Fall 2016 Math 2B
Suggested Homework Problems
Solutions

Antiderivatives

Exercise 2 : For all $x \in]-\infty, +\infty[$, the most general antiderivative of f is given by :

$$F(x) = \left(\frac{x^3}{3}\right) - 3\left(\frac{x^2}{2}\right) + 2x + C = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + C.$$

Exercise 4 : For all $x \in]-\infty, +\infty[$, the most general antiderivative of f is given by :

$$F(x) = 6\left(\frac{x^6}{6}\right) - 8\left(\frac{x^5}{5}\right) - 9\left(\frac{x^3}{3}\right) + C = x^6 - \frac{8}{5}x^5 - 3x^3 + C.$$

Exercise 8 : For all $x \in [0, +\infty[$, the most general antiderivative of f is given by :

$$F(x) = \frac{x^{4.4}}{4.4} - 2\left(\frac{x^{\sqrt{2}}}{\sqrt{2}}\right) + C = \frac{5}{22}x^{4.4} - \sqrt{2}x^{\sqrt{2}} + C.$$

Exercise 9 : For all $x \in]-\infty, +\infty[$, the most general antiderivative of f is given by :

$$F(x) = \sqrt{2}x + C.$$

Exercise 11 : For all $x \in [0, +\infty[$, the most general antiderivative of f is given by :

$$F(x) = 3\left(\frac{2}{3}x^{3/2}\right) - 2\left(\frac{3}{4}x^{4/3}\right) + C = 2x^{3/2} - \frac{3}{2}x^{4/3} + C.$$

Exercise 13 : The most general antiderivative of f is given by :

$$F(x) = \begin{cases} \frac{1}{5}x - 2\ln|x| + C_1, \text{ if x<0,} \\ \frac{1}{5}x - 2\ln|x| + C_2, \text{ if x>0.} \end{cases}$$

Exercise 15 : For all $t \in]0, +\infty[$, the most general antiderivative of g is given by:

$$G(t) = 2t^{1/2} + \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + C.$$

Exercise 17 : For all $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, the most general antiderivative of h is given by:

$$H(\theta) = -2\cos(\theta) + \tan(\theta) + C.$$

Exercise 19 : For all $x \in]-\infty, +\infty[$, the most general antiderivative of f is given by

$$F(x) = \frac{2^x}{\ln 2} + 4\cosh(x) + C.$$

Exercise 22 : We can rewrite the expression of f as follows :

$$f(x) = \frac{2x^2+5}{x^2+1} = \frac{2(x^2+1)+3}{x^2+1} = 2 + \frac{3}{x^2+1}.$$

Then $F(x) = 2x + 3\tan^{-1}(x) + C$.

Exercise 27 : For all $x \in]-\infty, +\infty[$, the most general antiderivative of f'' is given by : $f'(x) = x^2 + 3e^x + C$.

For all $x \in]-\infty, +\infty[$, the most general antiderivative of f' is given by :

$$f(x) = \frac{x^3}{3} + 3e^x + Cx + D.$$

Exercise 33 : For all $x \in]-\infty, +\infty[$, the most general antiderivative of f' is given by :

$$f(x) = 4\tan^{-1}(x) + C.$$

We plug in 1 into this expression to get

$$f(1) = 4\tan^{-1}(1) + C = 4\frac{\pi}{4} + C = \pi + C.$$

Thus the condition $f(1) = 0$ gives us $C = -\pi$ and for all $x \in]-\infty, +\infty[$

$$f(x) = 4\tan^{-1}(x) - \pi.$$

Exercise 38 : An antiderivative of f' has the following general form

$$f(t) = \begin{cases} \frac{3^t}{\ln 3} - 3\ln|t| + C_1, \text{ if t<0,} \\ \frac{3^t}{\ln 3} - 3\ln|t| + C_2, \text{ if t>0.} \end{cases}$$

We use the condition $f(-1) = 1$ to determine the constant C_1. Thanks to the general form of f, we have

$$f(-1) = \frac{3^{-1}}{\ln 3} - 3\ln|-1| + C_1 = \frac{1}{3\ln 3} + C_1.$$

So $C_1 = -\frac{1}{3\ln 3}$.

We use the condition $f(1) = 2$ to determine the constant C_2. Thanks to the general form of f, we have

$$f(1) = \frac{3^1}{\ln 3} - 3\ln|1| + C_2 = \frac{3}{\ln 3} + C_2.$$

So $C_2 = 2 - \frac{3}{\ln 3}$.

Thus

$$f(t) = \begin{cases} \frac{3^t}{\ln 3} - 3\ln|t| - \frac{1}{3\ln 3}, \text{ if t<0,} \\ \frac{3^t}{\ln 3} - 3\ln|t| + 2 - \frac{3}{\ln 3}, \text{ if t>0.} \end{cases}$$

Exercise 47 : An antiderivative of f'' has the following general form

$$f'(x) = -\frac{1}{x} + C$$

for all $x \in]0, +\infty[$.

