Fall 2014 Math 2B - First Midterm

This exam consists of 5 questions. Problems #1 and 4 are worth 15 points each, problem #2 is worth 10 points and problems #3 and 5 are worth 25 points each. Read directions for each problem carefully. Please show all work needed to arrive at your solutions. Clearly indicate your final answers.

Name :

Problem 1 : Determine whether the statement is true or false. If it is true, explain why. If it is false, give the right result or an example that disproves the statement.

(a) If f is continuous, $\frac{d}{dx}\left(\int_a^b f(x)\,dx\right) = f(x)$.

$\int_a^b f(x)dx$ is a number

False

So $\frac{d}{dx}\left(\int_a^b f(x)dx\right) = 0$

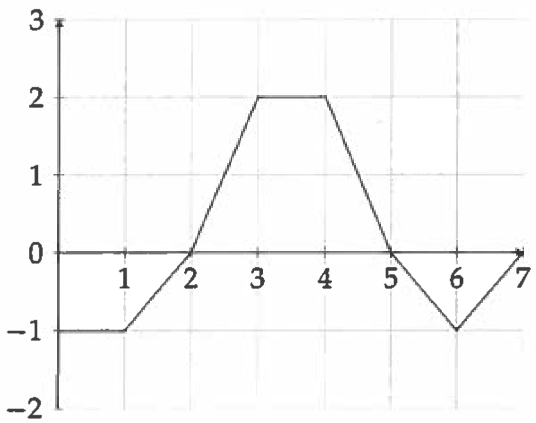
(b) If $\int_0^1 f(x)\,dx = 0$, then $f(x) = 0$ for $0 \leq x \leq 1$.

$\int_0^1 f(x)dx = 0$

but $f \neq 0$

False

(c) Below is the graph of the function g. Let $f(x) = \int_0^x g(t)\,dt$ for all $0 \leq x \leq 7$.



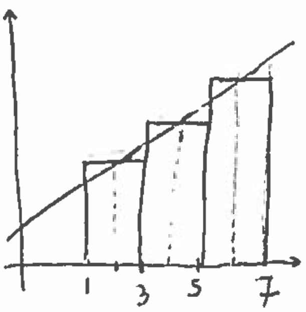
(i) f has a local maximum at $x = 5$.

True $f'(5) = 0$ and f' goes from positive values to negative values.

(ii) $f'(3) = \frac{5}{2}$. $f'(3) = g(3) = 2$ False

(iii) $f(2) = 0$. $f(2) = -\frac{3}{2}$ False

Problem 2 : (a) Use three rectangles and the midpoint rule to approximate the area under the graph $f(x) = \frac{1}{2}x + 1$ and above the x-axis from $x = 1$ to $x = 7$.



$\Delta x = \frac{7-1}{3} = 2$

$A \approx \Delta x f(2) + \Delta x f(4) + \Delta x f(6)$

$= 2(2 + 3 + 4) = \boxed{18}$

(b) Find an expression for the same area as a limit of a Riemann sum. (You do not need to evaluate it.)

$A = \lim_{n \to +\infty} \sum_{i=1}^{n} \Delta x f(x_i) = \lim_{n \to +\infty} \sum_{i=1}^{n} \frac{6}{n} \left(\frac{1}{2}\left(1 + \frac{6i}{n}\right) + 1 \right)$

$\Delta x = \frac{7-1}{n} = \frac{6}{n}$

$x_i = 1 + i\frac{6}{n}$

Problem 3 : (a) Evaluate the following indefinite integrals.

(i) $\int \frac{2+\ln x}{x}\,dx$

Let $u = 2+\ln x$

$du = \frac{dx}{x}$

$= \int u\,du = \frac{u^2}{2} + C = \boxed{\frac{(2+\ln x)^2}{2} + C}$

(ii) $\int \frac{-e^{-2t}+t}{e^{-2t}+t^2}\,dt$

Let $u = e^{-2t} + t^2$

$du = (-2e^{-2t} + 2t)\,dt$

$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln|e^{-2t}+t^2| + C}$

(b) Evaluate the following definite integrals.

(i) $\int_0^1 \frac{x^3}{(1+x^4)^5}\,dx$

Let $u = 1+x^4$

$du = 4x^3\,dx$

when $x=0$ $u=1$

$x=1$ $u=2$

$= \frac{1}{4}\int_1^2 \frac{du}{u^5} = -\frac{1}{4}\left[\frac{1}{4}\frac{1}{u^4}\right]_1^2 = \boxed{\frac{15}{256}}$

(ii) $\int_{\pi/4}^{\pi/2} (\cos^2(y) - \sin^2(y))\,dy$

$\cos^2 y - \sin^2 y = \cos(2y)$

$= \int_{\pi/4}^{\pi/2} \cos(2y)\,dy$

Let $u = 2y$ $du = 2dy$ when $y = \frac{\pi}{4}$ $u = \frac{\pi}{2}$

$y = \frac{\pi}{2}$ $u = \pi$

$= \frac{1}{2}\int_{\pi/2}^{\pi} \cos u\,du = \frac{1}{2}\left[\sin u\right]_{\frac{\pi}{2}}^{\pi} = \boxed{-\frac{1}{2}}$

