Practice Final 2

by Nikki Fider and Anton Butenko

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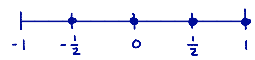
1. Compute the fourth Riemann Sum for $f(x) = e^x + \arctan(x) + 2$ from $x = -1$ to $x = 1$. Use right endpoints as your sample points.

$$n = 4, \quad a = -1, \quad b = 1, \quad \Delta x = \frac{1-(-1)}{4} = \frac{1}{2}$$



Sample points: $-\frac{1}{2}, 0, \frac{1}{2}, 1$

$$\sum_{i=1}^{4} f(x_i^*)\Delta x = \left[e^{-\frac{1}{2}} + \arctan(-\tfrac{1}{2}) + 2\right]\tfrac{1}{2} + \left[e^0 + \arctan(0) + 2\right]\tfrac{1}{2} + \left[e^{\frac{1}{2}} + \arctan(\tfrac{1}{2}) + 2\right]\tfrac{1}{2} + \left[e^1 + \arctan(1) + 2\right]\tfrac{1}{2}$$

$$= \left[e^{-\frac{1}{2}} - \arctan(\tfrac{1}{2}) + 2 + e^0 + \arctan(0) + 2 + e^{\frac{1}{2}} + \arctan(\tfrac{1}{2}) + 2 + e + \arctan(1) + 2\right]\tfrac{1}{2}$$

$$= \left[e^{-\frac{1}{2}} + e^{\frac{1}{2}} + e + 9 + \frac{\pi}{4}\right]\cdot\frac{1}{2}$$

2. Let $f(x)$ be continuous and $\int_0^{16} f(x)\mathrm{d}x = 6$. Find $\int_0^4 x\boxed{f(x^2)}\mathrm{d}x$. COMPOSITION probably: u-sub

$$u = x^2, \quad \tfrac{1}{2}du = x\,dx \quad \mapsto \quad \int_0^4 f(x^2)\,x\,dx = \frac{1}{2}\int_0^{16} f(u)\,du = \frac{1}{2}\cdot 6 = 3$$

3. Compute $\int_{-2}^{3} \left(x^2 - 4x + 3\right) \mathrm{d}x$ by evaluating the limit of its Riemann sums.

$$a = -2, \quad b = 3, \quad \Delta x = \frac{3-(-2)}{n} = \frac{5}{n}, \quad x_i^* = -2 + \frac{5i}{n}$$

n-th Riemann sum:

$$\sum_{i=1}^{n}\left[\left(-2+\frac{5i}{n}\right)^2 - 4\left(-2+\frac{5i}{n}\right) + 3\right]\frac{5}{n} = \sum_{i=1}^{n}\left[4 - \frac{20i}{n} + \frac{25i^2}{n^2} + 8 - \frac{20i}{n} + 3\right]\frac{5}{n}$$

$$= \sum_{i=1}^{n}\left[\frac{25i^2}{n^2} - \frac{40i}{n} + 15\right]\frac{5}{n}$$

$$= \frac{125}{n^3}\sum_{i=1}^{n} i^2 - \frac{200}{n^2}\sum_{i=1}^{n} i + \frac{75}{n}\sum_{i=1}^{n} 1$$

$$= \frac{125}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{200}{n^2}\cdot\frac{n(n+1)}{2} + \frac{75}{n}\cdot n$$

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[\left(-2+\frac{5i}{n}\right)^2 - 4\left(-2+\frac{5i}{n}\right) + 3\right]\frac{5}{n}\right] = \lim_{n\to\infty}\left[\frac{125}{6}\cdot\frac{n(n+1)(2n+1)}{n^3} - \frac{200}{2}\cdot\frac{n(n+1)}{n^2} + 75\right]$$

$$= \frac{125}{3} - 100 + 75 = \frac{125}{3} - 25 = \frac{50}{3}$$

4. Suppose a particle moves back and forth along a straight line with velocity $v(t)$, measured in feet per second, and and acceleration $a(t)$. What are the meanings of $\int_0^{60} v(t)\mathrm{d}t$, $\int_0^{60} |v(t)|\mathrm{d}t$ and $\int_0^{60} a(t)\mathrm{d}t$?

$\int_0^{60} v(t)\,dt$: the displacement (in feet) of the particle after travelling from 0seconds to 60 seconds

$\int_0^{60} |v(t)|\,dt$: the total distance travelled (in feet) "

$\int_0^{60} a(t)\,dt = v(t)\Big|_0^{60} = v(60) - v(0)$: the change in speed of the particle (in ft/s)

5. (a) Simplify $\frac{d}{dx}\left[\int_{x^2}^{10} \arctan(t) - 4\mathrm{d}t\right]$

$$= \frac{d}{dx}\left[- \int_{10}^{x^2} (\arctan(t) - 4)\,dt \right]$$

$$= -\frac{d}{dx}\left[\int_{10}^{x^2} \arctan(t) - 4\,dt \right]$$

$$= -\left[\arctan(x^2) - 4\right]\cdot 2x$$

MULTIPLICATION

Probably : integration by parts

(b) Given $h(2) = 1$, $h(5) = 8$, $h'(2) = 2$, $h'(5) = 7$. Evaluate $\int_2^5 \boxed{xh''(x)}\mathrm{d}x$.

