MATH 2B: SAMPLE MIDTERM #2

- This exam consists of 5 questions and 90 total points.
- Read the directions for each problem carefully and answer all parts of each problem.
- Please show all work needed to arrive at your solutions (unless instructed otherwise). Label graphs and define any notation used. Cross out incorrect scratch-work.
- No calculators or other forms of assistance are allowed. Do not check your cell phones during the exam.
- Clearly indicate your final answer to each problem.

1. (10 points each) Evaluate each of the following integrals.

a. $\int \frac{\ln t}{t^5}\, dt$

Integration by parts

Let $u = \ln t$ $\quad v = -\frac{1}{4}\frac{1}{t^4}$

$du = \frac{dt}{t}$ $\quad dv = \frac{dt}{t^5}$

$$\int \frac{\ln t}{t^5}\, dt = -\frac{1}{4}\frac{1}{t^4}\ln t + \frac{1}{4}\int \frac{1}{t^5}\, dt = \boxed{-\frac{1}{4}\frac{1}{t^4}\ln t - \frac{1}{16}\frac{1}{t^4} + C}$$

b. $\int e^x \sin(3x)\, dx$

Integration by parts Let $u = \sin 3x$ $\quad v = e^x$

$du = 3\cos 3x\, dx$ $\quad dv = e^x\, dx$

$$\int e^x \sin 3x\, dx = e^x \sin 3x - 3\int \cos 3x\, e^x\, dx$$

Integration by parts again Let $u = \cos 3x$ $\quad v = e^x$

$du = -3\sin 3x\, dx$ $\quad dv = e^x\, dx$

$$\int e^x \sin 3x\, dx = e^x \sin 3x - 3\left[\cos 3x\, e^x + 3\int \sin 3x\, e^x\, dx\right]$$

$$= e^x \sin 3x - 3\cos 3x\, e^x - 9\int \sin 3x\, e^x\, dx$$

Thus $\int e^x \sin 3x\, dx \boxed{= \frac{1}{10}\left[e^x \sin 3x - 3\cos 3x\, e^x\right]}$

c. $\int \sin^5\theta\,d\theta = \int \sin^4\theta \sin\theta\,d\theta = \int (1-\cos^2\theta)^2 \sin\theta\,d\theta$

We make the substitution $u = \cos\theta$

$du = -\sin\theta\,d\theta$

$$\int (1-\cos^2\theta)^2 \sin\theta\,d\theta = -\int (1-u^2)^2\,du = -\int 1+u^4-2u^2\,du$$

$$= -u - \frac{u^5}{5} + \frac{2u^3}{3} + C$$

$$\boxed{= -\cos\theta - \frac{\cos^5\theta}{5} + \frac{2}{3}\cos^3\theta + C}$$

d. $\displaystyle\int_{2\sqrt{2}}^{4} \frac{1}{x\sqrt{x^2-4}}\,dx$ We make the trig substitution $x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = 2\sqrt{\sec^2\theta - 1} = 2\sqrt{\tan^2\theta} = 2\tan\theta$$

when $x = 2\sqrt{2}$ $\quad 2\sec\theta = 2\sqrt{2}$ $\quad \sec\theta = \sqrt{2}$ $\quad \cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad \theta = \frac{\pi}{4}$

$x = 4$ $\quad 2\sec\theta = 4$ $\quad \sec\theta = 2$ $\quad \cos\theta = \frac{1}{2}$ $\quad \theta = \frac{\pi}{3}$

$$\int_{\pi/4}^{\pi/3} \frac{1}{2\sec\theta \cdot 2\tan\theta}\, 2\sec\theta\tan\theta\,d\theta = \frac{1}{2}\int_{\pi/4}^{\pi/3} d\theta$$

$$= \frac{1}{2}\left[\frac{\pi}{3} - \frac{\pi}{4}\right] = \boxed{\frac{\pi}{24}}$$

2. (15 points) Determine whether the following improper integrals are convergent or divergent. Evaluate those that are convergent.

a. $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x}}$

Let $t \geq 2$

$$\int_2^t \frac{dx}{\sqrt{x}} = \left[2\sqrt{x}\right]_2^t = 2\sqrt{t} - 4 \xrightarrow[t \to +\infty]{} +\infty$$

$\boxed{\text{Divergent}}$

b. $\displaystyle\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$

Discontinuity at $x = 3$. Let $0 \leq t < 3$.

$$\int_0^t \frac{1}{\sqrt{9-x^2}}\,dx$$

Trig sub $x = 3\sin\theta$ $dx = 3\cos\theta\, d\theta$

when $x = 0$ $3\sin\theta = 0$ $\sin\theta = 0$ $\theta = 0$

$x = t$ $3\sin\theta = t$ $\sin\theta = \frac{t}{3}$ $\theta = \sin^{-1}\left(\frac{t}{3}\right)$

$\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta} = 3\sqrt{1-\sin^2\theta} = 3\cos\theta$

$$= \int_0^{\sin^{-1}(\frac{t}{3})} \frac{\cancel{3\cos\theta}\, d\theta}{\cancel{3\cos\theta}}$$

$$= \left[\theta\right]_0^{\sin^{-1}(\frac{t}{3})} = \sin^{-1}\left(\frac{t}{3}\right)$$

$\lim_{t \to 3} \sin^{-1}\left(\frac{t}{3}\right) = \sin^{-1}(1) = \frac{\pi}{2}$

$\boxed{\text{Convergent} \quad \int_0^3 \frac{dx}{\sqrt{9-x^2}} = \frac{\pi}{2}}$

c. $\displaystyle\int_0^\infty \frac{dz}{z^2+3z+2}$

Let $t \geq 0$.

