Fall 2016 Math 2B
Suggested Homework Problems
Solutions

Sections	Topics covered	Problems
5.1	Area and distances	1, 2, 5, 13, 18, 21
5.2	The definite integral	1, 3, 5, 7, 18, 20, 21, 24, 34, 38, 40, 53, 58, 63, 64
5.3	Fundamental theorem of calculus	2, 8, 10, 12, 14, 16, 18, 22, 24, 26, 29, 30, 34, 36, 41, 43, 60, 62, 74, 75

Area and distances

Exercise 1 : Correction at the end of your book

Exercise 2 : **(a)** We have in our case

$$\Delta x = \frac{12-0}{6} = 2.$$

(i) The left endpoints are $x_0 = 0, x_1 = 0 + \Delta x = 2, x_2 = 4, x_3 = 6$, $x_4 = 8$ and $x_5 = 10$. Therefore

$$\begin{aligned} L_6 &= 2f(0) + 2f(2) + 2f(4) + 2f(6) + 2f(8) + 2f(10) \\ &= 2\cdot 9 + 2\cdot 8.8 + 2\cdot 8.2 + 2\cdot 7.3 + 2\cdot 6 + 2\cdot 4 \\ &= 86.6. \end{aligned}$$

(ii) The right endpoints are $x_1 = 0 + \Delta x = 2, x_2 = 4, x_3 = 6, x_4 = 8, x_5 = 10$ and $x_6 = 12$. Therefore

$$\begin{aligned} R_6 &= 2f(2) + 2f(4) + 2f(6) + 2f(8) + 2f(10) + 2f(12) \\ &= 2\cdot 8.8 + 2\cdot 8.2 + 2\cdot 7.3 + 2\cdot 6 + 2\cdot 4 + 2\cdot 1 \\ &= 70.6. \end{aligned}$$

(iii) The midpoints are $\bar{x}_1 = 1, \bar{x}_2 = 3, \bar{x}_3 = 5, \bar{x}_4 = 7, \bar{x}_5 = 9$ and $\bar{x}_6 = 11$. Therefore

$$\begin{aligned} M_6 &= 2f(1) + 2f(3) + 2f(5) + 2f(7) + 2f(9) + 2f(11) \\ &= 2\cdot 8.9 + 2\cdot 8.5 + 2\cdot 7.8 + 2\cdot 6.6 + 2\cdot 5 + 2\cdot 2.8 \\ &= 79.2. \end{aligned}$$

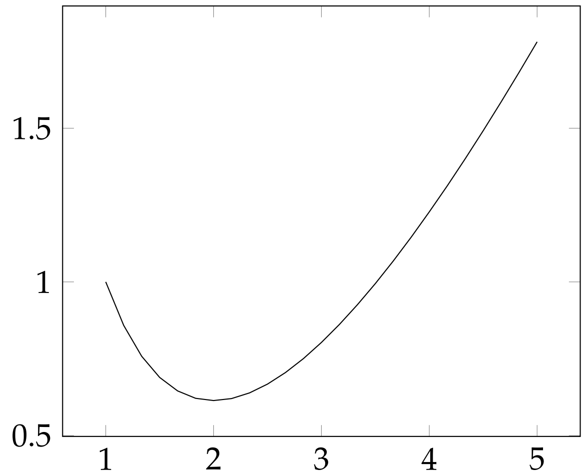
(b) Since f is decreasing, L_6 is an overestimate of the true area.

(c) Since f is decreasing, R_6 is an underestimate of the true area.

(d) Draw the rectangles. M_6.

Exercise 5 : Correction at the end of your book.

