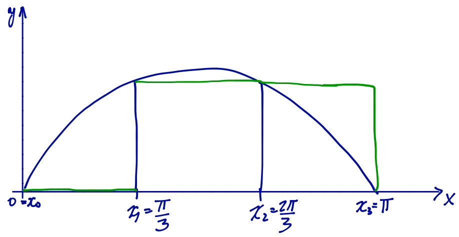
Practice Final 1

by Nikki Fider and Anton Butenko

December 3, 2014

1. Estimate the area under $f(x) = \sin(x)$ on the interval $[0, \pi]$ by computing the Riemann sum using three subintervals and left endpoints.



$n = 3, \quad a = 0, \quad b = \pi, \quad \Delta x = \frac{b-a}{n} = \frac{\pi - 0}{3} = \frac{\pi}{3}$

$x_0 = a = 0$ $\qquad f(0) = 0$

$x_1 = a + \Delta x = \frac{\pi}{3}$ $\qquad f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$x_2 = a + 2\Delta x = \frac{2\pi}{3}$ $\qquad f\left(\frac{2\pi}{3}\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

$x_3 = a + 3\Delta x = b = \pi$

$$L_3 = \sum_{i=0}^{2} f(x_i)\Delta x = \frac{\pi}{3}\left(0 + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = \frac{\pi\sqrt{3}}{3}$$

2. Evaluate the integral $\int_{-2\pi}^{2\pi} \sqrt{4\pi^2 - x^2}\sin(x)\,\mathrm{d}x$. Show all work.

Notice that $f(x) = \sqrt{4\pi^2 - x^2}\sin x\, dx$ is odd, because

$$f(-x) = \sqrt{4\pi^2 - (-x)^2}\sin(-x) = -\sqrt{4\pi^2 - x^2}\sin x = -f(x)$$

And $[-2\pi; 2\pi]$ is symmetric about 0, thus $\int_{-2\pi}^{2\pi} f(x) = 0$

3. Write the integral $\int_5^{10} (6x + \cos(x) - 1)\,\mathrm{d}x$ as a limit of Riemann sums using right endpoints..

$f(x) = 6x + \cos x - 1$

$a = 5, \; b = 10, \; \Delta x = \frac{10-5}{n} = \frac{5}{n}$

$x_i = a + i\Delta x = 5 + \frac{5i}{n}$ $\qquad f(x_i) = 6x_i + \cos(x_i) - 1 = 6\left(5 + \frac{5i}{n}\right) + \cos\left(5 + \frac{5i}{n}\right) - 1$

$$\int_5^{10} f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[6\left(5 + \frac{5i}{n}\right) + \cos\left(5 + \frac{5i}{n}\right) - 1\right] \cdot \frac{5}{n}$$

4. A function for the basal metabolism rate, in kcal/h, of a young man is $R(t)$, where t is the time in hours measured from 5 : 00 AM. What does the integral $\int_0^{24} R(t)\mathrm{d}t$ represent? What are the units?

Integral represents the amount of calories burnt by a young man during 24-hr period

its units are kcal

5. (a) Given $\int_1^{-3} h(x)\mathrm{d}x = 2$, $\int_{-3}^{1} 3f(t)\mathrm{d}t = 6$. Evaluate $\int_{-3}^{1} \left(4f(z) - \frac{1}{2}h(z)\right)\mathrm{d}z$.

$\int_1^{-3} h(x)dx = 2 \Rightarrow \int_{-3}^{1} h(x)dx = -2$ and $\int_{-3}^{1} 3f(t)\,dt = 6 \Rightarrow \int_{-3}^{1} f(t)dt = 2$

$\int_{-3}^{1} 4f(z) - \frac{1}{2}h(z)\,dz = 4\int_{-3}^{1} f(z)dz - \frac{1}{2}\int_{-3}^{1} h(z)dz = 4\cdot 2 - \frac{1}{2}\cdot(-2) = 9$

(b) Let $h(x) = \int_{e^x+x}^{x\ln x} t\mathrm{d}t$. Find $h'(x)$. Using the FTC and Chain Rule: $\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))\,g'(x) - f(h(x))\cdot h'(x)$

First, find $(x\ln x)' = \ln x + 1$ and $(e^x + x)' = e^x + 1$

Then apply the formula: $\frac{d}{dx}\int_{e^x+x}^{x\ln x} t\,dt = (x\ln x)\cdot(\ln x + 1) + (e^x + x)(e^x + 1)$