$u = x \quad v = h'(x)$
$du = 1dx \quad dv = h''(x)dx$ $\mapsto \; xh'(x)\Big|_2^5 - \int_2^5 h'(x)dx$

$$= \left(xh'(x) - h(x)\right)\Big|_2^5 = \left(5h'(5) - h(5)\right) - \left(2h'(2) - h(2)\right) = 35 - 8 - 4 + 1 = 24$$

6. (a) Find the most general antiderivative of $x^2 - 4x + 3$.

$$\tfrac{1}{3}x^3 - 4\cdot\tfrac{1}{2}x^2 + 3x + C$$

$$= \tfrac{1}{3}x^3 - 2x^2 + 3x + C$$

(b) Evaluate $\int_{-1}^{3} x^2 - 4x + 3\mathrm{d}x$

$$= \left[\tfrac{1}{3}x^3 - 2x^2 + 3x\right]\Big|_{-1}^{3} = (9 - 18 + 9) - (-\tfrac{1}{3} - 2 - 3) = 5\tfrac{1}{3}$$

COMPOSITION

Probably: u-sub

7. Suppose $g(-2) = 3$, $g(2) = 9$ and $\int_{36}^{108} f(z)\mathrm{d}z = 15$. Evaluate $\int_{-2}^{2} 4\boxed{f\left(12g(x)\right)}g'(x)\mathrm{d}x$.

$u = 12g(x)$

$\frac{1}{12}du = g'(x)dx$ $\mapsto$ $\frac{1}{12}\int_{36}^{108} 4f(u)\,du = \frac{1}{3}\int_{36}^{108} f(u)\,du = \frac{1}{3}\cdot 15 = 5$

8. Evaluate the integral $\int_0^1 \frac{1}{(5x+2)^{30}}\mathrm{d}x$. Show all work.

$u = 5x+2$

$\frac{1}{5}du = dx$ $\mapsto$ $\frac{1}{5}\int_2^7 \frac{1}{u^{30}}\,du = \frac{1}{5}\int_2^7 u^{-30}\,du = -\frac{1}{5}\cdot\frac{1}{29}u^{-29}\Big|_2^7 = -\frac{1}{145}\left(7^{-29} - 2^{-29}\right)$

9. Evaluate the integral $\int \tan(\frac{x}{2})\mathrm{d}x$. Show all work.

$= \int \frac{\sin(\frac{x}{2})}{\cos(\frac{x}{2})}\,dx$

$u = \cos(\frac{x}{2})$

$du = -\sin(\frac{x}{2})\cdot\frac{1}{2}\,dx$

$-2du = \sin(\frac{x}{2})\,dx$

$\mapsto$ $-2\int \frac{1}{u}\,du = -2\ln|u| + C = -2\ln|\cos(\frac{x}{2})| + C = 2\ln|\sec(\frac{x}{2})| + C$

10. Evaluate the integral $\int_e^{e^2} \ln(x)\mathrm{d}x$. Show all work.

$$= \int_e^{e^2} \ln(x) \cdot 1 \, dx$$

$u = \ln x \quad v = x$

$du = \frac{1}{x} dx \quad dv = 1\, dx$

$$\longmapsto \; x\ln x\Big|_e^{e^2} - \int_e^{e^2} \frac{x}{x} dx = x\ln x\Big|_e^{e^2} - \int_e^{e^2} 1 \, dx = (x\ln x - x)\Big|_e^{e^2} = e^2\ln(e^2) - e^2 - e\ln(e) + e = e^2$$

11. Evaluate the integral $\int \arctan(3x)\, \mathrm{d}x$. Show all work.

$$= \int \arctan(3x) \cdot 1 \, dx$$

$u = \arctan(3x) \quad v = x$

$du = \frac{1}{1+9x^2} \cdot 3 \, dx \quad dv = 1 \, dx$

$$\longmapsto \; x\arctan(3x) - 3\int \frac{x}{1+9x^2} dx$$

$u = 1 + 9x^2$

$\frac{1}{18} du = x\, dx$

$$\longmapsto \; x\arctan(3x) - \frac{3}{18}\int \frac{1}{u} du = x\arctan(3x) - \frac{1}{6}\ln|u| + C = x\arctan(3x) - \frac{1}{6}\ln|1+9x^2| + C$$

12. Evaluate the integral $\int_{-1}^{3} \frac{x^3}{\sqrt{36-x^2}}\mathrm{d}x$. Show all work.

$u = 36 - x^2$

$-\frac{1}{2} du = x\,dx$

$x^2 = 36 - u$

$$\longmapsto \; -\frac{1}{2}\int_{35}^{27} \frac{(36-u)}{\sqrt{u}} du = \frac{1}{2}\int_{27}^{35} u^{-\frac{1}{2}}(36-u)\, du = \frac{1}{2}\int_{27}^{35} 36u^{-\frac{1}{2}} - u^{\frac{1}{2}} \, du = \frac{1}{2}\left[72u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]\Big|_{27}^{35}$$

$$= \frac{72}{2}(35)^{\frac{1}{2}} - \frac{1}{3}(35)^{\frac{3}{2}} - \frac{72}{2}\cdot(27)^{\frac{1}{2}} + \frac{1}{3}\cdot(27)^{\frac{3}{2}} = \frac{73}{3}\sqrt{35} - 8\sqrt{3}$$

