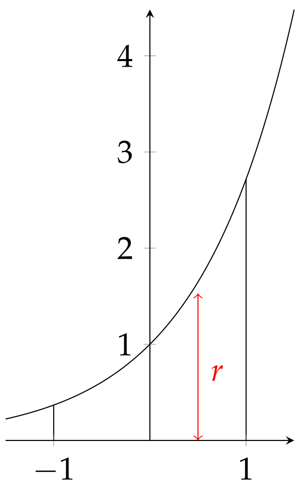
Fall 2016 Math 2B
Suggested homework problems solutions

Volumes

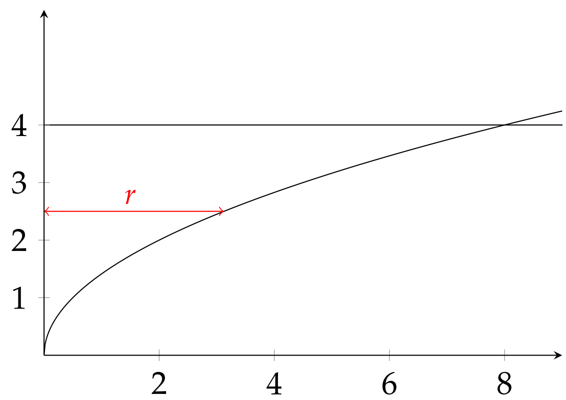
Problem 4 : The cross-section at x perpendicular to the x-axis is the disk with radius e^x, so its area is $A(x) = \pi(e^x)^2 = \pi e^{2x}$.

$$V = \int_{-1}^{1} A(x)\,dx = \pi \int_{-1}^{1} e^{2x}\,dx = \pi \left[\frac{1}{2}e^{2x}\right]_{-1}^{1} = \frac{\pi}{2}(e^2 - e^{-2}).$$



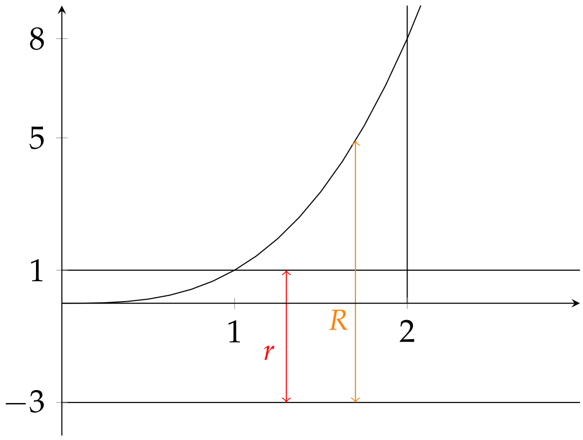
Problem 6 : A cross-section is a disk with radius $y^2/2$, so its area is $A(y) = \pi y^4/4$.

$$V = \frac{\pi}{4}\int_0^4 y^4\,dy = \frac{\pi}{4}\left[\frac{1}{5}y^5\right]_0^4 = \frac{\pi 4^4}{5}.$$



Problem 12 : A cross-section at x perpendicular to the x-axis is a washer with inner radius $r = 3 + 1 = 4$ and outer radius $R = 3 + x^3$ so its area is $A(y) = \pi((3 + x^3)^2 - 4^2) = \pi(x^6 + 6x^3 - 7)$.

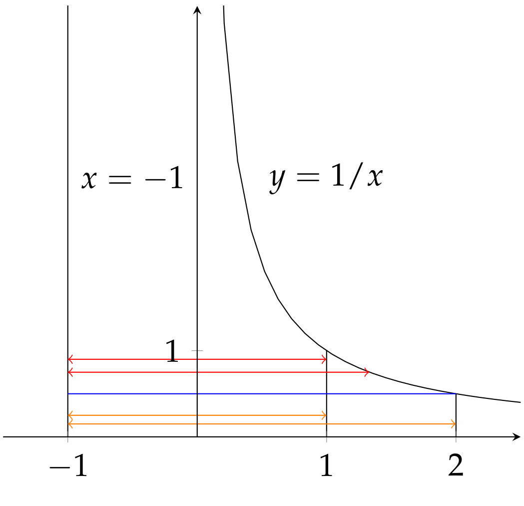
$$V = 4\pi \int_1^2 (x^6 + 6x^3 - 7)\, dx = \pi \left[\frac{x^7}{7} + \frac{3}{2}x^4 - 7x \right]_1^2 = \pi \frac{471}{14}.$$



Problem 16 : For $0 \leq y \leq 1/2$, a cross-section is a washer with inner radius 2 and outer radius 3, so its area $A(y) = \pi(3^2 - 2^2) = 5\pi$.