6. Evaluate $\int_0^1 \sqrt{v}\left(v^3 + 2\right)^2 \mathrm{d}v$

$\int_0^1 \sqrt{v}\cdot(v^6 + 4v^3 + 4)\,dv = \int_0^1 \left(v^{\frac{13}{2}} + 4v^{\frac{7}{2}} + 4v^{\frac{1}{2}}\right)dv = \left(\frac{2}{15}v^{\frac{15}{2}} + 4\cdot\frac{2}{9}\cdot v^{\frac{9}{2}} + 4\cdot\frac{2}{3}\cdot v^{\frac{3}{2}}\right)\Big|_0^1 = \frac{2}{15} + \frac{8}{9} + \frac{8}{3} = \frac{6 + 40 + 120}{45} = \frac{166}{45}$

7. Suppose $\int_1^{e^2} f(z)\mathrm{d}z = 10$. Evaluate $\int_0^1 e^{2x} f\left(e^{2x}\right)\mathrm{d}x$.

substitution

$z = e^{2x}$

$dz = 2e^{2x}dx$

$\frac{1}{2}dz = e^{2x}dx$

x	0	1
z	1	e^2

$= \int_1^{e^2} f(z)\frac{1}{2}dz = \frac{1}{2}\int_1^{e^2} f(z)dz = \frac{1}{2}\cdot 10 = 5$

8. Evaluate the integral $\int_1^2 \frac{x^{10}}{1+x^{22}}\mathrm{d}x$. Show all work.

substitution

$u = x^{11}$

$du = 11x^{10}dx$

$\frac{1}{11}du = x^{10}dx$

x	1	2
u	1	2048

$1+x^{22} = 1+u^2$

$= \int_1^{2048} \frac{1}{1+u^2}\frac{1}{11}du = \frac{1}{11}\arctan u \Big|_1^{2048} = \frac{1}{11}\arctan(2048) - \frac{\pi}{44}$

9. Evaluate the integral $\int \frac{\ln\left(e^{x\ln x}\right)}{x^2}\mathrm{d}x$. Show all work.

$\int \frac{\ln(e^{x\ln x})}{x^2}dx = \int \frac{x\ln x}{x^2}dx = \int \frac{\ln x}{x}dx =$

substitution

$u = \ln x$

$du = \frac{1}{x}dx$

$= \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$

10. Evaluate the integral $\int_0^{\frac{\pi}{2}} 3x^2\cos(x)\mathrm{d}x$. Show all work.

$$\left|\begin{array}{ll} \multicolumn{2}{c}{\text{by parts}} \\ u=3x^2 & dv=\cos x\,dx \\ du=6x & v=\sin x \end{array}\right| = 3x^2\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2} 6x\sin x\,dx = \left|\begin{array}{ll} \multicolumn{2}{c}{\text{by parts}} \\ u=6x & dv=\sin x\,dx \\ du=6dx & v=-\cos x \end{array}\right| =$$

$$= 3\left(\frac{\pi}{2}\right)^2\sin\frac{\pi}{2} - \left(-6x\cos x\Big|_0^{\pi/2} + \int_0^{\pi/2} 6\cos x\,dx\right) =$$

$$= \frac{3\pi^2}{4} - 6\sin x\Big|_0^{\pi/2} = \frac{3\pi^2}{4} - 6$$

11. Evaluate the integral $\int e^{2x}\sin(\pi x)\mathrm{d}x$. Show all work.

$$I = \int e^{2x}\sin\pi x\,dx = \left|\begin{array}{ll} \multicolumn{2}{c}{\text{by parts}} \\ u=e^{2x} & dv=\sin\pi x\,dx \\ du=2e^{2x}dx & v=\frac{-1}{\pi}\cos\pi x \end{array}\right| =$$

$$= -\frac{1}{\pi}e^{2x}\cos\pi x + \frac{2}{\pi}\int e^{2x}\cos\pi x = \left|\begin{array}{ll} \multicolumn{2}{c}{\text{by parts}} \\ u=e^{2x} & dv=\cos\pi x\,dx \\ du=2e^{2x}dx & v=\frac{1}{\pi}\sin\pi x \end{array}\right| =$$

$$= -\frac{1}{\pi}e^{2x}\cos\pi x + \frac{2}{\pi}\left[\frac{1}{\pi}e^{2x}\sin\pi x - \frac{2}{\pi}\int e^{2x}\sin\pi x\,dx\right]$$

$$I = -\frac{1}{\pi}e^{2x}\cos\pi x + \frac{2}{\pi^2}e^{2x}\sin\pi x - \frac{4}{\pi^2}I$$

$$\frac{\pi^2+4}{\pi^2}I = -\frac{1}{\pi}e^{2x}\cos\pi x + \frac{2}{\pi^2}e^{2x}\sin\pi x$$

$$I = \frac{e^{2x}}{\pi^2+4}\left(2\sin\pi x - \pi\cos\pi x\right) + C$$

12. Evaluate the integral $\int_0^1 \frac{1}{(x^2+1)^2}\mathrm{d}x$. Show all work.

$$\left|\begin{array}{l} \text{substitution} \\ x=\tan\theta \quad \begin{array}{c|c|c} x & 0 & 1 \\ \hline \theta & 0 & \pi/4 \end{array} \\ dx=\sec^2\theta\,d\theta \\ x^2+1=\tan^2\theta+1=\sec^2\theta \end{array}\right| = \int_0^{\pi/4}\frac{1}{(\sec^2\theta)^2}\cdot\sec^2\theta\,d\theta = \int_0^{\pi/4}\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/4}\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta = \left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/4} = \frac{\pi}{8} + \frac{1}{4}\sin\frac{\pi}{2} = \frac{\pi}{8} + \frac{1}{4}$$