13. Evaluate the integral $\int \sqrt{4x^2+25}\mathrm{d}x$. Show all work.

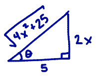
$$= \int \sqrt{(2x)^2 + 5^2}\, dx$$

$2x = 5\tan\theta$
$dx = \frac{5}{2}\sec^2\theta\, d\theta$

$$\longmapsto \frac{5}{2}\int \sqrt{5^2\tan^2\theta + 5^2}\,\sec^2\theta\, d\theta = \frac{25}{2}\int \sec^3\theta\, d\theta = \frac{25}{4}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big] + C$$

$$= \frac{25}{4}\left[\frac{2x\sqrt{4x^2+25}}{25} + \ln\left|\frac{\sqrt{4x^2+25}}{5} + \frac{2x}{5}\right|\right] + C$$

$\tan\theta = \frac{2x}{5}$ "$\frac{O}{A}$"



$\sec\theta = \frac{\sqrt{4x^2+25}}{5}$

$$\int \sec^3\theta\, d\theta = \int \sec\theta\sec^2\theta\, d\theta = \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\, d\theta = \sec\theta\tan\theta - \int(\sec^2\theta - 1)\sec\theta\, d\theta = \sec\theta\tan\theta - \int\sec^3\theta\, d\theta + \int\sec\theta\, d\theta$$

$u = \sec\theta$ $\quad v = \tan\theta$
$du = \sec\theta\tan\theta\, d\theta$ $\quad dv = \sec^2\theta\, d\theta$

$$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| - \int\sec^3\theta\, d\theta \longrightarrow \int\sec^3\theta\, d\theta = \tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

14. Evaluate the integral $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2(x)\mathrm{d}x$. Show all work.

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2(x) - 1\, dx = \tan(x) - x\,\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \sqrt{3} - \frac{\pi}{3} - \frac{\sqrt{3}}{3} + \frac{\pi}{6} = \sqrt{3} - \frac{\sqrt{3}}{3} - \frac{\pi}{6}$$

15. Evaluate the integral $\int \sin^2(x)\cos^4(x)\mathrm{d}x$. Show all work.

$$= \int \sin^2(x)\left(\cos^2(x)\right)^2 dx = \int\left[\tfrac{1}{2}(1-\cos(2x))\right]\left[\tfrac{1}{2}(1+\cos(2x))\right]^2 dx = \frac{1}{8}\int(1-\cos(2x))(1+\cos(2x))^2\, dx = \frac{1}{8}\int(1-\cos^2(2x))(1+\cos(2x))\, dx$$

$$= \frac{1}{8}\int 1 + \cos(2x) - \cos^2(2x) - \cos^3(2x)\, dx =$$

$$= \frac{1}{8}\int 1\, dx + \frac{1}{8}\int\cos(2x)\, dx - \frac{1}{8}\int\cos^2(2x)\, dx - \frac{1}{8}\int\cos^3(2x)\, dx$$

$$= \frac{1}{8}x + \frac{1}{16}\sin(2x) - \frac{1}{16}\int 1 + \cos(4x)\, dx - \frac{1}{8}\int(1 - \sin^2(2x))\cos(2x)\, dx$$

$u = \sin(2x)$, $\frac{1}{2}du = \cos(2x)dx$

$$= \frac{1}{8}x + \frac{1}{16}\sin(2x) - \frac{1}{16}x - \frac{1}{64}\sin(4x) - \frac{1}{16}\int 1 - u^2\, du$$

$$\frac{1}{16}\left(u - \tfrac{1}{3}u^3\right)$$

$$= \frac{1}{16}x + \frac{1}{16}\sin(2x) - \frac{1}{64}\sin(4x) - \frac{1}{16}\sin(2x) + \frac{1}{48}\sin^3(2x) + C$$

$$= \frac{1}{16}x - \frac{1}{64}\sin(4x) + \frac{1}{48}\sin^3(2x) + C$$

$$\frac{4x}{(x+1)(x^2+1)}$$ Not cts at $x=-1$

$''$

16. Evaluate the integral $\int_0^1 \frac{4x}{x^3+x^2+x+1} dx$. Show all work.

Scratchwork:

$$\frac{4x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$4x = A(x^2+1) + (Bx+C)(x+1)$$

$$4x = (A+B)x^2 + (B+C)x + (A+C)$$

$$\begin{aligned} A+B &= 0 \\ B+C &= 4 \\ A+C &= 0 \end{aligned} \longmapsto \begin{aligned} B-C &= 0 \\ B+C &= 4 \end{aligned} \longmapsto \begin{aligned} B&=2 \\ C&=2 \\ A&=-2 \end{aligned} \longmapsto \frac{4x}{(x+1)(x^2+1)} = \frac{2x+2}{x^2+1} - \frac{2}{x+1}$$