(c) Evaluate the following.

$$\frac{d}{dx}\left(\int_{1/x}^{\sin(3x)} \sqrt{t^2+1}\,dt\right)$$

$$\frac{d}{dx}\int_{1/x}^{\sin 3x} \sqrt{t^2+1}\,dt = \frac{d}{dx}\left(\int_{1/x}^{0}\sqrt{t^2+1}\,dt + \int_0^{\sin 3x}\sqrt{t^2+1}\,dt\right)$$

$$= \frac{d}{dx}\left(-\int_0^{1/x}\sqrt{t^2+1}\,dt + \int_0^{\sin 3x}\sqrt{t^2+1}\,dt\right)$$

We use the chain rule and get

$$= -\sqrt{\left(\tfrac{1}{x}\right)^2+1}\left(-\frac{1}{x^2}\right) + \sqrt{\sin^2(3x)+1}\,(3\cos 3x)$$

$$\boxed{= \frac{1}{x^2}\sqrt{\frac{1}{x^2}+1} + 3\cos 3x\sqrt{\sin^2(3x)+1}}$$

Problem 4 : A particle moves in a straight line and has acceleration given by $a(t) = -\cos t$ for all $t \geq 0$. Its initial velocity is $1/2$.

(a) Find the velocity $v(t)$ of the particle for all $t \geq 0$.

v has the general form $\quad v(t) = -\sin t + C$

We know $\quad v(0) = \frac{1}{2} = -\sin 0 + C = C$

Thus $\boxed{v(t) = -\sin t + \frac{1}{2}}$

(b) What is the total distance traveled by the particle between times $t = 0$ and $t = \pi/2$?

$$\int_0^{\pi/2} |v(t)|\,dt = \int_0^{\pi/2} \left|-\sin t + \tfrac{1}{2}\right| dt$$

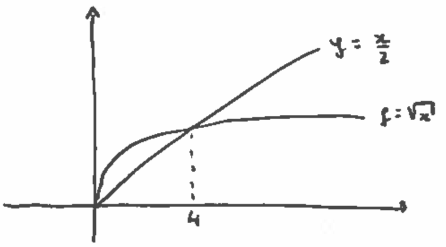
$-\sin t + \frac{1}{2} \geq 0 \qquad \frac{1}{2} \geq \sin t \qquad 0 \leq t \leq \frac{\pi}{6}$

$$= \int_0^{\pi/6} -\sin t + \tfrac{1}{2}\,dt + \int_{\pi/6}^{\pi/2} -\left(-\sin t + \tfrac{1}{2}\right)dt = \left[\cos t + \tfrac{1}{2}t\right]_0^{\pi/6} - \left[\cos t + \tfrac{1}{2}t\right]_{\pi/6}^{\pi/2}$$

$$= \frac{\sqrt{3}}{2} + \frac{\pi}{12} - 1 - \frac{\pi}{4} + \frac{\pi}{12} + \frac{\sqrt{3}}{2} = \boxed{\sqrt{3} - 1 - \frac{\pi}{12}}$$

Problem 5 : Let $\mathcal{R}$ be the region bounded by $y = \sqrt{x}$ and $y = x/2$.

(a) Find the area of $\mathcal{R}$.



Point of intersection :

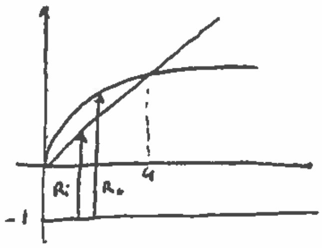
$\sqrt{x} = \frac{x}{2}$ $\qquad x = \frac{x^2}{4}$ $\qquad x^2 = 4x$

$x(x-4) = 0$ $\qquad \boxed{\begin{array}{l} x = 0 \\ x = 4 \end{array}}$

Area

$$\int_0^4 \sqrt{x} - \frac{x}{2}\,dx = \left[\frac{2}{3}x^{3/2} - \frac{x^2}{4}\right]_0^4 \boxed{= \frac{4}{3}}$$

(b) (i) Find the volume obtained by revolving the region $\mathcal{R}$ about the line $y = -1$.



Variable : x

Bounds : $0 \le x \le 4$

Cross section : Washer

$A(x) = \pi(R_o^2 - R_i^2) = \pi\left[(1+\sqrt{x})^2 - (1+\frac{x}{2})^2\right]$

$= \pi\left[\cancel{1} + \cancel{x} + 2\sqrt{x} - \cancel{1} - \frac{x^2}{4} - \cancel{x}\right]$

$= \pi\left[2\sqrt{x} - \frac{x^2}{4}\right]$

Volume

$$V = \pi\int_0^4 2\sqrt{x} - \frac{x^2}{4}\,dx = \pi\left[\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4 \boxed{= \frac{16}{3}\pi.}$$

(ii) Set up an integral to find the volume of the solid whose base is the region $\mathcal{R}$ and whose cross-sections perpendicular to the y-axis are squares with side lying in the xy-plane. (You do not need to evaluate it.)

Variable : y

Bounds : $0 \le y \le 2$

Cross section : square

$A(y) = a^2 = (2y - y^2)^2$

Volume :

$$\boxed{V = \int_0^2 (2y - y^2)^2\,dy}$$