13. Evaluate the integral $\int \frac{x^2}{\sqrt{1-9x^2}}\mathrm{d}x$. Show all work.

substitution

$x=\frac{1}{3}\sin\theta \qquad \theta=\sin^{-1}3x$

$dx=\frac{1}{3}\cos\theta\, d\theta$

$\sqrt{1-9x^2}=\sqrt{1-\sin^2\theta}=|\cos\theta|=\cos\theta$

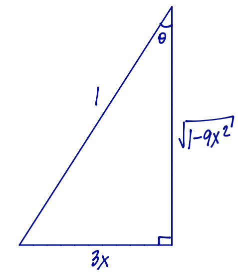
$$=\int \frac{\left(\frac{1}{3}\sin\theta\right)^2}{\cos\theta}\,\frac{1}{3}\cos\theta\, d\theta=\frac{1}{27}\int \sin^2\theta\, d\theta=$$

$$=\frac{1}{27}\int\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta=\frac{1}{54}\theta-\frac{1}{108}\sin 2\theta+C=\frac{1}{54}\sin^{-1}3x-\frac{1}{108}\sin(2\sin^{-1}3x)+C=\frac{1}{54}\sin^{-1}3x-\frac{1}{18}x\sqrt{1-9x^2}+C$$

To find $\cos(\underbrace{\sin^{-1}3x}_{\theta})$:

$\frac{op}{hyp}=\sin\theta=\frac{3x}{1}$

$\cos\theta=\frac{\sqrt{1-9x^2}}{1}$



To simplify use $\sin 2\theta=2\sin\theta\cos\theta$

$\sin(2\sin^{-1}3x)=2\sin(\sin^{-1}3x)\cos(\sin^{-1}3x)=2\,3x\cdot\cos(\sin^{-1}3x)$

14. Evaluate the integral $\int_0^{\frac{\pi}{2}}\sin^3(x)\cos^3(x)\mathrm{d}x$. Show all work.

$$\int_0^{\pi/2}\sin^3x\cos^2x\cos x\, dx=$$

substitution

$u=\sin x$

$du=\cos x\, dx$

x	0	$\pi/2$
u	0	1

$\cos^2x=1-\sin^2x=1-u^2$

$$=\int_0^1 u^3(1-u^2)\,du=\int_0^1 u^3-u^5\,du=\frac{1}{4}u^4-\frac{1}{6}u^6\Big|_0^1=\frac{1}{4}-\frac{1}{6}=\frac{1}{12}$$

15. Evaluate the integral $\int \sec^{20}(x)\tan^5(x)\mathrm{d}x$. Show all work.

$$\int \sec^{19}x\tan^4x\,\sec x\tan x\, dx=$$

substitution

$u=\sec x$

$du=\sec x\tan x\, dx$

$\tan^4x=(\tan^2x)^2=(\sec^2x-1)^2=(u^2-1)^2$

$$=\int u^{19}(u^2-1)^2\,du=\int u^{19}(u^4-2u^2+1)\,du=\int u^{23}-2u^{21}+u^{19}\,du=\frac{1}{24}u^{24}-\frac{1}{11}u^{22}+\frac{1}{20}u^{20}+C=$$

$$=\frac{1}{24}\sec^{24}x-\frac{1}{11}\sec^{22}x+\frac{1}{20}\sec^{20}x+C$$

16. Evaluate the integral $\int_0^1 \frac{2}{2x^2+3x+1} \mathrm{d}x$. Show all work.

Partial Fraction Decomposition:

$$\frac{2}{2x^2+3x+1} = \frac{2}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1} = \frac{A(x+1)+B(2x+1)}{(2x+1)(x+1)} = \frac{Ax+A+2Bx+B}{(2x+1)(x+1)} = \frac{(A+2B)x+(A+B)}{(2x+1)(x+1)} \Rightarrow \begin{cases} A+B=2 \\ A+2B=0 \end{cases} \begin{cases} B=-2 \\ A=4 \end{cases}$$

$$\frac{2}{2x^2+3x+1} = \frac{4}{2x+1} - \frac{2}{x+1}$$

$$\int_0^1 \left(\frac{4}{2x+1} - \frac{2}{x+1}\right)dx = \left(2\ln|2x+1| - 2\ln|x+1|\right)\Big|_0^1 = (2\ln 3 - 2\ln 2) - (2\ln 1 - 2\ln 1) = 2\ln\frac{3}{2}$$