$$\int_0^1 \frac{2x+2}{x^2+1} - \frac{2}{x+1}\, dx = \int_0^1 \frac{2x}{x^2+1} dx + \int_0^1 \frac{2}{x^2+1} dx - \int_0^1 \frac{2}{x+1} dx$$

$$= \ln|x^2+1|\Big|_0^1 + 2\arctan(x)\Big|_0^1 - 2\ln|x+1|\Big|_0^1$$

$$= \ln 2 - \ln 1 + 2\arctan(1) - 2\arctan(0) - 2\ln 2 + 2\ln 1$$

$$= -\ln 2 + \frac{\pi}{2}$$

17. Evaluate the integral $\int \frac{6x^2+8x+3}{x^3+x} dx$. Show all work.

$'' \quad \frac{6x^2+8x+3}{x(x^2+1)}$ Not cts at $x=0$

Scratchwork:

$$\frac{6x^2+8x+3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$6x^2+8x+3 = A(x^2+1) + (Bx+C)x$$

$$6x^2+8x+3 = (A+B)x^2 + Cx + A$$

$$\begin{aligned} A+B &= 6 \\ C &= 8 \\ A &= 3 \end{aligned} \longmapsto \begin{aligned} A&=3 \\ B&=3 \\ C&=8 \end{aligned} \longmapsto \frac{6x^2+8x+3}{x(x^2+1)} = \frac{3}{x} + \frac{3x+8}{x^2+1}$$

$$\int \frac{3}{x} + \frac{3x+8}{x^2+1} dx = \int \frac{3}{x} + \frac{3x}{x^2+1} + \frac{8}{x^2+1} dx$$

$$= 3\ln|x| + \frac{3}{2}\ln|x^2+1| + 8\arctan(x) + C$$

18.Determine whether the improper integral $\int_0^\infty \cos(x) dx$ is convergent or divergent. Evaluate the integral if convergent, or explain why it diverges. Show all work.

$$\lim_{t\to\infty} \int_0^t \cos(x)\, dx = \lim_{t\to\infty} \sin(x)\Big|_0^t = \lim_{t\to\infty} \sin(t) - \sin(0) = \lim_{t\to\infty} \sin(t)$$

This limit DOES NOT EXIST.

Thus, the integral DIVERGES.

19. Determine whether the improper integral $\int_{-2}^{0} \frac{1}{x^2+5x+6} dx$ is convergent or divergent. Evaluate the integral if convergent, or explain why it diverges. Show all work.

$$\frac{1}{x^2+5x+6} = \frac{1}{(x+3)(x+2)}$$

Not continuous at $x=-3$, $\underline{\underline{x=-2}}$

Need to look at $\lim_{t\to -2^+} \int_t^0 \frac{1}{x^2+5x+6} dx$

Scratchwork:

$$\frac{1}{(x+3)(x+2)} = \frac{A}{x+3} + \frac{B}{x+2}$$

$$1 = (A+B)x + (2A+3B)$$

$1 = 2A+3B$, $0 = A+B$ $\mapsto$ $B=1$, $A=-1$ $\mapsto$ $\frac{1}{(x+3)(x+2)} = \frac{1}{x+3} - \frac{1}{x+2}$

$$\lim_{t\to-2^+} \int_t^0 \frac{1}{x+3} - \frac{1}{x+2} dx = \lim_{t\to-2^+} \left(\ln|x+3| - \ln|x+2|\right)\Big|_t^0 = \lim_{t\to-2^+} \left(\ln\left|\frac{x+3}{x+2}\right|\right)\Big|_t^0 = \lim_{t\to-2^+} \left(\ln\frac{3}{2} - \ln\left|\frac{t+3}{t+2}\right|\right) = -\infty$$

So the integral DIVERGES

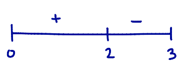
20. A particle moves along a line with velocity function $v(t) = t^3 - 7t^2 + 10t$, where v is measured in meters per second. Find (a) the displacement and (b) the distance traveled by the particle during the time interval $[0, 3]$.

a) $\int_0^3 t^3 - 7t^2 + 10t \, dt = \frac{1}{4}t^4 - \frac{7}{3}t^3 + 5t^2 \Big|_0^3 = \left(\frac{81}{4} - \frac{7\cdot 21}{3} + 45\right) - (0-0+0) = \frac{81}{4} - 63 + 45 = \frac{81}{4} - 18 = \frac{9}{4}$ m

b) $\int_0^3 |t^3 - 7t^2 + 10t| \, dt = \int_0^2 |t^3 - 7t^2 + 10t| \, dt + \int_2^3 |t^3 - 7t^2 + 10t| \, dt = \int_0^2 t^3 - 7t^2 + 10t \, dt - \int_2^3 t^3 - 7t^2 + 10t \, dt$

$= \left(\frac{1}{4}t^4 - \frac{7}{3}t^3 + 5t^2\right)\Big|_0^2 - \left(\frac{1}{4}t^4 - \frac{7}{3}t^3 + 5t^2\right)\Big|_2^3$

