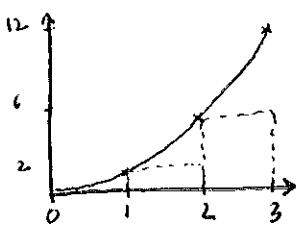
MATH 2B: SAMPLE MIDTERM #1

- This exam consists of 5 questions and 85 total points.
- Read the directions for each problem carefully and answer all parts of each problem.
- Please show all work needed to arrive at your solutions (unless instructed otherwise). Label graphs and define any notation used. Cross out incorrect scratch-work.
- No calculators or other forms of assistance are allowed. Do not check your cell phones during the exam.
- Clearly indicate your final answer to each problem.

1. (15 points)

a. Estimate the area under the graph of $f(x) = x^2 + x$ from $x = 0$ to $x = 3$ using 3 approximating rectangles and left endpoints.

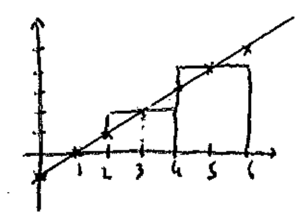


$\Delta x = \frac{3-0}{3} = 1$ Left endpoints $0, 1, 2$

$A \approx \Delta x f(0) + \Delta x f(1) + \Delta x f(2)$

$= 0 + 2 + 6 \boxed{= 8}$

b. Estimate the area under the graph of $f(x) = x - 1$ from $x = 0$ to $x = 6$ using 3 rectangles and midpoints.



$\Delta x = \frac{6-0}{3} = 2$ midpoints $1, 3, 5$

$A \approx \Delta x f(1) + \Delta x f(3) + \Delta x f(5)$

$= 2 \cdot 0 + 2 \cdot 2 + 2 \cdot 4$

$\boxed{= 12}$

c. Find an expression for the area under the graph of $f(x) = x^2 + x$ from $x = 2$ to $x = 5$ as a limit of Riemann sums. (You do not need to evaluate the limit.)

$\Delta x = \frac{5-2}{n} = \frac{3}{n}$

$f(x) = x^2 + x$

$x_i = a + i\Delta x = 2 + \frac{3i}{n}$

$A = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

$= \lim_{n \to +\infty} \sum_{i=1}^{n} \frac{3}{n} \left[\left(2 + \frac{3i}{n}\right)^2 + \left(2 + \frac{3i}{n}\right) \right]$

2. (15 points) Evaluate each of the following indefinite integrals.

a. $\int x\sqrt{3x^2-1}\,dx$

$u = 3x^2 - 1$

$du = 6x\,dx$

$\int x\sqrt{3x^2-1}\,dx = \frac{1}{6}\int \sqrt{u}\,du = \frac{1}{6}\cdot\frac{2}{3}u^{3/2} + C$

$\boxed{= \frac{1}{9}(3x^2-1)^{3/2} + C}$

b. $\displaystyle\int \frac{1-\sin^2(x)}{\cos x}\,dx$

$\int \frac{1-\sin^2 x}{\cos x}\,dx = \int \frac{\cos^2 x}{\cos x}\,dx = \int \cos x\,dx \boxed{= \sin x + C}$

c. $\int \sin(7\theta+5)\,d\theta$

$u = 7\theta + 5$

$du = 7\,d\theta$

$\int \sin(7\theta+5)\,d\theta = \frac{1}{7}\int \sin u\,du = -\frac{1}{7}\cos u + C \boxed{= -\frac{1}{7}\cos(7\theta+5) + C}$

3. (15 points)

a. Find the average value of the function $f(x) = \tan^3(x)\sec^2(x)$ on the interval $[0, \frac{\pi}{4}]$.

$$\frac{4}{\pi}\int_0^{\pi/4} \tan^3 x \sec^2 x\, dx = \frac{4}{\pi}\int_0^1 u^3\, du = \frac{4}{\pi}\left[\frac{u^4}{4}\right] = \frac{1}{\pi}.$$

$u = \tan x$

$du = \sec^2 x\, dx$

when $x = 0$ $\quad u = \tan 0 = 0$

$x = \frac{\pi}{4}$ $\quad u = \tan\frac{\pi}{4} = 1$

b. A particle moves along a line so that its velocity at time t is $v(t) = |2 - t|$. Find the displacement of the particle during the time period $0 \le t \le 3$.

$$\int_0^3 |2-t|\,dt$$

$|2-t| = 2-t$ when $2-t \geq 0$ that is $2 \geq t$

$-(2-t)$ when $2-t \leq 0$ that is $2 \leq t$

$$\int_0^3 |2-t|\,dt = \int_0^2 2-t\,dt + \int_2^3 t-2\,dt$$

$$= \left[2t - \frac{t^2}{2}\right]_0^2 + \left[\frac{t^2}{2} - 2t\right]_2^3$$

$$= 4 - 2 + \frac{9}{2} - 6 - 2 + 4$$

$$\boxed{= \frac{5}{2}}$$

4. (20 points)

a. Complete the blanks in the following statement of the Fundamental Theorem of Calculus.

