Fall 2016 Math 2B
Suggested homework problems solutions

Sections	Topics covered	Problems
5.4	Indefinite integrals	6, 8, 10, 15, 16, 18, 26, 30, 31, 33, 34, 44, 46, 52, 54, 60, 67
5.5	Substitution rule	4, 8, 9, 10, 11, 16, 18, 19, 21, 28, 32, 34, 35, 38, 54, 59, 71, 73, 82, 88
6.1	Area between curves	1, 3, 5, 7, 10, 14, 17, 19, 22, 25, 58

Indefinite integrals and net change theorem

Problem 6 :

$$\int \sqrt[4]{x^5}\,dx = \int x^{5/4}\,dx = \frac{4}{9}x^{9/4} + C.$$

Problem 8 :

$$\int (u^6 - 2u^5 - u^3 + \frac{2}{7})\,du = \frac{u^7}{7} - \frac{u^6}{3} - \frac{u^4}{4} + \frac{2}{7}u + C.$$

Problem 10 :

$$\int \sqrt{t}(t^2 + 3t + 2)\,dt = \int (t^{5/2} + 3t^{3/2} + 2t^{1/2})\,dt = \frac{2}{7}t^{7/2} + \frac{6}{5}t^{5/2} + \frac{4}{3}t^{3/2} + C.$$

Problem 15 :

$$\int (2 + \tan^2\theta)\,d\theta = \int (1 + 1 + \tan^2\theta)\,d\theta = \theta + \tan\theta + C.$$

Problem 16 :

$$\int \sec t(\sec t + \tan t)\,dt = \tan t + \sec t + C.$$

Problem 18 :

$$\int \frac{\sin(2x)}{\sin x}\,dx = \int \frac{2\cos x\sin x}{\sin x}\,dx = 2\int \cos x\,dx = 2\sin x + C.$$

Problem 26 :

$$\int_{-1}^{1} t(1-t)^2\,dt = \int_{-1}^{1} (t-2t^2+t^3)\,dt = \left[\frac{1}{2}t^2 - \frac{2}{3}t^3 + \frac{1}{4}t^4\right]_{-1}^{1} = -\frac{4}{3}.$$

Problem 30 :

$$\int_0^1 \frac{4}{1+p^2}\,dp = 4\,[\arctan p]_0^1 = \pi.$$

Problem 31 :

$$\int_0^1 x(\sqrt[3]{x}+\sqrt[4]{x})\,dx = \int_0^1 (x^{4/3}+x^{5/4})\,dx = \left[\frac{3}{7}x^{7/3} + \frac{4}{9}x^{9/4}\right]_0^1 = \frac{55}{63}.$$

Problem 33 :

$$\int_1^2 (\frac{x}{2} - \frac{2}{x})\,dx = \left[\frac{x^2}{4} - 2\ln|x|\right]_1^2 = \frac{3}{4} - 2\ln 2.$$

Problem 34 :

$$\int_0^1 (5x-5^x)\,dx = \left[\frac{5}{2}x^2 - \frac{5^x}{\ln 5}\right]_0^1 = \frac{5}{2} - \frac{4}{\ln 5}.$$

Problem 44 :

$$|2x-1| = \begin{cases} 1-2x, & \text{if } 0 \le x < 1/2, \\ 2x-1, & \text{if } x \ge 1/2. \end{cases}$$

Thus

$$\int_0^2 |2x-1|\,dx = \int_0^{1/2} (1-2x)\,dx + \int_{1/2}^2 (2x-1)\,dx = \left[x-x^2\right]_0^{1/2} + \left[x^2-x\right]_{1/2}^2 = \frac{5}{2}.$$

Problem 46 :

$$\int_0^{3\pi/2} |\sin x|\,dx = \int_0^{\pi} \sin x\,dx + \int_{\pi}^{3\pi/2} (-\sin x)\,dx = [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{3\pi/2} = 3.$$

Problem 52 : By the net change theorem, we have

$$\int_a^b I(t)\,dt = \int_a^b Q'(t)\,dt = Q(b) - Q(a).$$

It represents the change in the charge Q from time $t = a$ and time $t = b$.

Problem 54 : By the net change theorem, we have

$$\int_0^{15} n'(t)\,dt = n(15) - n(0) = n(15) - 100.$$

It represents the increase in the bee population in 15 weeks. So $100 + \int_0^{15} n'(t)\,dt = n(15)$ is the total bee population after 15 weeks.