$t(t^2 - 7t + 10) = 0$
$t(t-5)(t-2) = 0$
$t = 0, 2, 5$



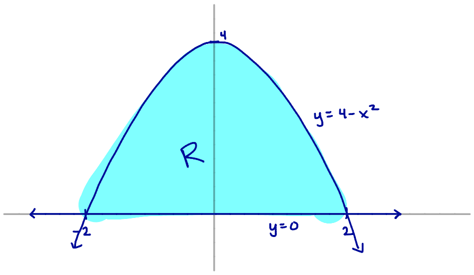
$= \left[\left(4 - \frac{56}{3} + 20\right) - (0-0+0)\right] - \left[\left(\frac{81}{4} - 63 + 45\right) - \left(4 - \frac{56}{3} + 20\right)\right]$

$= 8 - \frac{112}{3} + 40 - \frac{81}{4} + 63 - 45$

$= 66 - \frac{112}{3} - \frac{81}{4}$

$= \frac{101}{12}$

21-22. Let R be the region bounded by the graphs of $y = 4 - x^2$ and $y = 0$.
(a) Sketch the region R and find its area.



$$A = \int_{-2}^{2} (4-x^2) - (0)\, dx$$

$$= \int_{-2}^{2} 4 - x^2\, dx$$

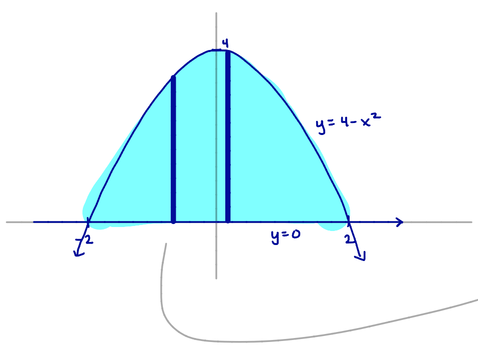
$$= 4x - \frac{1}{3}x^3 \Big|_{-2}^{2}$$

$$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$

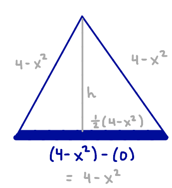
$$= 16 - \frac{16}{3}$$

$$= \frac{32}{3}$$

(b) Set up an integral to compute the volume of the solid whose cross-sections perpendicular to x-axis are equilateral triangles. Do not evaluate the integral!



Cross sections $\perp$ to x-axis look like:



$$h^2 + \left(\tfrac{1}{2}(4-x^2)\right)^2 = 4 - x^2$$

$$\Rightarrow \quad h = \frac{\sqrt{3}}{2}(4-x^2)$$

Area of each cross-section:

$$A(x) = \frac{1}{2}\, b(x)\, h(x)$$

$$= \frac{1}{2}(4-x^2) \cdot \frac{\sqrt{3}}{2}(4-x^2)$$

$$V = \int_a^b A(x)\, dx = \int_{-2}^{2} \frac{\sqrt{3}}{4}(4-x^2)^2\, dx$$

23. Find the exact length of the curve $y = 2\ln\left(\sin \frac{1}{2}x\right)$ for $\frac{\pi}{3} \leq x \leq \pi$.

$a = \frac{\pi}{3},\ b = \pi$

$L = \int_a^b \sqrt{[f'(x)]^2 + 1}\ dx$

$f(x) = 2\ln(\sin(\frac{1}{2}x))$

$f'(x) = \frac{2}{\sin(\frac{1}{2}x)} \cdot \cos(\frac{1}{2}x) \cdot \frac{1}{2}$

$= \cot(\frac{1}{2}x)$

$[f'(x)]^2 = \cot^2(\frac{1}{2}x)$

$[f'(x)]^2 + 1 = \cot^2(\frac{x}{2}) + 1$

$= \csc^2(\frac{1}{2}x)$

$L = \int_{\frac{\pi}{3}}^{\pi} \sqrt{\csc^2(\frac{1}{2}x)}\ dx$

$= \int_{\frac{\pi}{3}}^{\pi} \csc(\frac{1}{2}x)\ dx$

$= -\ln|\csc(\frac{x}{2}) + \cot(\frac{x}{2})|\ \Big|_{\frac{\pi}{3}}^{\pi}$

$= -\ln|\csc(\frac{\pi}{2}) + \cot(\frac{\pi}{2})| + \ln|\csc(\frac{\pi}{6}) + \cot(\frac{\pi}{6})|$

$= -\ln|1 + 0| + \ln|2 + \sqrt{3}|$

$= \ln|2 + \sqrt{3}|$

24. Find the average of the function $f(x) = x^{-2}e^{\frac{1}{x}}$ over the interval $[1, 3]$. Show all work.

$a = 1,\ b = 3$

$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$

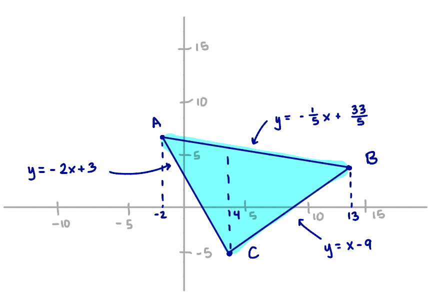
$f_{ave} = \frac{1}{3-1}\int_1^3 x^{-2}e^{\frac{1}{x}}\,dx$

$u = \frac{1}{x}$
$-du = x^{-2}dx$

$\longmapsto\ -\frac{1}{2}\int_1^{\frac{1}{3}} e^u\,du = \frac{1}{2}\int_{\frac{1}{3}}^{1} e^u\,du = \frac{1}{2}e^u\Big|_{\frac{1}{3}}^{1} = \frac{1}{2}e - \frac{1}{2}e^{\frac{1}{3}}$

25. Using integration find the area of the triangle with vertices $A = (-2, 7)$, $B = (13, 4)$, $C = (4, -5)$ and sides AB : $x + 5y = 33$, $BC : x - y = 9$, $CA : 2x + y = 3$.