17. Evaluate the integral $\int \frac{x^5+x-1}{x^3+1} \mathrm{d}x$. Show all work.

$$\begin{array}{r|l} & x^2 \\ x^3+1 & x^5+x-1 \\ & x^5+x^2 \\ \hline & -x^2+x-1 \end{array}$$

Also recall $x^3+1 = (x+1)(x^2-x+1)$

$$\int x^2 - \frac{x^2-x+1}{(x+1)(x^2-x+1)}dx = \int x^2\,dx - \int \frac{1}{x+1}dx = \frac{1}{3}x^3 - \ln|x+1| + C$$

18. Determine whether the improper integral $\int_0^\infty re^{-3r}dr$ is convergent or divergent. Evaluate the integral if convergent, or explain why it diverges. Show all work.

$$\int_0^\infty re^{-3r}dr = \left| \begin{array}{ll} \text{by parts} & \\ u=r & dv=e^{-3r}dr \\ du=dr & v=-\frac{1}{3}e^{-3r} \end{array} \right| =$$

$$= -\frac{1}{3}re^{-3r}\Big|_0^\infty + \frac{1}{3}\int_0^\infty e^{-3r}dr = -\frac{1}{3}re^{-3r}\Big|_0^\infty - \frac{1}{9}e^{-3r}\Big|_0^\infty = \left(\frac{1}{3}re^{-3r} - \frac{1}{9}e^{-3r}\right)\Big|_0^\infty =$$

$$= \left[\lim_{t\to\infty}\left(\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t}\right)\right] - \left(\cancel{\frac{1}{3}0e^{-3\cdot 0}}^{\,0} - \frac{1}{9}\cancel{e^{-3\cdot 0}}^{\,1}\right) = \left[\lim_{t\to\infty}\left(\frac{t}{3e^{3t}} - \frac{1}{9e^{3t}}\right)\right] + \frac{1}{9}$$

$$= \left[\lim_{t\to\infty} \frac{3t-1}{9e^{3t}}\right] + \frac{1}{9} \overset{LH}{=} \left[\lim_{t\to\infty} \cancel{\frac{3}{27e^{3t}}}^{\,0}\right] + \frac{1}{9} = \frac{1}{9}$$

Converges to $\frac{1}{9}$

19. Determine whether the improper integral $\int_0^{\frac{\pi}{2}} \sec^2\theta d\theta$ is convergent or divergent. Evaluate the integral if convergent, or explain why it diverges. Show all work.

$$\int_0^{\pi/2} \sec^2\theta d\theta = \lim_{t\to\pi/2} \tan\theta\Big|_0^t = \lim_{t\to\pi/2} \tan t - \tan 0 = DNE$$

diverges

20. A particle moves along a line with velocity function $v(t) = \cos t$, where v is measured in feet per hour. Find (a) the displacement and (b) the distance traveled by the particle during the time interval $[0, \frac{2\pi}{3}]$.

(a) $\int_0^{2\pi/3} \cos t\, dt = \sin t\Big|_0^{2\pi/3} = \sin 2\pi/3 - \sin 0 = \frac{\sqrt{3}}{2}$

Why $\frac{\pi}{2}$? Because $\cos t$ changes sign at $\frac{\pi}{2}$, that is $\cos t \geq 0$, $t \leq \frac{\pi}{2}$; $\cos t < 0$, $t > \frac{\pi}{2}$

(b) $\int_0^{2\pi/3} |\cos t|\, dt = \int_0^{\pi/2} |\cos t|\, dt + \int_{\pi/2}^{2\pi/3} |\cos t|\, dt$

meaning $|\cos t| = \begin{cases} \cos t, & t \leq \frac{\pi}{2} \\ -\cos t, & t > \frac{\pi}{2} \end{cases}$

$= \int_0^{\pi/2} \cos t\, dt + \int_{\pi/2}^{2\pi/3} -\cos t\, dt =$

$= \sin t\Big|_0^{\pi/2} - \sin t\Big|_{\pi/2}^{2\pi/3} =$

$= \sin\frac{\pi}{2} - \sin 0 - \sin\frac{2\pi}{3} + \sin\frac{\pi}{2} = 1 - 0 - \frac{\sqrt{3}}{2} + 1 = 2 - \frac{\sqrt{3}}{2}$

21-22. Let R be the region bounded by the graphs of $x = 2y^2$ and $x = 4 + y^2$.