For all $x \in]0, +\infty[$, the most general antiderivative of f' is given by :

$$f(x) = -\ln|x| + Cx + D = -\ln(x) + Cx + D$$

We use the conditions $f(1) = 0$ and $f(2) = 0$ to determine the constants C and D. Thanks to the general form of f, we have

$$f(1) = -\ln(1) + C \times 1 + D = C + D, \quad f(2) = -\ln(2) + C \times 2 + D.$$

The first condition gives us $C = -D$.
We plug this in the second and get : $-\ln(2) - D = 0$.

So we have finally : $C = \ln(2)$, $D = -\ln(2)$ and $f(x) = -\ln(x) + \ln(2)x - \ln(2)$ for all $x > 0$.

Exercise 62 : The velocity has the following general form :

$$v(t) = 3\sin(t) + 2\cos(t) + C,$$

for all $t > 0$. We use the initial condition $v(0) = 4$ on the velocity to determine C. Thanks to the general form of v, we have $v(0) = 3\sin(0) + 2\cos(0) + C = 2 + C$. So $C = 2$ and

$$v(t) = 3\sin(t) + 2\cos(t) + 2,$$

for all $t > 0$. The position has the following general form :

$$s(t) = -3\cos(t) + 2\sin(t) + 2t + D,$$

for all $t > 0$. We use the initial condition $s(0) = 0$ on the position to determine D. Thanks to the general form of s, we have $s(0) = -3\cos(0) + 2\sin(0) + 2 \times 0 + D = -3 + D$. So $D = 3$ and

$$s(t) = -3\cos(t) + 2\sin(t) + 2t + 3,$$

for all $t > 0$.

Exercise 68 : Let s_1 and s_2 be the altitudes of the first and second ball. Let v_1 and v_2 be the velocities of the first and second ball. Both balls have a negative constant acceleration equal to -32. The velocities have the following general forms :

$$v_1(t) = -32t + C_1, \text{for all t>0},$$

$$v_2(t) = -32t + C_2, \text{for all t>1}.$$

We use the initial conditions $v_1(0) = 48$ and $v_2(1) = 24$ to determine C_1 and C_2. Thanks to the general forms of v_1 and v_2, we have :

$$v_1(0) = C_1,$$

$$v_2(1) = -32 + C_2.$$

So $C_1 = 48$ and $C_2 = 56$, and

$$v_1(t) = -32t + 48, \text{for all t>0},$$

$$v_2(t) = -32t + 56, \text{for all t>1}.$$

The positions of the two balls have the following general forms :

$$s_1(t) = -32\left(\frac{t^2}{2}\right) + 48t + D_1, \text{for all t>0},$$
$$s_2(t) = -32\left(\frac{t^2}{2}\right) + 56t + D_2, \text{for all t>1}.$$

We use the initial conditions $s_1(0) = 432$ and $s_2(1) = 432$ to determine D_1 and D_2. Thanks to the general forms of s_1 and s_2, we have :

$$s_1(0) = D_1,$$
$$s_2(1) = -\frac{32}{2} + 56 + D_2 = 40 + D_2.$$

So $D_1 = 432$ and $D_2 = 432 - 40 = 392$, and

$$s_1(t) = -32\left(\frac{t^2}{2}\right) + 48t + 432, \text{for all t>0},$$
$$s_2(t) = -32\left(\frac{t^2}{2}\right) + 56t + 392, \text{for all t>1}.$$

The balls pass each other if there exits $t > 1$ such that $s_1(t) = s_2(t)$ that is :

$$-32\left(\frac{t^2}{2}\right) + 48t + 432 = -32\left(\frac{t^2}{2}\right) + 56t + 392.$$

Or $48t + 432 = 56t + 392$, that is $8t = 40$ and $t = 5$.

The balls pass each other at $t = 5s$.

Exercise 78 : (a) The acceleration of the rocket verifies :

$$a(t) = 60t, \quad \text{for all } 0 \le t \le 3,$$

$$a(t) = -32, \quad \text{for all } 3 \le t \le 17.$$

Therefore the velocity has the following general form :

$$v(t) = 30t^2 + C_1, \quad \text{for all } 0 \le t \le 3,$$

$$v(t) = -32t + C_2, \quad \text{for all } 3 \le t \le 17.$$

We use the initial condition $v(0) = 0$ to determine C_1, and get $C_1 = 0$. We use the continuity of the velocity at time $t = 3$ to find C_2. The first expression of the velocity gives us $v(3) = 30 \times 3^2 = 270$. The second expression of the velocity gives us $v(3) = -32 \times 3 + C_2$. Therefore we have $C_2 = 32 \times 3 + 270$ and

$$v(t) = 30t^2, \quad \text{for all } 0 \le t \le 3,$$

$$v(t) = -32(t-3) + 270, \quad \text{for all } 3 \le t \le 17.$$

At $t = 17$s, the second expression gives us $v(17) = -32 \times 14 + 270 = -178$ ft/s. Then the rocket slows linearly to -18 ft/s in 5s. So for $17 < t < 22$,