Problem 60 : (a) Displacement

$$\int_2^4 (t^2 - 2t - 3)\,dt = \left[\frac{1}{3}t^3 - t^2 - 3t\right]_2^4 = \frac{2}{3}.$$

(b) Distance traveled

$t \mapsto t^2 - 2t - 3$ is a polynomial with degree two and a positive dominant coefficient. It is therefore negative between its roots and positive outside.

$$|t^2 - 2t - 3| = |(t+1)(t-3)| = \begin{cases} t^2 - 2t - 3, & \text{if } x \leq -1 \text{ or } 3 \leq x, \\ -t^2 + 2t + 3, & \text{if } -1 \leq x \leq 3. \end{cases}$$

$$\begin{aligned} \int_2^4 |t^2 - 2t - 3|\,dt &= \int_2^3 (-t^2 + 2t + 3)\,dt + \int_3^4 (t^2 - 2t - 3)\,dt \\ &= \left[-\frac{1}{3}t^3 + t^2 + 3t\right]_2^3 + \left[\frac{1}{3}t^3 - t^2 - 3t\right]_3^4 = 4. \end{aligned}$$

Problem 67 : From the Net Change Theorem, the increase in cost if the production level is raised from 2000 yards to 4000 yards is $C(4000) - C(2000) = \int_{2000}^{4000} C'(x)\,dx$.

$$\int_{2000}^{4000} C'(x)\,dx = \int_{2000}^{4000} (3 - \frac{1}{100}x + \frac{6}{10^6}x^2)\,dx = \left[3x - \frac{1}{200}x^2 + \frac{2}{10^6}x^3\right]_{2000}^{4000} = 58000.$$

Substitution rule

Problem 4 : Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$. So

$$\int \sin^2\theta\cos\theta\,d\theta = \int u^2\,du = \frac{u^3}{3} + C = \frac{\sin^3\theta}{3} + C.$$

Problem 8 : Let $u = x^3$. Then $du = 3x^2\,dx$, and $x^2\,dx = \frac{1}{3}du$. So

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C.$$

Problem 9 : Let $u = 1 - 2x$. Then $du = -2\,dt$. So

$$\int (1-2x)^9\,dx = -\frac{1}{2}\int u^9 du = -\frac{u^{10}}{20} + C = -\frac{(1-2x)^{10}}{20} + C.$$

Problem 10 : Let $u = 1 + \cos t$. Then $du = -\sin t\,dt$. So

$$\int \sin t\sqrt{1+\cos t}\,dt = -\int \sqrt{u}du = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}(1+\cos t)^{3/2} + C.$$

Problem 11 : Let $u = \pi t/2$. Then $du = \frac{\pi}{2}\,dt$. So

$$\int \cos(\pi t/2)\,dx = \frac{2}{\pi}\int \cos u\,du = \frac{2}{\pi}\sin u + C = \frac{2}{\pi}\sin(\pi t/2) + C.$$

Problem 16 : Let $u = -5r$. Then $du = -5\,dr$. So

$$\int e^{-5r}\,dr = -\frac{1}{5}\int e^u\,du = -\frac{1}{5}e^u + C = -\frac{1}{5}e^{-5r} + C.$$

Problem 18 : Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$. So

$$\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx = 2\int \sin(u)\,du = -2\cos u + C = -2\cos(\sqrt{x}) + C.$$

Problem 19 : Let $u = 3ax + bx^3$. Then $du = (3a + 3bx^2)\,dx = 3(a + bx^2)\,dx$. So

$$\int \frac{a+bx^2}{\sqrt{3ax+bx^3}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{u}}\,du = \frac{2}{3}u^{1/2} + C = \frac{2}{3}\sqrt{3ax+bx^3} + C.$$

Problem 21 : Let $u = \ln x$. Then $du = dx/x$. So

$$\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C.$$

Problem 28 : Let $u = \cos t$. Then $du = -\sin t\,dt$. So

$$\int e^{\cos t}\sin t\,dt = -\int e^u\,du = -e^u + C = -e^{\cos t} + C$$

Problem 32 : Let $u = x^2 + 4$. Then $du = 2\,dx$. So

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+4) + C.$$

Problem 34 : Let $u = \frac{\pi}{x}$. Then $du = \frac{\pi}{x^2}\,dx$. So

$$\int \frac{\cos(\pi/x)}{x^2}\,dx = -\frac{1}{\pi}\int \cos u\,du = -\frac{1}{\pi}\sin u + C = -\frac{1}{\pi}\sin(\frac{\pi}{x}) + C.$$