Exercise 6 : (a)



(b) We have in our case

$$\Delta x = \frac{5-1}{4} = 1.$$

The right endpoints are $x_1 = 1 + \Delta x = 2, x_2 = 3, x_3 = 4$, and $x_4 = 5$. Therefore

$$\begin{aligned} R_4 &= f(2) + f(3) + f(4) + f(5) \\ &= (2 - 2\ln(2)) + (3 - 2\ln(3)) + (4 - 2\ln(4)) + (5 - 2\ln(5)) \\ &\approx 4.4. \end{aligned}$$

The midpoints are $m_1 = 1.5, m_2 = 2.5, m_3 = 3.5$, and $m_4 = 4.5$. Therefore

$$\begin{aligned} M_4 &= f(1.5) + f(2.5) + f(3.5) + f(4.5) \\ &= (1.5 - 2\ln(1.5)) + (2.5 - 2\ln(2.5)) + (3.5 - 2\ln(3.5)) + (4.5 - 2\ln(4.5)) \\ &\approx 3.8. \end{aligned}$$

(c) We have in our case

$$\Delta x = \frac{5-1}{8} = 0.5.$$

The right endpoints are $x_1 = 1 + \Delta x = 1.5, x_2 = 2, x_3 = 2.5, ...,$ and $x_8 = 5$. Therefore

$$\begin{aligned} R_8 &= \frac{1}{2}\left[f(1.5) + f(2) + ... + f(5)\right] \\ &= \frac{1}{2}\left[(1.5 - 2\ln(1.5)) + (2 - 2\ln(2)) + ... + (5 - 2\ln(5))\right] \\ &\approx 4.1. \end{aligned}$$

The midpoints are $m_1 = 1.25, m_2 = 1.75, ...,$ and $m_8 = 4.75$. Therefore

$$\begin{aligned} M_8 &= \frac{1}{2}\left[f(1.25) + f(1.75) + ... + f(4.75)\right] \\ &= \frac{1}{2}\left[(1.25 - 2\ln(1.25)) + (1.75 - 2\ln(1.75)) + ... + (4.75 - 2\ln(4.75))\right] \\ &\approx 3.9. \end{aligned}$$

Exercise 13 : Since v is an increasing function, L_6 will give us a lower estimate and R_6 an upper estimate.

$$L_6 = 0.5\left[0 + 6.2 + 10.8 + 14.9 + 18.1 + 19.4\right] = 34.7.$$

$$R_6 = 0.5\left[6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2\right] = 44.8.$$

Exercise 18 : We use M_6 to get an estimate. Here

$$\Delta t = \frac{30 - 0}{6} = 5 = \frac{5}{3600} = \frac{1}{720}.$$

Therefore

$$M_6 = \frac{1}{720}\left[v(2.5) + v(7.5) + v(12.5) + v(17.5) + v(22.5) + v(27.5)\right] \approx 0.725km.$$

Exercise 21 : Here

$$\Delta x = \frac{3 - 1}{n} = \frac{2}{n},$$

and,

$$x_i = 1 + i\Delta x = 1 + \frac{2i}{n},$$

for $1 \leq i \leq n$. Therefore

$$R_n = \Delta x \sum_{i=1}^{n} f(x_i) = \frac{2}{n}\sum_{i=1}^{n} \frac{2(1 + 2i/n)}{(1 + 2i/n)^2 + 1},$$

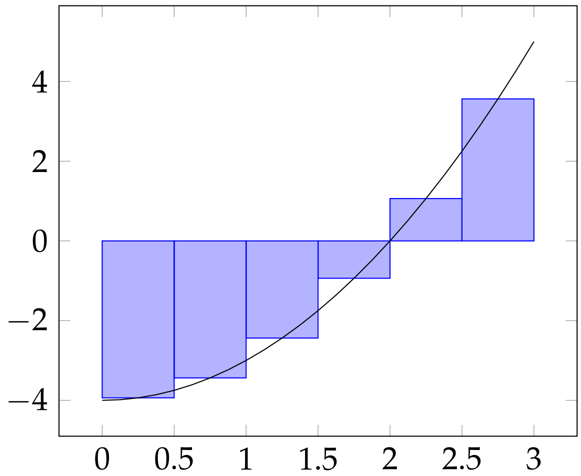
and,

$$\mathcal{A} = \lim_{n\to\infty} R_n.$$

Definite integral

Exercise 1 : Correction at the end of your book.