$$\int_0^t \frac{dz}{z^2+3z+2}$$

① Proper ② $z^2 + 3z + 2 = (z+1)(z+2)$

③ $\dfrac{1}{(z+1)(z+2)} = \dfrac{A}{z+1} + \dfrac{B}{z+2} = \dfrac{(A+B)z + 2A + B}{(z+1)(z+2)}$

$\begin{cases} A+B = 0 \\ 2A+B = 1 \end{cases}$ $\begin{cases} B = -A \\ 2A - A = A = 1 \end{cases}$ $\begin{cases} A = 1 \\ B = -1 \end{cases}$

④ $\displaystyle\int_0^t \frac{dz}{z^2+3z+2} = \int_0^t \frac{dz}{z+1} - \int_0^t \frac{dz}{z+2} = \left[\ln|z+1|\right]_0^t - \left[\ln|z+2|\right]_0^t$

$$= \ln t+1 - \ln t+2 + \ln 2 = \ln\left(\frac{t+1}{t+2}\right) + \ln 2$$

$\dfrac{t+1}{t+2} \xrightarrow[t \to +\infty]{} 1$ Thus $\ln\left(\dfrac{t+1}{t+2}\right) \xrightarrow[t \to +\infty]{} 0$

$\boxed{\text{Convergent.} \quad \int_0^{+\infty} \frac{dz}{z^2+3z+2} = \ln 2}$

3. (10 points) Find the length of the curve $f(x) = x^3 + \frac{1}{12x}$ on the interval $[\frac{1}{2}, 2]$

$f'(x) = 3x^2 - \frac{1}{12} \frac{1}{x^2}$

$\sqrt{1+f'(x)^2} = \sqrt{1+(3x^2 - \frac{1}{12x^2})^2} = \sqrt{1+9x^4 - \frac{1}{2} + \frac{1}{144x^4}} = \sqrt{9x^4 + \frac{1}{2} + \frac{1}{144x^4}} = \sqrt{(3x^2 + \frac{1}{12x^2})^2}$

$= 3x^2 + \frac{1}{12x^2}$

$L = \int_{\frac{1}{2}}^{2} \sqrt{1+f'(x)^2}\, dx = \int_{\frac{1}{2}}^{2} 3x^2 + \frac{1}{12x^2}\, dx = \left[x^3 - \frac{1}{12x}\right]_{\frac{1}{2}}^{2} = 8 - \frac{1}{24} - \frac{1}{8} + \frac{1}{6}$

$\boxed{= 8}$

4. (10 points) Determine whether each of the following statements is true or false. Briefly justify your answers.

a. True/False: If $\{a_n\}$ is decreasing and $a_n > 0$ for all n, then a_n is convergent.

Yes. $(a_n)_{n \geq 0}$ is decreasing and has a lower bound. So it is convergent.

b. True/False: If $f(x) \leq g(x)$ and $\int_0^\infty g(x)\, dx$ diverges, then $\int_0^\infty f(x)\, dx$ also diverges.

False $\frac{1}{x^2+1} \leq x$ $\int_0^{+\infty} x\, dx$ diverges $\int_0^{+\infty} \frac{dx}{x^2+1}$ converges

c. True/False: The integral $\displaystyle\int_1^\infty \frac{1}{x^\pi}\, dx$ converges.

True. Since $\pi > 1$.

d. True/False: $\int_0^3 e^{x^2}\, dx = \int_0^5 e^{x^2}\, dx + \int_5^3 e^{x^2}\, dx$.

True. $\int_0^5 e^{x^2} dx + \int_5^3 e^{x^2} dx = \int_0^5 e^{x^2} dx - \int_3^5 e^{x^2} dx = \int_0^3 e^{x^2} dx$

5. (15 points) Determine whether each of the following sequences is convergent or divergent. If a sequence is convergent, find its limit.

a. $a_n = n\sin(\frac{1}{n})$

Let $f(x) = x \sin\left(\frac{1}{x}\right)$. $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$

$= \lim_{x\to+\infty} \frac{-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)}{-\frac{1}{x^2}} = \lim_{x\to+\infty} \cos\left(\frac{1}{x}\right) = \cos(0) = 1$

$\boxed{\text{Thus } \lim_{n\to+\infty} a_n = 1 \quad \text{convergent}}$

b. $a_n = \sin^{-1}(\frac{3n}{3n+8})$

$\lim_{n\to+\infty} \frac{3n}{3n+8} = \lim_{n\to+\infty} \frac{\cancel{3n}}{\cancel{3n}} = 1$

$\sin^{-1}$ is continuous at 1.

Thus $\lim_{n\to+\infty} a_n = \sin^{-1}(1) \boxed{= \frac{\pi}{2}}$ $\boxed{\text{convergent}}$

c. $a_n = -5 + (0.9)^n$

$|0.9| < 1$ so $0.9^n \xrightarrow[n\to+\infty]{} 0$

$\boxed{\lim_{n\to+\infty} a_n = -5 \quad \text{convergent}}$

d. $a_n = 4 + (-1)^n$

$(-1)^n$ oscillates and has no limit.

$\boxed{\text{Thus } (a_n)_{n\geq 0} \text{ is divergent.}}$

e. $a_n = \frac{n^2+2n-12}{n+2}$

$\lim_{n\to+\infty} \frac{n^2+2n-12}{n+2} = \lim_{n\to+\infty} \frac{n^2}{n} = +\infty$ $\boxed{\text{divergent}}$