\ 0\le y\le x;\ 0\le z\le y;\ 0\le t\le z\}$.

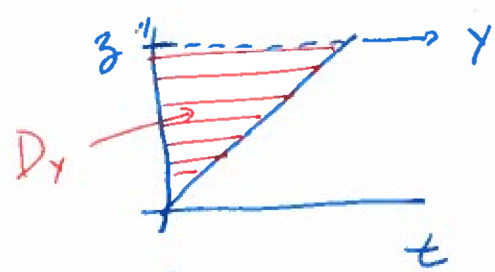
then we will change the parametrization of E_x step by step. In this case we freeze one variable and we swap the other two.

fix y:

$E_x = \{(t,y,z)\ 0\le y\le x$ and $(t,z)\in D_y\}$

where $D_y = \{(t,z):\ 0\le z\le y;\ 0\le t\le z\}$.

↳



so we can rewrite D_y as

$D_y = \{(t,z) \quad 0 \le t \le y\, ; \ t \le z \le y\}.$

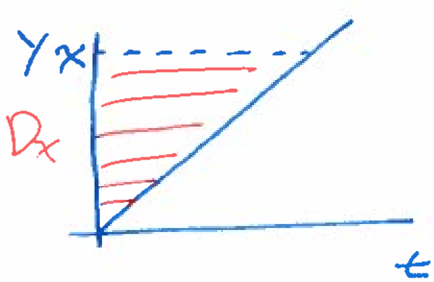
then

$E_x = \{(t,y,z) \quad 0 \le y \le x, \ 0 \le t \le y\, ; \ t \le z \le y\}.$

fix z:

$E_x = \{(t,y,z) \quad (t,y) \in D_x \text{ and } t \le z \le y\}.$

$D_x = \{(t,y) \quad 0 \le y \le x, \ 0 \le t \le y\}$



we can easily reparametrize D_x as.

$D_x = \{(t,y) \quad 0 \le t \le x\, ; \ t \le y \le x\}.$

then $E_x = \{(t,y,z) \quad 0 \le t \le x\, ; \ t \le y \le x\, ; \ t \le z \le y\}.$

thus we rewrite the integral

$$\int_0^x \int_0^y \int_0^z f(t)\,dt\,dz\,dy = \int_0^x \int_t^x \int_t^y f(t)\,dz\,dy\,dt.$$

Now we just need to compute the integral

$$\int_0^x \int_t^x \underbrace{\int_t^y f(t)\,dz}_{=\, f(t)\cdot(y-t)}\,dy\,dt$$

$$= \int_0^x \underbrace{\int_t^x f(t)\cdot(y-t)\,dy}\,dt$$

$$= f(t)\cdot\left[\frac{y^2}{2} - ty\right]_{y=t}^{y=x}$$

$$= f(t)\left(\frac{x^2-t^2}{2} - t(x-t)\right)$$

$$= f(t)\left(\frac{x^2 - \cancel{t^2} - 2tx + \cancel{2}t^2}{2}\right)$$

$$= \frac{1}{2} f(t)(x-t)^2$$

$$= \frac{1}{2}\int_0^x (x-t)^2 f(t)\,dt$$, which is the expression we were looking for.