Fundamental Theorem of Calculus:
Suppose f is continuous on $[a, b]$. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) =$ $f(x)$ and $\int_a^b f(x)\,dx =$ F(b) - F(a), where F is any antiderivative of f.

b. Use the Fundamental Theorem of Calculus to evaluate the following.

i. $\frac{d}{dy}\int_2^y \frac{\sin(t)}{t^2+3}\,dt$ $\boxed{= \frac{\sin y}{y^2 + 3}}$

ii. $\frac{d}{dx}\int_x^{x^4} \sqrt{t}\,dt$ $= 4x^3\sqrt{x^4} - \sqrt{x}$

$\boxed{= 4x^5 - \sqrt{x}}$

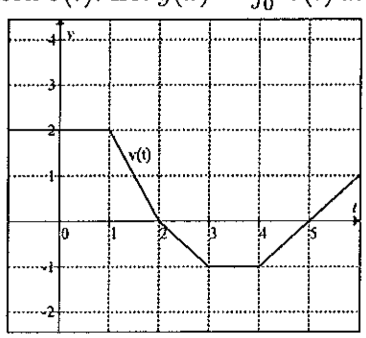
c. Answer each of the following questions. No work or explanation is needed.

i. If $f(t)$ is measured in dollars per year and t in years, what are the units of $\int_0^{10} f(t)\,dt$? dollars

ii. True/False: All continuous functions have derivatives. false

iii. True/False: All continuous functions have antiderivatives. true

iv. Below is the graph of a function $v(t)$. Let $g(x) = \int_0^x v(t)\,dt$.

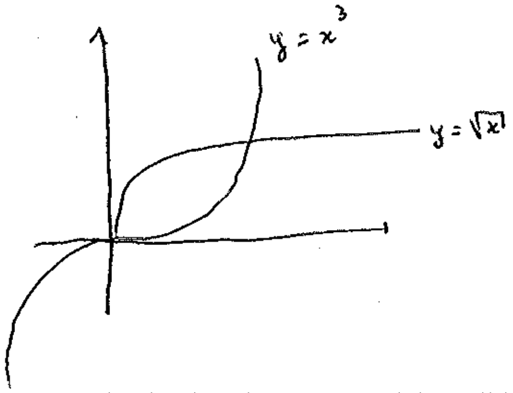


Find each of the following:

$g(0) =$ 0, $g(2) =$ 3, $g'(1) =$ 2, $g'(4) =$ -1

5. (20 points) Let S be the region bounded by $y = x^3$ and $y = \sqrt{x}$.

a. Find the area of the region S.



Point of intersection:

$x^3 = \sqrt{x}$

$x = 0, 1$

$$A = \int_0^1 \sqrt{x} - x^3 \, dx = \left[\frac{2}{3} x^{3/2} - \frac{x^4}{4} \right]_0^1$$

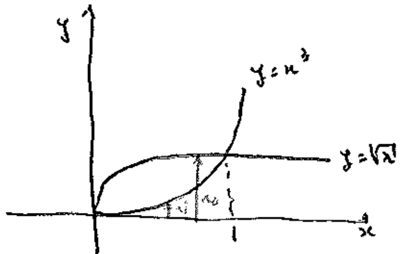
$$= \frac{2}{3} - \frac{1}{4} = \boxed{\frac{5}{12}}.$$

b. i. Find the volume of the solid obtained by revolving the region S about the x-axis.

the x-axis is horizontal. Thus we integrate with respect to x.

$0 \le x \le 1$

Crossection at x perpendicular to the x-axis



washer

$A(x) = \pi (r_o^2 - r_i^2)$

$= \pi (x - x^6)$

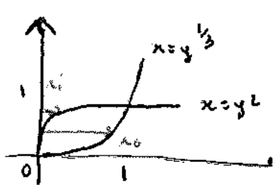
$r_i = x^3$

$r_o = \sqrt{x}$

$$V = \int_0^1 \pi (x - x^6) \, dx = \pi \left[\frac{x^2}{2} - \frac{x^7}{7} \right]_0^1$$

$$= \pi \left(\frac{1}{2} - \frac{1}{7} \right) = \boxed{\frac{5\pi}{14}}$$

ii. Set up an integral to find the volume obtained by revolving S about the y-axis. (You do not need to evaluate the integral.)



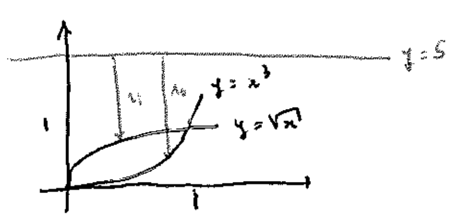
Variable: y

Crossection at y perpendicular to the y axis

Washer $\quad r_i = y^2 \quad r_o = y^{1/3}$

$$\boxed{V = \pi \int_0^1 y^{\frac{2}{3}} - y^4 \, dy}$$

iii. Set up an integral to find the volume obtained by revolving S about the line $y = 5$. (You do not need to evaluate the integral.)



Variable: x

Crossection at x perpendicular to the x axis

Washer $\quad r_i = 5 - \sqrt{x} \quad r_o = 5 - x^3$

$$\boxed{V = \pi \int_0^1 (5 - x^3)^2 - (5 - \sqrt{x})^2 \, dx}$$