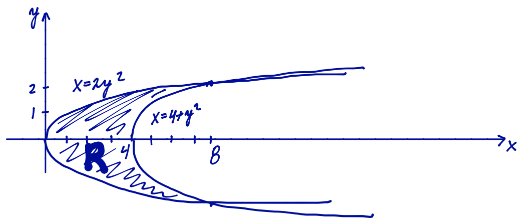
(a) Sketch the region R and find its area.

$2y^2 = 4 + y^2$

$y^2 = 4$

$y = \pm 2$

$x = 8$

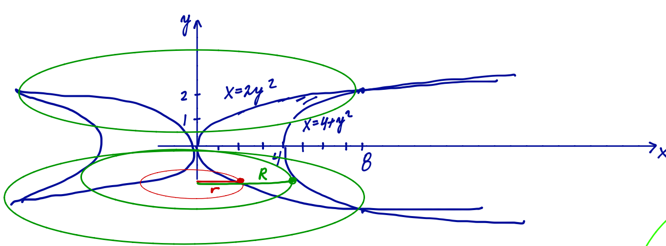


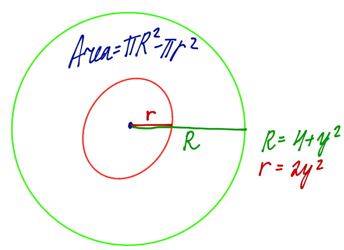
Area: $\int_{-2}^{2} (4+y^2)-(2y^2)\,dy = \int_{-2}^{2} 4-y^2\,dy = 4y - \frac{1}{3}y^3\Big|_{-2}^{2}$

$= \left(4\cdot 2 - \frac{1}{3}2^3\right) - \left(4(-2) - \frac{1}{3}(-2)^3\right) =$

$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = 8 - \frac{8}{3} + 8 - \frac{8}{3} = 16 - \frac{16}{3} = \frac{32}{3}$

(b) Set up an integral to compute the volume of the solid generated by revolving the region R (from part (a)) about the y-axis. Do not evaluate the integral!

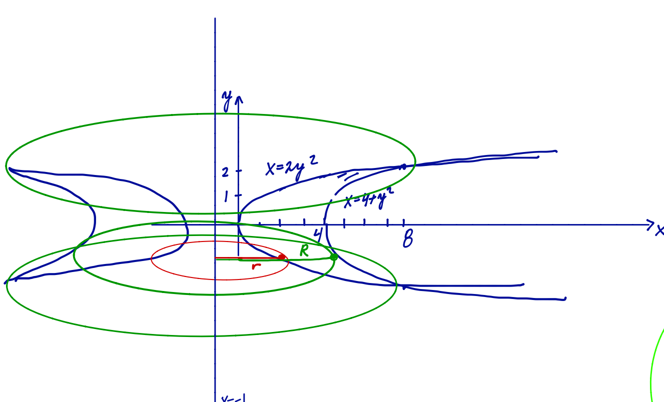


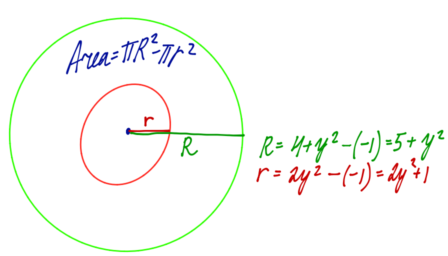


Volume $= \int_a^b A(y)\,dy$

$\pi \int_{-2}^{2} (4+y^2)^2 - (2y^2)^2\,dy$

(c) Set up an integral to compute the volume of the solid generated by revolving the region R (from part (a)) about the line $x = -1$. Do not evaluate the integral!





Volume $= \int_a^b A(y)\,dy$

$\pi \int_{-2}^{2} (5+y^2)^2 - (2y^2+1)^2\,dy$

23. Find the exact length of the curve $y = \frac{1}{4}x^2 - \ln\sqrt{x}$ for $+1 \le x \le 2$.

Recall: $L = \int_a^b \sqrt{1+[y'(x)]^2}\,dx$

$$y' = \frac{1}{2}x - \frac{1}{2x}$$

$$1+[y']^2 = 1+\left(\frac{1}{2}x - \frac{1}{2x}\right)^2 = 1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2} = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{1}{2}x + \frac{1}{2x}\right)^2$$

$$L = \int_1^2 \sqrt{\left(\frac{1}{2}x+\frac{1}{2x}\right)^2}\,dx = \int_1^2 \left|\frac{1}{2}x+\frac{1}{2x}\right| dx = \int_1^2 \left(\frac{1}{2}x+\frac{1}{2x}\right) dx = \frac{1}{4}x^2 + \frac{1}{2}\ln|x| \Big|_1^2 =$$

$$= \left(\frac{1}{4}2^2 + \frac{1}{2}\ln 2\right) - \left(\frac{1}{4}1^2 + \frac{1}{2}\ln 1\right) = \frac{3}{4} + \frac{1}{2}\ln 2$$