$$v(t) = \frac{-18 + 178}{5}(t - 17) - 178 = 32(t - 17) - 178.$$

We have therefore :

$$v(t) = 30t^2, \quad \text{for all } 0 \le t \le 3,$$

$$v(t) = -32(t-3) + 270, \quad \text{for all } 3 \le t \le 17,$$

$$v(t) = 32(t-17) - 178, \quad \text{for all } 17 \le t \le 22$$

$$v(t) = -18, \quad \text{for all } t \ge 22.$$

The position has then the following general expression :

$$s(t) = 10t^3 + D_1, \quad \text{for all } 0 \le t \le 3,$$

$$s(t) = -16(t-3)^2 + 270t + D_2, \quad \text{for all } 3 \le t \le 17,$$

$$s(t) = 16(t-17)^2 - 178t + D_3, \quad \text{for all } 17 \le t \le 22$$

$$s(t) = -18t + D_4, \quad \text{for all } t \ge 22.$$

We use the initial condition $s(0) = 0$ to determine D_1, and get $D_1 = 0$. We use the continuity of the position at time $t = 3$,time $t = 17$ and time $t = 22$ to find D_2, D_3 and D_4. The first expression of the position gives us $s(3) = 10 \times 3^3 = 270$. The second expression of the position gives us $s(3) = 270 \times 3 + D_2$. Therefore we have $D_2 = -270 \times 3 + 270$. and for all $3 \geq t \geq 17$,

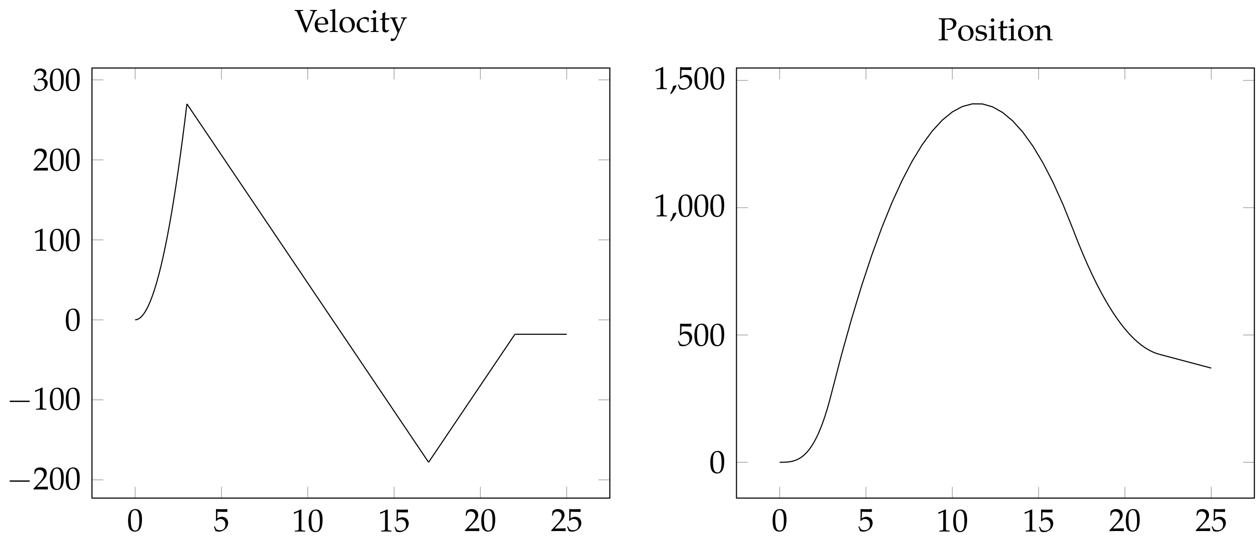
$$s(t) = -16(t-3)^2 + 270(t-3) + 270.$$

The second expression of the position gives us $s(17) = -16 \times 14^2 + 270 \times 14 + 270 = 914$. Hence $D_3 = 178 \times 17 + 914$. and $17 \geq t \geq 22$,

$$s(t) = 16(t-17)^2 - 178(t-17) + 914.$$

The third expression of the velocity gives us $s(22) = 424$ and thus for all $t \leq 22$,

$$s(t) = -18(t-22) + 424.$$



(b) The rocket reaches its maximum height when its velocity is zero for the first time. Let t_m be the time for which the rocket reach this maximum height. We know that $v(3) = 270$ and $v(17) = -178$. So $3 \leq t_m \leq 17$, and t_m verifies

$$-32(t_m - 3) + 270 = 0.$$

We hence obtain $t_m = 231/32 \approx 11.4$s. The maximum height that the rocket reaches is then $s(t_m) = -16(t_m - 3)^2 + 270(t_m - 3) + 270 \approx 1409.1$ft.

(c) The rocket lands when its position reaches zero. Let t_l be the landing time of the rocket. We know that $s(22) > 0$ so $t_l > 0$ and we have

$$-18(t_l - 22) + 424 = 0.$$

That is $t_l = 424/18 + 22 \approx 45.6$s.