Exercise 3 :



Here

$$\Delta x = \frac{3-0}{6} = \frac{1}{2}.$$

Since we are using midpoints, $x_i^* = \dfrac{x_{i-1}+x_i}{2}$.

$$\begin{aligned} M_6 &= \Delta x \sum_{i=1}^{6} f(x_i) = \frac{1}{2}\left[f(\frac{1}{4}) + f(\frac{3}{4}) + f(\frac{5}{4}) + f(\frac{7}{4}) + f(\frac{9}{4}) + f(\frac{11}{4}) \right] \\ &= -\frac{49}{16}. \end{aligned}$$

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the three rectangles below the x-axis, that is the net area of the rectangles with respect to the x-axis.

Exercise 5 : Correction at the end of your book.

Exercise 7 : Correction at the end of your book.

Exercise 18 : $\displaystyle\int_2^5 x\sqrt{1+x^3}\,dx.$

Exercise 20 : $\displaystyle\int_1^3 \frac{x}{x^2+4}\,dx.$

Exercise 21 : Here

$$\Delta x = \frac{5-2}{n} = \frac{3}{n},$$

and for $0 \le i \le n$,

$$x_i = 2 + i\Delta x = 2 + \frac{3i}{n}.$$

Therefore

$$\begin{aligned} R_n &= \Delta x \sum_{i=1}^{n} f(x_i) = \frac{3}{n}\sum_{i=1}^{n}\left[4 - 2(2+\frac{3i}{n})\right] \\ &= \frac{3}{n}\sum_{i=1}^{n} -\frac{6i}{n} = -\frac{18}{n^2}\sum_{i=1}^{n} i \\ &= -\frac{18}{n^2}\frac{n(n+1)}{2} = -9\frac{n(n+1)}{n^2} = -9(1+\frac{1}{n}) \end{aligned}$$

And then

$$\int_2^5 (4-2x)\,dx = \lim_{n\to\infty} R_n = -9.$$

Exercise 24 : Here

$$\Delta x = \frac{2-0}{n} = \frac{2}{n},$$

and for $0 \le i \le n$,

$$x_i = 0 + i\Delta x = \frac{2i}{n}.$$

Therefore

$$\begin{aligned} R_n &= \Delta x \sum_{i=1}^{n} f(x_i) = \frac{2}{n}\sum_{i=1}^{n}\left[2\left(\frac{2i}{n}\right) - \left(\frac{2i}{n}\right)^3\right] \\ &= \frac{2}{n}\sum_{i=1}^{n}\frac{4i}{n} - \frac{2}{n}\sum_{i=1}^{n}\frac{8i^3}{n^3} = \frac{8}{n^2}\sum_{i=1}^{n} i - \frac{16}{n^4}\sum_{i=1}^{n} i^3 \\ &= \frac{8}{n^2}\frac{n(n+1)}{2} - \frac{16}{n^4}\frac{n^2(n+1)^2}{4} \\ &= 4\frac{n(n+1)}{n^2}\left[1 - \frac{n(n+1)}{n^2}\right] = 4\left(1+\frac{1}{n}\right)\left[1-\left(1+\frac{1}{n}\right)\right] \\ &= -4\left(1+\frac{1}{n}\right)\frac{1}{n}. \end{aligned}$$

And then

$$\int_0^2 (2x - x^3)\,dx = \lim_{n\to\infty} R_n = 0.$$

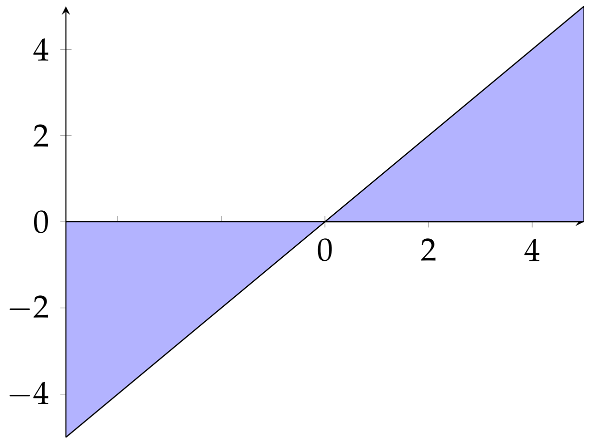
Exercise 34 : **(a)** $\displaystyle\int_0^2 g(x)\,dx = \frac{4*2}{2} = 4$. (Area of a triangle)