24. Find the average of the function $f(x) = \frac{x^4+x^2+\ln(e^x)}{x^2}$ over the interval $[2,3]$. Show all work.

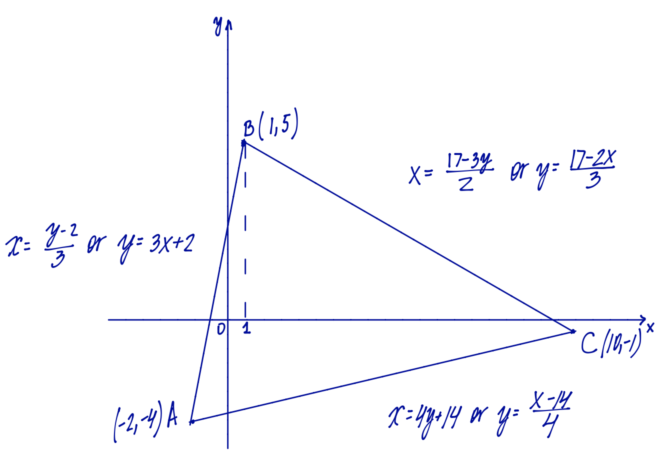
Recall: $f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$$f_{av} = \frac{1}{3-2}\int_2^3 \frac{x^4+x^2+x}{x^2}\,dx = \int_2^3 x^2+1+\frac{1}{x}\,dx =$$

$$= \left(\frac{1}{3}x^3 + x + \ln|x|\right)\Big|_2^3 = (9+3+\ln 3) - \left(\frac{8}{3}+2+\ln 2\right) =$$

$$= \frac{22}{3} + \ln\frac{3}{2}$$

25. Using integration find the area of the triangle with vertices $A=(-2,-4)$, $B=(1,5)$, $C=(10,-1)$ and sides $AB: 3x-y=-2$, $BC: 2x+3y=17$, $CA: x-4y=14$.



$$A = \int_{-2}^{1} 3x+2 - \frac{x-14}{4}\,dx + \int_1^{10} \frac{17-2x}{3} - \frac{x-14}{4}\,dx =$$

$$= \left(\frac{3}{2}x^2 + 2x - \frac{1}{8}x^2 + \frac{7}{2}x\right)\Big|_{-2}^{1} + \left(\frac{17}{3}x - \frac{1}{3}x^2 - \frac{1}{8}x^2 + \frac{7}{2}x\right)\Big|_1^{10} =$$

$$= \left(\frac{3}{2}+2-\frac{1}{8}+\frac{7}{2}\right) - \left(\frac{3}{2}\cdot 4 - 2\cdot 2 - \frac{1}{8}\cdot 4 + \frac{7}{2}(-2)\right) + \left(\frac{17}{3}10 - \frac{1}{3}10^2 - \frac{1}{8}10^2 + \frac{7}{2}10\right) - \left(\frac{17}{3} - \frac{1}{3} - \frac{1}{8} + \frac{7}{2}\right) =$$

$$= \left(7-\frac{1}{8}\right) - \left(-5-\frac{1}{2}\right) + \left(\frac{70}{3} - \frac{25}{2} + 35\right) - \left(\frac{16}{3} - \frac{1}{8} + \frac{7}{2}\right) =$$

$$= 7 - \frac{1}{8} + 5 + \frac{1}{2} + \frac{70}{3} - \frac{25}{2} + 35 - \frac{16}{3} + \frac{1}{8} - \frac{7}{2} =$$

$$= 47 + \frac{1}{2} - 16 + 18 = 49\frac{1}{2} = \frac{99}{2}$$

26. For the sequence $b_n = n^{-1}\sin\left(\frac{\pi}{2n}\right)$ determine if the sequence is (a) monotone, (b) bounded, and (c) what conclusion can you make based on (a) and (b)?

a) $f(x) = \dfrac{\sin\frac{\pi}{2x}}{x}$ $\quad f'(x) = \dfrac{\cos\frac{\pi}{2x}\cdot\left(-\frac{\pi}{2x^2}\right)\cdot x - \sin\frac{\pi}{2x}}{x^2} = \dfrac{-\pi\cos\frac{\pi}{2x} - 2x\sin\frac{\pi}{2x}}{x^3} < 0$ since when $x>0$ denominator is positive but numerator is negative

thus $\{b_n\}$ is decreasing

b) b_n is bounded above by b_1, since it's decreasing, thus $b_1 > b_2 > b_3 \ldots$

b_n is bounded below by 0, since $\dfrac{\sin\pi/2n}{n} > 0$ because both numerator and denominator are positive