Problem 35 : Let $u = \cot x$. Then $du = -\csc^2 x\,dx$. So

$$\int \sqrt{\cot x}\csc^2 x \ = -\int \sqrt{u}\,du = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}(\cot x)^{3/2} + C.$$

Problem 38 : Let $u = 1 + \tan t$. Then $du = \sec^2 t\,dt$. So

$$\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}} \ = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{1+\tan t} + C.$$

Problem 54 : Let $u = 3t - 1$. Then $du = 3\,dt$. When $t = 0, u = -1$, when $t = 1, u = 2$. So

$$\int_0^1 (3t-1)^{50}\,dx = \frac{1}{3}\int_{-1}^{2} u^{50}\,du = \frac{1}{3}\left[\frac{u^{51}}{51}\right]_{-1}^{2} = \frac{2^{51}+1}{153}.$$

Problem 59 : Let $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2}\,dx$. When $x = 1, u = 1$, when $x = 2, u = 1/2$. So

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = -\int_1^{1/2} e^u\,du = -\left[e^u\right]_1^{1/2} = e - \sqrt{e}.$$

Problem 71 : Let $u = e^z + z$. Then $du = (e^z + 1)\,dz$. When $z = 0, u = 1$, when $z = 1$, $u = e + 1$. Thus,

$$\int_0^1 \frac{e^z+1}{e^z+z}\,dz = \int_1^{e+1} \frac{1}{u}\,du = \left[\ln|u|\right]_1^{e+1} = \ln(e+1).$$

Problem 73 : Let $u = 1 + \sqrt{x}$. So $du = \frac{1}{2\sqrt{x}}\,dx$ and $dx = 2(u-1)\,du$. When $x = 0, u = 1$, when $x = 1, u = 2$. Thus

$$\int_0^1 \frac{dx}{(1+\sqrt{x})^4} = \int_1^2 \frac{2(u-1)}{u^4}\,du = 2\int_1^2 (\frac{1}{u^3} - \frac{1}{u^4})\,du = \frac{1}{6}.$$

Problem 82 : Let $a = 450.268$, and $b = 1.12567$. Then $r(t) = ae^{bt}$. Since we start from 400 bacteria, the population will be :

$$n(3) = 400 + \int_0^3 r(t)\,dt = 400 + \left[\frac{a}{b}e^{bt}\right]_0^3 = 400 + \frac{a}{b}(e^{3b} - 1) \approx 11713.$$

Problem 88 : Let $u = x^2$. Then $du = 2\,dx$. So

$$\int_0^3 xf(x^2)\,dx = \frac{1}{2}\int_0^9 f(u)\,du = 2.$$

Area between curves

Problem 1 : Solution at the end of your book.

Problem 3 : Solution at the end of your book.

Problem 5 : Solution at the end of your book.

Problem 7 : Solution at the end of your book.

Problem 10 : By observation, $y = \sin x$ and $y = 2x/\pi$ intersect at $(0,0)$ and $(\pi/2, 1)$, for $x \geq 0$. So

$$A = \int_0^{\pi/2} (\sin x - \frac{2\pi}{x})\,dx = \left[-\cos x - \frac{1}{\pi}x^2\right]_0^{\pi/2} = 1 - \frac{\pi}{4}.$$

Problem 14 : Points of intersection

$$x^2 = 4x - x^2 \Leftrightarrow 2x^2 - 4x = 0 \Leftrightarrow 2x(x-2) = 0 \Leftrightarrow x = 0 \text{ or } x = 2.$$

So

$$A = \int_0^2 [(4x - x^2) - x^2]\,dx = \int_0^2 (4x - 2x^2)\,dx = \left[2x^2 - \frac{2}{3}x^3\right]_0^2 = \frac{8}{3}.$$

Problem 17 : Points of intersection

$$2y^2 = 4 + y^2 \Leftrightarrow y^2 = 4 \Leftrightarrow y = -2 \text{ or } y = 2.$$

So

$$A = \int_{-2}^2 [(4 + y^2) - 2y^2]\,dx = 2\int_{-2}^2 (4 - y^2)\,dx = \left[4y - \frac{1}{3}y^3\right]_{-2}^2 = \frac{32}{3}.$$

Problem 19 : By inspection, the curves intersect at $x = \pm\frac{1}{2}$. So by symmetry

$$A = 2\int_0^{1/2} (\cos(\pi x) - 4x^2 + 1)\, dx = 2\left[\frac{1}{\pi}\sin(\pi x) - \frac{4}{3}x^3 + x\right]_0^{1/2} = \frac{2}{\pi} + \frac{2}{3}.$$

Problem 22 : We sketch the two curves. There is a symmetry about $O(0,0)$.