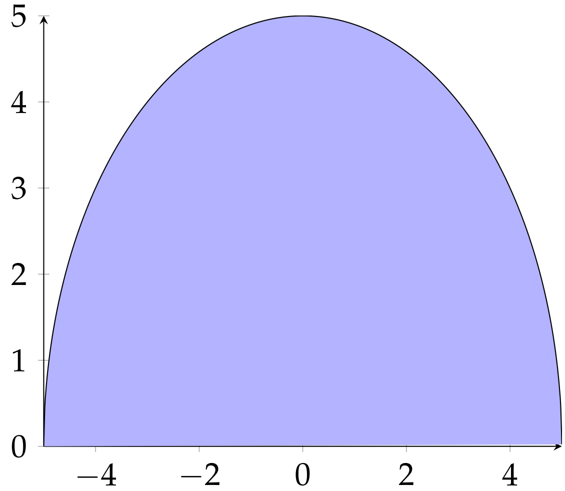
(b) $\displaystyle\int_2^6 g(x)\,dx = -\frac{1}{2}\pi 2^2 = -2\pi$. (Negative of the area of a semi-circle)

(c) $\displaystyle\int_6^7 g(x)\,dx = \frac{1*1}{2} = \frac{1}{2}$. (Area of a triangle)

Exercise 38 :

$$\int_{-5}^{5}(x-\sqrt{25-x^2})\,dx = \int_{-5}^{5} x\,dx - \int_{-5}^{5}\sqrt{25-x^2}\,dx.$$

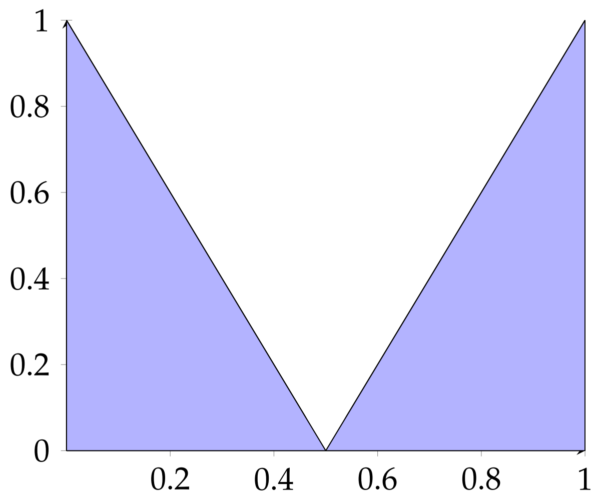




By symmetry, the value of the first integral is 0, since the blue area above the x-axis equals the blue area under the x-axis.

The second integral can be interpreted as one half the area of a circle with radius 5, that is $\frac{25\pi}{2}$. Thus the value of the original integral is $0 - \frac{25\pi}{2} = -\frac{25\pi}{2}$.

Exercise 40 :



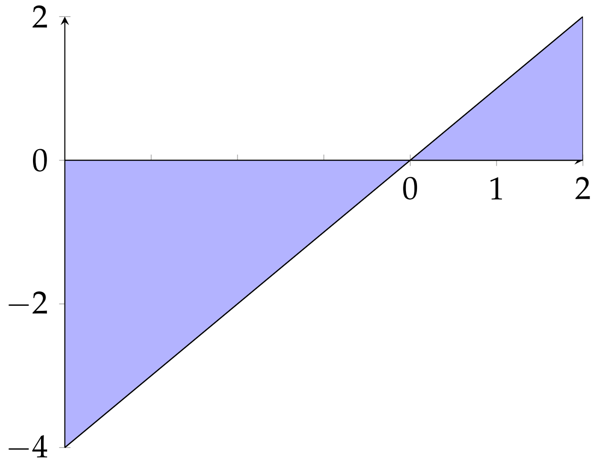
The integral can be interpreted as the sum of the two blue areas, that is

$$2 * \frac{1 * 0.5}{2} = 0.5.$$

Exercise 53 : We have

$$\int_{-4}^{2}(f(x)+2x+5)\,dx = \int_{-4}^{2} f(x)\,dx + 2\int_{-4}^{2} x\,dx + \int_{-4}^{2} 5\,dx.$$

The first integral is equal to $-3+3-3 = -3$. The second integral is equal to $-\dfrac{4*4}{2} + \dfrac{2*2}{2} = -6$.