b_n is bounded below and above, thus bounded

c) since b_n is decreasing and bounded it converges

27. Use the Squeeze Theorem to show that the sequence $c_n = \frac{4+\sin(n)}{3n+1}$ converges.

$$0 = \lim_{n\to\infty} \frac{3}{3n+1} \leq \lim_{n\to\infty} \frac{4+\sin n}{3n+1} \leq \lim_{n\to\infty} \frac{5}{3n+1} = 0$$

by Squeeze Theorem, $\lim_{n\to\infty} \dfrac{4+\sin n}{3n+1} = 0$

28. Determine the general term formula for the sequence $\left\{\frac{1}{10}, \frac{1}{15}, \frac{1}{20}, \frac{1}{25}, \frac{1}{30}\ldots\right\}$. Use the formula to find the 100^{th} term.

Notice : $a_1 = \frac{1}{10} = \frac{1}{2\cdot 5}$; $a_2 = \frac{1}{15} = \frac{1}{3\cdot 5}$; $a_3 = \frac{1}{20} = \frac{1}{4\cdot 5}$

Therefore: $a_n = \dfrac{1}{(n+1)5}$

So $a_{100} = \dfrac{1}{101\cdot 5} = \dfrac{1}{505}$

For each of the following sequences $\{a_n\}_{n=1}^{\infty}$, compute the $\lim_{n\to\infty} a_n$. If a limit doesn't exist, expain why not. Show all work.

29. $a_n = (-2)^n$

$\lim_{n\to\infty} (-2)^n$ DNE , because $a_n = (-2)^n$ is not bounded

30. $a_n = \arctan\left(\frac{n^5+4}{1-n^3}\right)$

Recall: $\lim_{n\to\infty} \frac{n^5+4}{1-n^3} = -\infty$ and $\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}$

$\lim_{n\to\infty} \arctan\left(\frac{n^5+4}{1-n^3}\right) = -\frac{\pi}{2}$

31. $a_n = \sin(2\pi n)$

$a_n = \{ \sin(2\pi)=0, \sin(4\pi)=0, \sin 6\pi = 0 \ldots \}$

$\lim_{n\to\infty} a_n = 0$

32. Find the sum $\sum_{n=2}^{\infty} \frac{3^n+5^n}{7^{n+1}} = \sum_{n=2}^{\infty} \frac{3^n}{7^{n+1}} + \sum_{n=2}^{\infty} \frac{5^n}{7^{n+1}}$

Note that $\sum_{n=2}^{\infty} \frac{3^n}{7^{n+1}} = \sum_{n=2}^{\infty} \frac{1}{7}\left(\frac{3}{7}\right)^n$ is a geom. series with $r=\frac{3}{7}$ and first term $n=2$: $\frac{1}{7}\left(\frac{3}{7}\right)^2 = \frac{9}{343}$, thus $\sum = \frac{9/343}{1-3/7}$

Similarly, $\sum_{n=2}^{\infty} \frac{5^n}{7^{n+1}} = \sum_{n=2}^{\infty} \frac{1}{7}\left(\frac{5}{7}\right)^n$ is a geom. series with $r=\frac{5}{7}$ and first term $n=2$: $\frac{1}{7}\left(\frac{5}{7}\right)^2 = \frac{25}{343}$, thus $\sum = \frac{25/343}{1-5/7}$

Finally, $\frac{9/343}{1-3/7} + \frac{25/343}{1-5/7} = \frac{1}{49}\left(\frac{9}{4} + \frac{25}{2}\right) = \frac{59}{196}$

33. Use the Divergence Test to determine that the series $\sum_{n=1}^{\infty} \arctan\left(\frac{1-n^2}{n}\right)$ is divergent. Show all work.

$\lim_{n\to\infty} \arctan\left(\frac{1-n^2}{n}\right) = -\frac{\pi}{2} \neq 0$ therefore by divergence test series divergent.

34. Use the Alternating Series Test to determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n}$ is convergent or divergent. Show all work.

$$\left.\begin{array}{l} 1)\ \lim_{n\to\infty} \frac{1}{4n} = 0 \\ 2)\ \frac{1}{4n} \text{ is decreasing, b/c } f(x)=\frac{1}{4x} \text{ and } f'(x) = -\frac{1}{4x^2} < 0 \\ 3)\ \frac{1}{4n} > 0 \text{, positive} \end{array}\right\}$$ by Alt. Series Test series convergent.