$y = -\frac{1}{5}x + \frac{33}{5}$ $\qquad y = x - 9$ $\qquad y = -2x + 3$



$\int_{-2}^{4} (-\frac{1}{5}x + \frac{33}{5}) - (-2x+3)\,dx + \int_4^{13} (-\frac{1}{5}x + \frac{33}{5}) - (x-9)\,dx$

$= \int_{-2}^{4} \frac{9}{5}x + \frac{18}{5}\,dx + \int_4^{13} -\frac{6}{5}x + \frac{78}{5}\,dx$

$= \left(\frac{9}{10}x^2 + \frac{18}{5}x\right)\Big|_{-2}^{4} + \left(-\frac{6}{10}x^2 + \frac{78}{5}x\right)\Big|_4^{13}$

$= \left(\frac{144}{10} + \frac{72}{5} - \frac{36}{10} + \frac{36}{5}\right) + \left(-\frac{(169)(6)}{10} + \frac{(78)(13)}{5} + \frac{96}{10} - \frac{312}{5}\right)$

$= \frac{1}{5}\left[72 + 72 - 18 + 36 - (169)(3) + (78)(13) + 48 - 312\right]$

$= \frac{405}{5}$

$= 81$

26. For the sequence $a_n = \frac{-n^3}{n^2+1}$ determine if the sequence is (a) monotone, (b) bounded, and (c) what conclusion can you make based on (a) and (b)?

a) Consider $f(x) = \frac{-x^3}{x^2+1}$

$$f'(x) = \frac{-3x^2(x^2+1) - (-x^3)(2x)}{x^2+1} = \frac{-3x^4 - 3x^2 + 2x^3}{x^2+1} = \frac{\underbrace{x^2}_{\text{positive}}\underbrace{(-3x^2-3x+2)}_{\text{negative when } x \geq 1}}{\underbrace{x^2+1}_{\text{positive}}}$$

So $f'(x) \leq 0$ when x is bigger than or equal to 1

So $f(x)$ is monotone (in fact, it is decreasing)

So $\{a_n\}$ is monotone.

b) $\{a_n\}$ is decreasing, so as n gets LARGER, the a_n's get SMALLER. So all the a_n's must be smaller than a_1 $(a_1 = -\frac{1}{2})$

$\Longrightarrow$ $\{a_n\}$ is bounded from above.

However, note that as $n \to \infty$, $a_n \to -\infty$. So there isn't a number which is smaller than all the a_n's.

$\Longrightarrow$ $\{a_n\}$ is NOT bounded from below

Thus, $\{a_n\}$ is NOT BOUNDED.

c) Nothing.

27. Use the Squeeze Theorem to show that the sequence $b_n = \frac{(-1)^{n+1}}{n^2}$ converges.

$$-1 \leq (-1)^{n+1} \leq 1$$

$$\Downarrow$$

$$\underbrace{\frac{-1}{n^2}}_{\to 0 \text{ as } n\to\infty} \leq \frac{(-1)^{n+1}}{n^2} \leq \underbrace{\frac{1}{n^2}}_{\to 0 \text{ as } n\to\infty}$$

So $\frac{(-1)^{n+1}}{n^2} \longrightarrow 0$ as $n \to \infty$

28. Determine the general term formula for the sequence $\{2, -3, \frac{9}{2}, -\frac{27}{4}, \frac{81}{8} \ldots\}$. Use the formula to find the 50^{th} term.

$\times -\frac{3}{2}$ $\times -\frac{3}{2}$ $\times -\frac{3}{2}$ $\times -\frac{3}{2}$

$a_1 = 2$
$a_2 = 2(-\frac{3}{2})^1$
$a_3 = 2(-\frac{3}{2})^2$
$a_4 = 2(-\frac{3}{2})^3$
...
$a_n = 2(-\frac{3}{2})^{n-1}$
...
$a_{50} = 2(-\frac{3}{2})^{50-1} = -2(\frac{3}{2})^{49}$

29-31. For each of the following sequences $\{b_n\}_{n=1}^{\infty}$, compute the $\lim_{n\to\infty} b_n$. If a limit doesn't exist, expain why not. Show all work.

(a) $b_n = e^{-n} \cdot n$

$\lim_{n\to\infty} \frac{n}{e^n} = \text{"}\frac{\infty}{\infty}\text{"}$ (indeterminate!)

$\lim_{n\to\infty} \frac{n}{e^n} = \lim_{x\to\infty} \frac{x}{e^x} \underset{\text{L'Hôpital}}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$ CONVERGES!