The third integral is equal to $5(2-(-4)) = 30$. Thus the original integral equals $-3+2*(-6)+30 = 15$.

Exercise 58 : For all $\dfrac{\pi}{6} \leq x \leq \dfrac{\pi}{3}$, we have

$$\sin(\frac{\pi}{6}) \leq \sin(x) \leq \sin(\frac{\pi}{3}),$$

because $x \mapsto \sin(x)$ is an increasing function on $[0, \dfrac{\pi}{2}]$.

Therefore for all $\dfrac{\pi}{6} \leq x \leq \dfrac{\pi}{3}, \dfrac{1}{2} \leq \sin(x) \leq \dfrac{\sqrt{3}}{2}$.
We apply the Property 8 and get :

$$\frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin(x)\, dx \leq \frac{\sqrt{3}}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right).$$

This gives us our result.

Exercise 63 : Let $f(x) = xe^{-x}$ for all $0 \leq x \leq 2$. f is differentiable on $[0,2]$ and we have, for all $0 \leq x \leq 2$,

$$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x).$$

For all $0 \leq x \leq 2$, $e^{-x} > 0$. Therefore we have the following table of variations for f:

	0 1 2	
f′	+	-
f	↗	↘

We have also $f(0) = 0$ and $f(2) = 2e^{-2} > 0$. So this absolute minimum value of f on $[0,2]$ is 0 and the absolute maximum is $f(1) = e^{-1}$. For all $0 \leq x \leq 2$, we thus have :

$$0 \leq xe^{-x} \leq e^{-1}.$$

By Property 8, this gives us :

$$0 \leq \int_0^2 xe^{-x}\,dx \leq 2e^{-1}.$$

Exercise 64 : Let $f(x) = x - 2\sin(x)$ for all $\pi \leq x \leq 2\pi$. f is differentiable on $[\pi, 2\pi]$ and we have, for all $\pi \leq x \leq 2\pi$,

$$f'(x) = 1 - 2\cos(x).$$

On $[\pi, 2\pi]$, $f'(x) = 0 \Leftrightarrow \cos(x) = \dfrac{1}{2} \Leftrightarrow x = \dfrac{5\pi}{3}$. Therefore we have the following table of variations for f:

	π $\quad 5\pi/3 \quad$ 2π	
f′	+	-
f	$\nearrow$	$\searrow$

We have also $f(\pi) = \pi$ and $f(2\pi) = 2\pi$. So this absolute minimum value of f on $[0,2]$ is π and the absolute maximum is

$$f(\frac{5\pi}{3}) = \frac{5\pi}{3} - 2\sin(\frac{5\pi}{3}) = \frac{5\pi}{3} - 2\sin(-\frac{\pi}{3}) = \frac{5\pi}{3} + \sqrt{3}.$$

For all $\pi \leq x \leq 2\pi$, we thus have :

$$\pi \leq f(x) \leq \frac{5\pi}{3} + \sqrt{3}.$$

By Property 8, this gives us :

$$\pi\,(2\pi - \pi) \leq \int_\pi^{2\pi} f(x)\,dx \leq \left(\frac{5\pi}{3} + \sqrt{3}\right)(2\pi - \pi)\,,$$