Points of intersection

$$x^3 = x \Leftrightarrow x^3 - x = 0 \Leftrightarrow x(x+1)(x-1) = 0 \Leftrightarrow x = -1 \text{ or } x = 0 \text{ or } x = 1.$$

We compute the area of the region for $0 \leq x \leq 1$ and double it to have the total area.

For all $0 \leq x \leq 1$, we have $x^3 \leq x$. Thus the area is given by

$$\int_0^1 (x - x^3)\, dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{4}.$$

Thus the total area between the two curves is $\frac{1}{2}$.

Problem 25 : We sketch the two curves. There is a symmetry about the y-axis. We compute the area of the region for $x \geq 0$ and double it to have the total area.

Points of intersection for $x \geq 0$

$$x^4 = 2 - x \Leftrightarrow x^4 + x - 2 = 0 \Leftrightarrow (x-1)(x^3 + x^2 + x - 2) = 0.$$

The point that interests us is $x = 1$.

For all $0 \leq x \leq 1$, we have $x^4 \leq 2 - x$. Thus the area is given by

$$\int_0^1 (2 - x - x^4)\, dx = \left[2x - \frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \frac{13}{10}.$$

Thus the total area between the two curves is $\frac{13}{5}$.

Problem 28 : Points of intersection

$$\frac{1}{4}x^2 = -x + 3 \Leftrightarrow x^2 + 4x - 12 = 0 \Leftrightarrow (x+6)(x-2) = 0 \Leftrightarrow x = -6 \text{ or } x = 2.$$

and

$$2x^2 = -x + 3 \Leftrightarrow 2x^2 + x - 3 = 0 \Leftrightarrow (2x+3)(x-1) = 0 \Leftrightarrow x = -\frac{3}{2} \text{ or } x = 1.$$

So

$$\begin{aligned} A &= \int_0^1 (2x^2 - \frac{1}{4}x^2)\, dx + \int_1^2 [(-x+3) - \frac{1}{4}x^2]\, dx \\ &= \int_0^1 \frac{7}{4}x^2\, dx + \int_1^2 (-\frac{1}{4}x^2 - x + 3)\, dx \\ &= \left[\frac{7}{12}x^3\right]_0^1 + \left[-\frac{1}{12}x^3 - \frac{1}{2}x^2 + 3x\right]_1^2 = \frac{3}{2}. \end{aligned}$$

Problem 58 : **(a)** We want to choose a such that

$$\int_1^a \frac{1}{x^2}\,dx = \int_a^4 \frac{1}{x^2}\,dx \Rightarrow \left[-\frac{1}{x}\right]_1^a = \left[-\frac{1}{x}\right]_a^4 \Rightarrow -\frac{1}{a}+1 = -\frac{1}{4}+\frac{1}{a} \Rightarrow a = \frac{8}{5}.$$

(b) The area under the curve $y = 1/x^2$ from $x = 1$ to $x = 4$ is $3/4$. (Take $a = 4$ in the first integral in part (a)). Now the line $y = b$ must intersect with the curve $x = 1/\sqrt{y}$ and not the line $x = 4$, since the area under the line $y = 1/4^2$ from $x = 1$ to $x = 4$ is only $3/16$, which is less than half of $3/4$. We want to choose b so that the upper area in the diagram is half of the total area under the curve $y = 1/x^2$ from $x = 1$ to $x = 4$. This implies that

$$\int_b^1 \left(\frac{1}{\sqrt{y}} - 1\right) dy = \frac{1}{2}\frac{3}{4} \Rightarrow [2\sqrt{y} - y]_b^1 = \frac{3}{8} \Rightarrow b - 2\sqrt{b} + \frac{5}{8} = 0.$$

Letting $c = \sqrt{b}$, we get $c^2 - 2c + \frac{5}{8} = 0$ and thus $c = 1 \pm \frac{\sqrt{6}}{4}$. But $c = \sqrt{b} < 1$. Then $c = 1 - \frac{\sqrt{6}}{4}$ and $b = c^2 \approx 0.15$.