(b) $b_n = \cos\left(\frac{\pi n}{4}\right)$

$\{b_n\} = \{0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1, \ldots\}$

DOES NOT CONVERGE (DIVERGES)

(c) $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0$ CONVERGES!

32. Find the sum $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$. $\quad = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \longrightarrow 1 = (A+B)n + A \longrightarrow \begin{matrix} A = 1 \\ A+B=0 \end{matrix} \longrightarrow \begin{matrix} A=1 \\ B=-1 \end{matrix} \longrightarrow \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^2+n} = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= \lim_{N\to\infty} \left[\underset{n=1}{\left(\frac{1}{1} - \frac{1}{2}\right)} + \underset{n=2}{\left(\frac{1}{2} - \frac{1}{3}\right)} + \underset{n=3}{\left(\frac{1}{3} - \frac{1}{4}\right)} + \dots + \underset{n=N-2}{\left(\frac{1}{N-2} - \frac{1}{N-1}\right)} + \underset{n=N-1}{\left(\frac{1}{N-1} - \frac{1}{N}\right)} + \underset{n=N}{\left(\frac{1}{N} + \frac{1}{N+1}\right)} \right]$$

$$= \lim_{N\to\infty} \left[1 + \frac{1}{N+1} \right]$$

$$= 1$$

33. Use the Divergence Test to determine that the series $\sum_{n=1}^{\infty} (-1)^n \cos(\pi n)$ diverges. Show all work.

Consider the sequence $\{b_n\} = \{(-1)^n \cos(\pi n)\} = \{1, 1, 1, 1, 1, 1, 1, \dots\}$

$\lim_{n\to\infty} b_n \neq 0$, so the series diverges.

34. Use the Alternating Series Test to determine whether the series $\sum_{n=1}^{\infty} \frac{n\cos(\pi n)}{n^5+1}$ is convergent or divergent. Show all work.

Note: $\{\cos(\pi n)\} = \{-1, 1, -1, 1, -1, 1, \dots\}$

$= \{(-1)^n\}$

So we can rewrite the series as: $\sum_{n=1}^{\infty} \frac{n(-1)^n}{n^5+1}$

$b_n = \frac{n}{n^5+1}$

positive ✓

goes to 0 as $n\to\infty$ ✓

decreasing? $\hookrightarrow$ $f(x) = \frac{x}{x^5+1}$

$f'(x) = \frac{(x^5+1) - x(5x^4)}{(x^5+1)^2} = \frac{(-4x^5+1)}{(x^5+1)^2}$ neg when $x \geq 1$ pos

So $f'(x) \leq 0$ when $x \geq 0$

So b_n is decreasing ✓

By the AST, the series CONVERGES.

35. Use the Direct or Limit Comparison Test to determine whether the series $\sum_{n=1}^{\infty} \frac{n^2-1}{n^3-n-2}$ is convergent or divergent. Show all work.

LIMIT COMPARISON TEST:

$??? \to \sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n}$ Harmonic $\therefore$ DIVERGENT

$\frac{n^2-1}{n^3-n-2}$: non-neg for all n? YES

n^2-1 positive for $n \geq 1$ ✓

$n^3-n-2 \geq 0 \iff n^3-n \geq 2 \iff n(n^2-1) \geq 2$ true for all $n \geq 2$ (good enough!) ✓

$\frac{1}{n}$: positive for all n? YES

$$\lim_{n\to\infty} \left(\frac{n^2-1}{n^3-n-2} \cdot \frac{n}{1} \right) = \lim_{n\to\infty} \left(\frac{n^3-n}{n^3-n-2} \right) = 1$$

So $\sum_{n=1}^{\infty} \frac{n^2-1}{n^3-n-2}$ also diverges.

36. Use the Ratio or Root Test to determine whether the following series is convergent or divergent. Show all work.

(a) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ ← RATIO TEST

Note: $\frac{(2n)!}{(n!)^2}$ is never 0 ✓

Scratchwork:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2(n+1))!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!}\right| = \left|\frac{(2n+2)!}{(2n)!} \cdot \frac{(n!)^2}{((n+1)!)^2}\right| = \cdots = \left|\frac{(2n+2)(2n+1)}{1} \cdot \frac{1}{(n+1)(n+1)}\right| = \left|\frac{(2n+2)(2n+1)}{(n+1)^2}\right| = \frac{(2n+2)(2n+1)}{(n+1)^2}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} = 4$$ ← larger than 1!

So the series DIVERGES by the Ratio Test.

(b) $\sum_{n=1}^{\infty} \frac{2^n(-1)^n}{n^n}$ ← ROOT TEST

Scratchwork:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{2^n(-1)^n}{n^n}\right|} = \sqrt[n]{\left|\frac{2\cdot(-1)}{n}\right|^n} = \frac{2}{n}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2}{n} = 0$$ ← smaller than 1!