that is,

$$\pi^2 \leq \int_\pi^{2\pi} f(x)\,dx \leq \frac{5\pi^2}{3} + \sqrt{3}\pi.$$

Fundamental theorem of Calculus

Exercice 2 : (a) $g(0) = \int_0^0 g(x)\,dx = 0.$

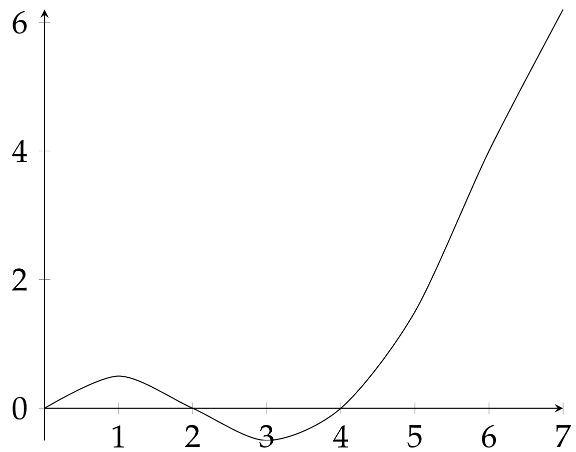
$$g(1) = \int_0^1 g(x)\,dx = \frac{1*1}{2} = \frac{1}{2}.$$
$$g(2) = \int_0^2 g(x)\,dx = \int_0^1 g(x)\,dx + \int_1^2 g(x)\,dx = g(1) - \frac{1*1}{2} = 0.$$
$$g(3) = \int_0^3 g(x)\,dx = g(2) + \int_2^3 g(x)\,dx = 0 - \frac{1*1}{2} = -\frac{1}{2}.$$
$$g(4) = g(3) + \frac{1*1}{2} = 0.$$
$$g(5) = g(4) + \frac{3}{2} = \frac{3}{2}.$$
$$g(6) = g(5) + \frac{5}{2} = 4.$$

(b) $g(7) = g(6) + \int_6^7 g(x)\,dx \approx 4 + 2.2 = 6.2.$

(c) f is the derivative of g. So the sign of f gives us the variations of g.

	0 1	3	7
f	+	-	+
g	↗	↘	↗

A change of sign of f gives a local minimum or maximum for g. When f goes from positive to negative values, we have a maximum for g at the point where f =0. When f goes from negative values to positive values, we have a minimum for g at the point where f=0. In our example, g has a local maximum at x=1 and a minimum at x=3. From the answers to part (a) and (b) g is maximum at x=7.



Exercice 8 : For $x \in (-\infty, \infty)$, $g'(x) = \ln(1 + x^2)$.

Exercice 10 : For $u \in (0, \infty)$, $h'(u) = \dfrac{\sqrt{u}}{u+1}$.

Exercice 12 : For $y \in (-\infty, \infty)$, $R'(y) = -y^3 \sin y$.

Exercice 14 : Let $g(x) = \int_1^x \frac{z^2}{z^4+1}\, dz$ for $x \in (-\infty, \infty)$.

The function h defined for $x \in [0, \infty)$ can be written as $h(x) = g(\sqrt{x})$.
Let $u(x) = \sqrt{x}$, for $x \geq 0$. We then have $h(x) = g(u(x))$, for all $x \geq 0$.
For all $x \in (0, \infty)$, we have by the chain rule

$$\begin{aligned}
\frac{d}{dx}h(x) &= \frac{d}{dx}g(u(x)) = \left(\frac{d}{du}g(u)\right)\left(\frac{du}{dx}\right) \\
&= \frac{u^2}{u^4+1} \times \frac{1}{2\sqrt{x}} \\
&= \frac{x}{x^2+1} \times \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Exercice 16 : For $x \in (-\infty, \infty)$, we have by the chain rule

$$y'(x) = 4x^3 \cos^2(x^4).$$

Exercice 18 : For $x \in (-\infty, \infty)$, we have by the chain rule

$$y'(x) = -\cos x \sqrt{1 + \sin^2 x}.$$

Exercice 22 :

$$\int_0^1 (1 - 8v^3 + 16v^7)\, dv = \left[v - 2v^4 + 2v^8\right]_0^1 = 1.$$

Exercice 24 :

$$\int_1^8 x^{-2/3} = \left[3x^{1/3}\right]_1^8 = 3.$$

Exercice 26 :

$$\int_{-5}^5 e\, dx = [ex]_{-5}^5 = 10e.$$

Exercice 29 :

$$\int_1^4 \frac{2+x^2}{\sqrt{x}}\, dx = \int_1^4 \frac{2}{\sqrt{x}} + x^{3/2}\, dx = \left[4\sqrt{x} + \frac{2}{5}x^{5/2}\right]_1^4 = \frac{82}{5}.$$

Exercice 30 :

$$\int_{-1}^2 (3u-2)(u+1)\, du = \int_{-1}^2 (3u^2 + u - 2)\, du = \left[u^3 + \frac{u^2}{2} - 2u\right]_{-1}^2 = \frac{9}{2}.$$

