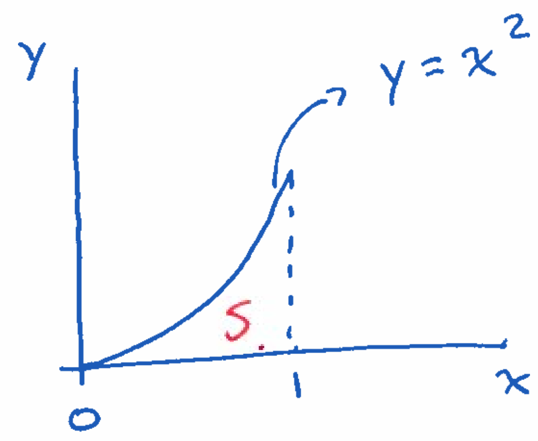
Example :

Let S be



i) Express S as a limit using the right end points

$$A = \lim_{n \to +\infty} R_n = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

now $f(x) = x^2$

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$x_i = a + i\Delta x = 0 + i\frac{1}{n} = \frac{i}{n}$$

then $$A = \lim_{n \to +\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n}$$

$$= \lim_{n \to +\infty} \sum_{i=1}^{n} \frac{i^2}{n^3}$$

n is independent of i

$$= \lim_{n \to +\infty} \frac{1}{n^3} \cdot \sum_{i=1}^{n} i^2$$

ii) Show that $\sum_{i=1}^{n} i = \frac{(n+1)n}{2}$

we have that

$$\sum_{i=1}^{n} i = 1+2+3+\ldots+n$$

but we can reorder it as

$$\sum_{i=1}^{n} i = n+(n-1)+(n-2)+\ldots+2+1.$$

then

$$\sum_{i=1}^{n} i + \sum_{i=1}^{n} i = \begin{array}{l} 1+2+3+\ldots+n \\ n+(n-1)+(n-2)+\ldots+1. \end{array}$$

$$= \underbrace{(n+1)+(n+1)+\ldots+(n+1)}_{n \text{ times.}}$$

$$2\left(\sum_{i=1}^{n} i\right) = n(n+1)$$

$$\Rightarrow \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

(iii) Compute $\displaystyle\sum_{i=1}^{n} (i+1)^3 - \sum_{i=1}^{n} i^3$

we expand both sums

$$\sum_{i=1}^{n} (i+1)^3 = 2^3 + 3^3 + 4^3 + \dots + (n+1)^3$$

$$\sum_{i=1}^{n} i^3 = 1 + 2^3 + 3^3 + \dots + n^3$$

So reordering

$$\begin{aligned}
\sum_{i=1}^{n} (i+1)^3 - \sum_{i=1}^{n} i^3 &= (n+1)^3 \\
&\quad + n^3 - n^3 \to 0 \\
&\quad + (n-1)^3 - (n-1)^3 \\
&\quad\ \vdots \\
&\quad + 3^3 - 3^3 \\
&\quad + 2^3 - 2^3 \\
&\quad - 1 \\
&= (n+1)^3 - 1
\end{aligned}$$

This is normally called a telescopic sum.

iv) Using the last parts show that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

We have that

$$\sum_{i=1}^{n} (i+1)^3 - \sum_{i=1}^{n} i^3 = (n+1)^3 - 1.$$

but

$$\sum_{i=1}^{n} (i+1)^3 - \sum_{i=1}^{n} i^3 = \sum_{i=1}^{n} (i+1)^3 - i^3$$

$$= \sum_{i=1}^{n} (3i^2 + 3i + 1)$$

$$= \sum_{i=1}^{n} 3i^2 + \sum_{i=1}^{n} 3i + \sum_{i=1}^{n} 1$$

$$= 3 \cdot \sum_{i=1}^{n} i^2 + 3 \underbrace{\sum_{i=1}^{n} i}_{= \frac{n(n+1)}{2}} + \underbrace{\sum_{i=1}^{n} 1}_{= n}$$

Therefore.

$$3 \cdot \sum_{i=1}^{n} i^2 + \frac{3n(n+1)}{2} + n = (n+1)^3 - 1.$$

$$\Rightarrow \quad \sum_{i=1}^{n} i^2 = (n+1)^3 - 1 - \frac{3n(n+1)}{2} - n$$

then

$$3\left(\sum_{i=1}^{n} i^2\right) = n^3 + 3n^2 + 3n + 1 - 1 - \frac{3n^2}{2} - \frac{3n}{2} - n.$$

$$= n^3 + 3n^2\left(1 - \frac{1}{2}\right) + \left(3 - \frac{3}{2} - 1\right)n$$

$$= n^3 + \frac{3}{2}n^2 + \frac{5}{2}n = n\left(n^2 + \frac{3}{2}n + \frac{1}{2}\right)$$

$$= \frac{n}{2}\cdot\left(2n^2 + 3n + 1\right)$$

$$= \frac{n}{2}(2n+1)(n+1)$$

then $\sum_{i=1}^{n} i^2 = \frac{1}{3}\cdot\frac{n}{2}(2n+1)(n+1)$

$$= \frac{n(2n+1)(n+1)}{6}$$

$\blacksquare$

*) Using the fact that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

compute the limit in part i)

we had that

$$A = \lim_{n \to +\infty} \frac{1}{n^3} \cdot \sum_{i=1}^{n} i^2$$

using the result of the sum.

$$A = \lim_{n \to +\infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n \to +\infty} \frac{2n^3 + 3n^2 + n}{6n^3}$$

$$= \lim_{n \to +\infty} \frac{2n^3}{6n^3} + \frac{3n^2}{6n^3} + \frac{n}{6n^3}$$

$$= \lim_{n \to +\infty} \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

$$= \frac{1}{3} + \underbrace{\lim_{n \to +\infty} \frac{1}{2n}}_{\to 0} + \underbrace{\lim_{n \to +\infty} \frac{1}{6n^2}}_{\to 0}$$

$$= \boxed{\frac{1}{3}}$$