For $1/2 \leq y \leq 1$, a cross-section is a washer with inner radius 2 and outer radius $1 + 1/y$, so its area $A(y) = \pi((1 + 1/y)^2 - 2^2) = \pi(1/y^2 + 2/y - 3)$.

$$V = 5\pi \int_0^{1/2} dy + \pi \int_{1/2}^1 (\frac{1}{y^2} + \frac{2}{y} - 3)\, dy = \frac{5\pi}{2} + \left[-\frac{1}{y} + 2\ln y - 3y \right]_{1/2}^1 = 2\pi(1 + \ln 2).$$



Problem 48 :

$$V = \frac{\pi h}{3}(R^2 + Rr + r^2).$$

Problem 49 :

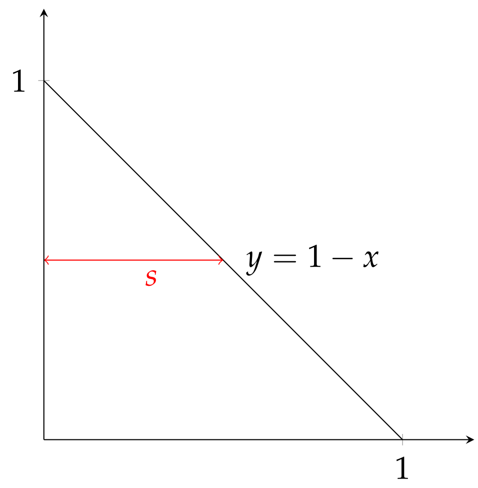
$$V = \frac{\pi h^2}{3}(3r - h).$$

Problem 56 : The cross-section at y perpendicular to the y-axis has length $1 - y$. The corresponding equilateral triangle with side s has area

$$A(y) = s^2(\frac{\sqrt{3}}{4}) = (\frac{\sqrt{3}}{4})(1-y)^2.$$

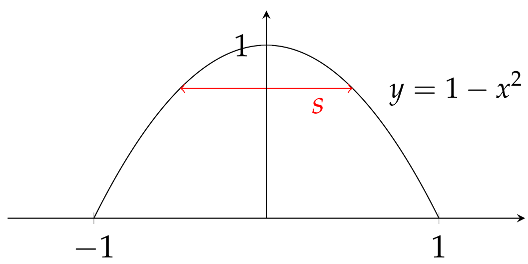
Therefore

$$V = \int_0^1 A(y)\,dy = (\frac{\sqrt{3}}{4}) \int_0^1 (1-y)^2\,dy = (\frac{\sqrt{3}}{4}) \left[\frac{(1-y)^3}{3}\right]_0^1 = \frac{\sqrt{3}}{12}.$$



Problem 58 : The cross-section at y perpendicular to the y-axis has length $2\sqrt{1-y}$. The corresponding square with side s has area $A(y) = (2\sqrt{1-y})^2 = 4(1-y)$. Therefore

$$V = \int_0^1 A(y)\,dy = 4\int_0^1 (1-y)\,dy = 2.$$



Problem 68 : We consider two cases : one in which the ball is not completly submerged and the other in which it is.

Case 1 : $0 \le h \le 10$ The ball will not be completly submerged, and so a cross-section of the water at height x parallel to the surface will be a washer of inner radius $r = \sqrt{5^2 - (5-x)^2}$ and outer radius $R = \sqrt{15^2 - (15-x)^2}$. So

$$A(x) = \pi(R^2 - r^2) = 20\pi x.$$

The volume of water when it has depth h is then

$$V = \int_0^h A(x)\,dx = 10\pi h^2.$$

Case 2 : $10 \le h \le 15$ In this case, we find the volume by simply subtracting the volume displaced by the ball from the total volume inside the bowl underneath the surface of the water. The total volume underneath the water is just the volume of a cap of the bowl, so we use the formula from Exercise 49:

$$V_{cap} = \frac{1}{3}\pi h^2(45 - h).$$

The volume of the small sphere is

$$V_{ball} = \frac{4}{3}\pi(5)^3,$$

so the total volume is

$$V_{total} = V_{cap} - V_{ball} = \frac{\pi}{3}(45h^2 - h^3 - 500).$$

Average value of a function

Problem 2 :

$$f_{ave} = \frac{1}{4-0}\int_0^4 \sqrt{x}\,dx = \frac{1}{4}\left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{4}{3}.$$

Problem 4 :

$$g_{ave} = \frac{1}{3-1}\int_1^3 \frac{t}{\sqrt{3+t^2}}\,dt = \frac{1}{2}\left[(3+t^2)^{1/2}\right]_1^3 = \sqrt{3} - 1.$$

Problem 5 : Let $u = \sin t$. We have $du = \cos t\,dt$. When $t = 0$, $u = 0$. When $t = \pi/2$, $u = 1$.

$$f_{ave} = \frac{1}{\pi/2 - 0}\int_0^{\pi/2} e^{\sin t}\cos t\,dt = \frac{2}{\pi}\int_0^1 e^u\,du = \frac{2}{\pi}\left[e^u\right]_0^1 = \frac{2}{\pi}(e-1).$$

Problem 10 : (a)

$$f_{ave} = \frac{1}{3-1}\int_1^3 \frac{1}{x}\,dx = \frac{1}{2}\left[\ln|x|\right]_1^3 = \frac{1}{2}\ln 3.$$

(b)

$$f(c) = f_{ave} \Leftrightarrow \frac{1}{c} = \frac{1}{2}\ln 3 \Leftrightarrow c = \frac{2}{\ln 3}.$$