Exercice 34 :

$$\int_0^3 (2\sin(x) - e^x)\,dx = [-2\cos(x) - e^x]_0^3 = 3 - 2\cos 3 - e^3.$$

Exercice 36 :

$$\int_1^{18} \sqrt{\frac{3}{z}}\,dz = \sqrt{3}\int_1^{18} \frac{1}{\sqrt{z}}\,dz = \sqrt{3}\left[2z^{1/2}\right]_1^{18} = 2\sqrt{3}(3\sqrt{2} - 1).$$

Exercice 41 :

$$\int_0^4 2^s\,ds = \left[\frac{2^s}{\ln 2}\right]_0^4 = \frac{15}{\ln 2}.$$

Exercice 43 :

$$\int_0^\pi f(x)\,dx = \int_0^{\pi/2} \sin x\,dx + \int_{\pi/2}^\pi \cos x\,dx = [-\cos x]_0^{\pi/2} + [\sin x]_{\pi/2}^\pi = 0.$$

Exercice 60 : For all $x \in (-\infty, +\infty)$, we have :

$$\begin{aligned} g(x) = \int_{1-2x}^{1+2x} t\sin t\,dt &= \int_{1-2x}^{0} t\sin t\,dt + \int_0^{1+2x} t\sin t\,dt \\ &= -\int_0^{1-2x} t\sin t\,dt + \int_0^{1+2x} t\sin t\,dt. \end{aligned}$$

We use the chain rule to differentiate g and find that for all $x \in (-\infty, +\infty)$,

$$\begin{aligned} g'(x) &= -(1-2x)\sin(1-2x) \times (-2) + (1+2x)\sin(1+2x) \times 2 \\ &= 2(1-2x)\sin(1-2x) + 2(1+2x)\sin(1+2x). \end{aligned}$$

Exercice 62 : For all $x \in [0, +\infty)$, we have :

$$\begin{aligned} F(x) = \int_{\sqrt{x}}^{2x} \arctan t\,dt &= \int_{\sqrt{x}}^{1} \arctan t\,dt + \int_1^{2x} \arctan t\,dt \\ &= -\int_1^{\sqrt{x}} \arctan t\,dt + \int_1^{2x} \arctan t\,dt. \end{aligned}$$

We use the chain rule to differentiate F and find that for all $x \in (0, +\infty)$,

$$F'(x) = -\frac{1}{2\sqrt{x}}\arctan(\sqrt{x}) + 2\arctan(2x).$$

Exercice 74 : (a) By the fundamental theorem of Calculus part 1, we know that f is the derivative of g. So g admits local minima and maxima when f equals to zero and change sign. At $x = 2$ and $x = 6$, f changes from positive to negative, so g has local maxima at these points. At $x = 4$ and $x = 8$, f changes from negative to positive so g has local minima at these points.

(b) We can see from the graph that

$$\left|\int_0^2 f(t)\,dt\right| > \left|\int_2^4 f(t)\,dt\right| > \left|\int_4^6 f(t)\,dt\right| > \left|\int_6^8 f(t)\,dt\right| > \left|\int_8^{10} f(t)\,dt\right|.$$

We have also :

$$g(2) = \left|\int_0^2 f(t)\,dt\right|,$$

$$g(6) = \int_0^6 f(t)\,dt = g(2) - \left|\int_2^4 f(t)\,dt\right| + \left|\int_4^6 f(t)\,dt\right|,$$

and

$$g(10) = \int_0^{10} f(t)\,dt = g(6) - \left|\int_6^8 f(t)\,dt\right| + \left|\int_8^{10} f(t)\,dt\right|.$$

Thus, $g(2) > g(6) > g(10)$, and so the absolute maximum of $g(x)$ occurs at $x = 2$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'' = f'$, which is negative when f is decreasing that is on $[1,3], [5,7]$ and $[9,10]$. So g is concave downward on these intervals.

Exercice 75 :

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^4}{n^5}+\frac{i}{n^2} = \lim_{n\to\infty}\frac{1-0}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^4+\frac{i}{n} = \int_0^1 x^4 + x\,dx = \left[\frac{x^5}{5}+\frac{x^2}{2}\right]_0^1 = \frac{7}{10}.$$