1. Find all the critical points of the function

$$f(x, y) = \sin x + y^2 + 2y \cos x + 1$$

and classify them (maximum, minimum or saddle point)

2. Determine if the following statements are true or false. If they are true, explain why, otherwise provide a counter-example.

(a)

$$\int_{-1}^{2} \int_{0}^{6} x^2 \sin(x - y) dx dy = \int_{0}^{6} \int_{-1}^{2} x^2 \sin(x - y) dy dx.$$

(b)

$$\int_{0}^{1} \int_{0}^{x} \sqrt{x + y^2} dy dx = \int_{0}^{x} \int_{0}^{1} \sqrt{x + y^2} dx dy.$$

(c)

$$\int_{1}^{2} \int_{3}^{4} x^2 e^y dy dx = \left(\int_{1}^{2} x^2 dx \right) \left(\int_{3}^{4} e^y dy \right).$$

(d)

$$\int_{-1}^{1} \int_{0}^{1} e^{x^2 + y^2} \sin(y) dx dy = 0.$$

(e) If f is continuous on $[0, 1]$, then

$$\int_{0}^{1} \int_{0}^{1} f(x) f(y) dx dy = \left[\int_{0}^{1} f(x) dx \right]^2.$$

(f)

$$\int_{1}^{4} \int_{0}^{1} (x^2 + \sqrt{y}) \sin(x^2 y^2) dx dy \leq 9.$$

(g) If f has a local minimum at (a, b) and f is differentiable at (a, b), then $\nabla f(a, b) = 0$

(h) If $(2, 1)$ is a critical point of f and

$$(\partial_{xx} f(2, 1)) (\partial_{yy} f(2, 1)) < (\partial_{xy} f(2, 1))^2$$

then f has a saddle point at $(2, 1)$.

(i) If $f(x, y)$ has two local maxima, then f must have a local minimum.

(j) There exists a function f with continuous second-order partial derivatives such that

$$\partial_x f(x, y) = x + y^2 \text{ and } \partial_y f(x, y) = x - y^2. \tag{1}$$

(k)

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y}. \tag{2}$$

3. (a) Evaluate

$$\iint_D \frac{1}{(x^2 + y^2)^{n/2}} dA,$$

where n is an integer, and D is the region bounded by the circles with center in the origin and radii r and R, such that $0 < r < R$. (Hint: use polar coordinates.)

(b) For what values of n does the integral above have a limit when $r \to 0^+$.

4. Show the following identities:

   (a)
   $$\int_0^1 \int_0^1 \frac{1}{1-xy} dxdy = \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{3}$$
   Hint: expand the integrand as a geometric series, and interchange the order of the sum and integrals.

   (b)
   $$\int_0^1 \int_0^1 \int_0^1 \frac{1}{1-xyz} dxdydz = \sum_{n=1}^{\infty} \frac{1}{n^3} \tag{4}$$

5. Suppose that a function F is defined as
   $$F(x,y) = f(x, g(x)k(y), h(x,y)),$$
   where f, g, h, k are twice differentiable functions. Find
   $$\frac{\partial^2 F}{\partial x \partial y}$$
   in terms of the partial derivatives of f, g, h, k.

6. Let $g : \mathbb{R}^2 \to \mathbb{R}$ a function twice differentiable, and let f be a function defined on $\Omega = \{(x,y) \in \mathbb{R}^2, y \neq 0\}$ twice differentiable such that $f(x,y) = g(xy, x/y)$.

   Suppose, that f satisfies
   $$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$
   and define $u = xy$ and $v = \frac{x}{y}$. Show that g satisfies
   $$\left(uv + \frac{u}{v}\right) \frac{\partial^2 g}{\partial u^2} + 2(1 - v^2) \frac{\partial^2 g}{\partial u \partial v} + \frac{v}{u}(1 + v^2) \frac{\partial^2 g}{\partial v^2} + 2\frac{v^2}{u} \frac{\partial g}{\partial v} = 0.$$
   Hint: use the equation that f satisfies, and compute the partial derivatives of f using the chain rule on g.

7. A disk or radius 1 is rotating in the counter-clockwise direction at a constant angular speed ω. A particle starts at the center of the disk and moves toward the edge along a fixed radius so that it position at time $t \geq 0$, is given by $\mathbf{r}(t) = t\mathbf{R}(t)$, where
   $$\mathbf{R}(t) = \cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}.$$

   (a) Show that the velocity $\mathbf{v}$ of the particle is
   $$\mathbf{v}(t) = \cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j} + t\mathbf{v}_d$$

where $\mathbf{v}_d = \mathbf{R}'(t)$ is the velocity of a point at the edge of the disk.