So the series CONVERGES (ABSOLUTELY) by the Root Test

37. Use any Convergence Test to determine whether the following series is convergent or divergent. Show all work.

(a) $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$

INTEGRAL TEST

$f(x) = \frac{1}{x \ln x}$ on $[3, \infty)$

continuous ✓

positive ✓

decreasing ✓ $\quad f'(x) = -[x \ln x]^{-2}\left[\ln x + \frac{x}{x}\right] = -\frac{(\ln x + 1)}{(x \ln x)^2}$ always negative on $[3, \infty)$

$\int_3^\infty \frac{1}{x \ln x}\,dx$ conv or div? $\lim_{t \to \infty} \int_3^t \frac{1}{x \ln x}\,dx = \lim_{t \to \infty} \int^t \frac{1}{u}\,du = \lim_{t \to \infty} \ln|u| \Big|_{x=3}^{\infty} = \lim_{t \to \infty} \ln|\ln(x)| \Big|_{x=3}^{t} = \lim_{t \to \infty} \left[\ln|\ln(t)| - \ln|\ln(3)|\right] = \infty$

The integral diverges.

Thus, the series diverges.

(b) $\sum_{n=2}^{\infty} \frac{\ln n}{n^3 - 5}$

COMPARISON TEST

Note: $\ln(n) \le n$ for all n

$n^3 - 5 \geq 0$ for all $n \geq 2$

$n^3 - 5 \le n^3$ for all n

So $\frac{\ln(n)}{n^3 - 5} \le \frac{n}{n^3 - 5}$

$\sum \frac{n}{n^3-5} \underset{???}{\leftrightsquigarrow} \sum \frac{n}{n^3} = \sum \frac{1}{n^2}$ p-series, $p = 2$

$\therefore$ CONVERGENT

$\frac{n}{n^3-5}$ positive ✓

$\frac{1}{n^2}$ positive ✓

$\lim_{n \to \infty} \left| \frac{n}{n^3-5} \cdot \frac{n^2}{1} \right| = \lim_{n \to \infty} \left| \frac{n^3}{n^3 - 5} \right| = 1$

So $\sum \frac{n}{n^3-5}$ also CONVERGES, by the LCT

Thus, $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^3 - 5}$ also converges.

(c) $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

COMPARISON TEST

Note: $\frac{n!}{2^n} = \frac{n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1}{2 \cdot 2 \cdot 2 \cdots 2 \cdot 2 \cdot 2}$ ← seems like it should diverge (want to bound it from BELOW)

$= \frac{n}{2} \cdot \frac{n-1}{2} \cdot \frac{n-2}{2} \cdots \frac{3}{2} \cdot \frac{2}{2} \cdot \frac{1}{2}$

$\geq \frac{n}{2} \cdot 1 \cdot 1 \cdots 1 \cdot 1 \cdot \frac{1}{2}$

$= \frac{n}{4}$

$\underset{\text{pos}}{\frac{n!}{2^n}} \geq \underset{\text{pos}}{\frac{n}{4}}$

$\sum_{n=1}^{\infty} \frac{n}{4}$ Diverges $\left(\frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^{-1}} \quad \text{p-series } p \le 1 \right)$

So $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ also Diverges.

38. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{\sqrt[3]{n}}$.

RATIO TEST: Want $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

Scratchwork:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+2} x^{n+1}}{\sqrt[3]{n+1}} \cdot \frac{\sqrt[3]{n}}{(-1)^{n+1} x^n} \right| = |x| \sqrt[3]{\frac{n}{n+1}}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} |x| \sqrt[3]{\frac{n}{n+1}} = |x| \quad \ldots$$

Want $|x| < 1$ $\quad R = 1$

$-1 < x < 1$

$x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (-1)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} -\frac{1}{n^{1/3}}$ p-series $p < 1$ DIVERGES

$x = 1$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (1)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$ AST $\rightarrow$ CONVERGES

I: $(-1, 1]$

39. Find the first three nonzero terms of the Taylor series expansion of $f(x) = \sin^2(x)$ about $x = \pi$.

k-th term: $\frac{1}{k!} f^{(k)}(\pi)(x-\pi)^k$

$k=0$	$f^{(0)}(x) = \sin^2(x)$	$f^{(0)}(\pi) = 0$	
$k=1$	$f^{(1)}(x) = 2\sin(x)\cos(x) = \sin(2x)$	$f^{(1)}(\pi) = 0$	
$k=2$	$f^{(2)}(x) = 2\cos(2x)$	$f^{(2)}(\pi) = 2$	$\leftarrow \frac{1}{2!}(2)(x-\pi)^2$
$k=3$	$f^{(3)}(x) = -4\sin(2x)$	$f^{(3)}(\pi) = 0$	
$k=4$	$f^{(4)}(x) = -8\cos(2x)$	$f^{(4)}(\pi) = -8$	$\leftarrow \frac{1}{4!}(-8)(x-\pi)^4$
$k=5$	$f^{(5)}(x) = 16\sin(2x)$	$f^{(5)}(\pi) = 0$	
$k=6$	$f^{(6)}(x) = 32\cos(2x)$	$f^{(6)}(\pi) = 32$	$\leftarrow \frac{1}{6!}(32)(x-\pi)^6$

40. Find the Maclaurin series for $f(x) = x\cos(x)$.

Maclaurin series for $\cos(x)$:

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

Thus, $x \cdot \cos(x) = x \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \sum_{k=0}^{\infty} x \cdot \frac{(-1)^k}{(2k)!} x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k+1}$