(b) Show that the acceleration $\mathbf{a}$ of the particle is given by

$$\mathbf{a} = 2\mathbf{v}_d + t\mathbf{a}_d$$

where, $\mathbf{a}_d = \mathbf{R}''(t)$ is the the acceleration of a point on the edge of the disk. In this case, $2\mathbf{v}_d$ corresponds to the Coriolis acceleration.

(c) Compute the Coriolis acceleration of a moving particle that moves on a rotating disk according to the equation

$$\mathbf{r}(t) = e^{-t}\cos(\omega t)\mathbf{i} + e^{-t}\sin(\omega t)\mathbf{j}.$$

8. Suppose that you have two planets: one of mass M situated at $\mathbf{y} \in \mathbb{R}^3$ and the other of mass m located at $\mathbf{x} \in \mathbb{R}^3$. Following the gravitation law we have that the force on the planet located at x is given by

$$\mathbf{F} = -\gamma \frac{Mm}{|\mathbf{x} - \mathbf{y}|^3}(\mathbf{x} - \mathbf{y}).$$

If $\mathbf{x}(t)$ is the position in the space of a planet of mass m that moves under the action of the gravitational force due to a mass M located in the origin, then show that the energy, defined as

$$E(t) = \frac{m}{2}\left|\frac{d\mathbf{x}(t)}{dt}\right|^2 - \gamma \frac{Mm}{|\mathbf{x}(t)|},$$

is preserved, i.e., show that $E'(t) = 0$.

Hint: you need to use Newton's second law $\mathbf{F} = m\mathbf{a}$, where $\mathbf{a} = \mathbf{x}''(t)$. Moreover, you may want to recall that $|\mathbf{x}(t)| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.

9. (a) Maximize $\sum_{i=1}^n x_i y_i$ subject to the constraints $\sum_{i=1}^n x_i^2 = 1$ and $\sum_{i=1}^n y_i^2 = 1$.
Hint: computing the derivatives may seem overwhelming, this can be easily tackled by exploiting the symmetry in the problem. In this case, you may want to treat x first and then treat y, and then put them together in the equation for the Lagrange multipliers.

(b) Let $\mathbf{a} = \langle a_1, a_2, ..., a_n\rangle$ and $\mathbf{b} = \langle b_1, b_2, ..., b_n\rangle$ two vectors in $\mathbb{R}^n$.
Put

$$x_i = \frac{a_i}{\sqrt{\sum_j^n a_i^2}} \quad \text{and} \quad y_i = \frac{b_i}{\sqrt{\sum_j^n b_i^2}}$$

and using the last question show that

$$\mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a}||\mathbf{b}|.$$

This inequality is known as the Cauchy-Schwarz inequality.
Hint: recall that $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a_i b_i$ and $|\mathbf{a}| = \sqrt{\sum_j^n a_i^2}$

10. If $[\![x]\!]$ denotes the greatest integer in x, evaluate the integral

$$\iint_R [\![x + y]\!] dA$$

where $R = [1,3] \times [2,5]$.

11. Evaluate the integral
$$\int_0^1 \int_0^1 e^{\max(x^2,y^2)} dxdy,$$
where $\max(x^2, y^2)$ means the larger of the numbers x^2 and y^2.

12. If f is continuous, show that
$$\int_0^x \int_0^y \int_0^z f(t)dtdzdy = \frac{1}{2} \int_0^x (x-t)^2 f(t)dt$$

13. (Example from Statistical physics) We define the entropy of a system as the function $S : \mathbb{R}^n \to \mathbb{R}$ defined as
$$S(x_1, x_2, ..., x_n) = -\sum_{i=k}^{n} x_k \ln x_k.$$
we aim to maximize this function.
   (a) Find the domain of S, and compute its partial derivatives.
   (b) Find a critical point and show that it is a maximum. (You will need to use a higher dimension version of the second derivative test.)
   (c) Find a critical point with the restriction that $\sum_{k=1}^{n} x_k = 1$. Show that it is a maximum.
   (d) Let $E_1 < E_2 < E_3 < ...E_n$ and E be given reals number (i.e. they are constant). In the next questions we want to maximize S under the restrictions
$$\sum_{k=1}^{n} x_k = 1 \quad \text{and} \quad \sum_{k=1}^{n} x_k E_k = E$$
write the equations for the Lagrange multipliers. (You will obtain $n+2$ equations)
   (e) Argue that the solution of the system of equations has the form $x_i = e^{-\beta E_i}/Z$ where $Z = \sum_{i=1}^{n} e^{-\beta E_i}$, and its known as the partition function.
   (f) Write the equation to find *beta* (do not try to find it).
   (g) Argue that the solution to the system maximizes S under the constraints.