35. Use the Direct or Limit Comparison Test to determine whether the series $\sum_{n=1}^{\infty} \frac{5^n}{n3^n}$ is convergent or divergent. Show all work.

$$\frac{5^n}{3^n} > 1 \Rightarrow \frac{5^n}{n3^n} > \frac{1}{n} \Rightarrow \sum_{n=1}^{\infty} \frac{5^n}{n3^n} > \sum_{n=1}^{\infty} \frac{1}{n} \cdot \text{diverges } (p=1)$$

$\hookrightarrow$ diverges by comparison test

36. Use the Ratio or Root Test to determine whether the following series is convergent or divergent. Show all work.

(a) $\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{(n+1)!}$

Ratio test: $\lim_{n\to\infty} \left| \frac{\frac{(-5)^{2(n+1)}}{(n+2)!}}{\frac{(-5)^{2n}}{(n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{(-5)^{2n+2}(n+1)!}{(-5)^{2n}\,(n+2)!} \right| = \lim_{n\to\infty} \left| \frac{(-5)^2}{n+2} \right| = 0 < 1$

thus $\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{(n+1)!}$ converges by Ratio test

(b) $\sum_{n=1}^{\infty} \left(\frac{3n}{2n+1}\right)^{5n}$

Root test: $\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n}{2n+1}\right)^{5n}} = \lim_{n\to\infty} \left(\frac{3n}{2n+1}\right)^5 = \left(\frac{3}{2}\right)^5 > 1$.

thus $\sum_{n=1}^{\infty} \left(\frac{3n}{2n+1}\right)^{5n}$ diverges by root test

37. Use any Convergence Test to determine whether the following series is convergent or divergent. Show all work.

(a) $\sum_{n=3}^{\infty} \frac{(\ln n)^3}{n} = \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^3}$

$f(x) = \frac{1}{x(\ln x)^3}$ is positive, decreasing, continuous for $x \geq 3$

$$\int_3^{\infty} \frac{1}{x(\ln x)^3}\,dx = \left| \begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \\ \begin{array}{c|c|c} x & 3 & \infty \\ \hline u & \ln 3 & \infty \end{array} \end{array} \right| = \int_{\ln 3}^{\infty} \frac{du}{u^3} = \lim_{t\to\infty} -\frac{1}{2u^2}\Big|_{\ln 3}^{t} = \lim_{t\to\infty} \frac{1}{2(\ln 3)^2} - \frac{1}{2t^2} = \frac{1}{2(\ln 3)^2}$$

thus $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^3}$ converges by integral test

(b) $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1 < \infty$$ therefore $\sum_{n=2}^{\infty} \frac{1}{n^2-1} \sim \sum_{n=2}^{\infty} \frac{1}{n^2}$

but we know $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges $(p = 2 > 1)$

thus $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges by Limit Comp. Test

(c) $\sum_{n=1}^{\infty} \frac{e^n}{n^2+\ln n}$

Div. Test: $$\lim_{n\to\infty} \frac{e^n}{n^2+\ln n} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{e^x}{x^2+\ln x} \overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{2x+\frac{1}{x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{2-\frac{1}{x^2}} = \infty$$

thus $\sum_{n=1}^{\infty} \frac{e^n}{n^2+\ln n}$ diverges.

38. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{3^n(x+4)^n}{\sqrt{n}}$.

$$\lim_{n\to\infty}\left|\frac{3^{n+1}(x+4)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{3^n(x+4)^n}\right| = |x+4|\cdot\lim_{n\to\infty}\frac{3\sqrt{n}}{\sqrt{n+1}} = 3\cdot|x+4|<1$$

$$\Rightarrow |x+4|<\frac{1}{3} \Rightarrow -\frac{1}{3}<x+4<\frac{1}{3} \Rightarrow -\frac{13}{3}<x<-\frac{11}{3}$$

$x=-\frac{13}{3}$: $\sum_{n=1}^{\infty}\frac{3^n\left(-\frac{1}{3}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ - converges by Alt. Series. Test

$x=-\frac{11}{3}$: $\sum_{n=1}^{\infty}\frac{3^n\left(\frac{1}{3}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ - diverges p-series $p=\frac{1}{2}<1$

$\Rightarrow \left[-\frac{13}{3}, -\frac{11}{3}\right)$ interval of convergence with center at -4 an radius $\frac{1}{3}$

39. Find the first three nonzero terms of the Taylor series expansion of $f(x)=\ln(x)$ about $x=e$.

$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$ Taylor series with coeff's $a_n=\frac{f^{(n)}(c)}{n!}$

$f(x)=\ln x$ $\quad f(e)=\ln e=1$ $\quad a_0=1$

$f'(x)=\frac{1}{x}$ $\quad f'(e)=\frac{1}{e}$ $\quad a_1=\frac{1}{e}$

$f''(x)=-\frac{1}{x^2}$ $\quad f''(e)=-\frac{1}{e^2}$ $\quad a_2=-\frac{1}{2e^2}$

$f'''(x)=\frac{2}{x^3}$ $\quad f'''(e)=\frac{2}{e^3}$ $\quad a_3=\frac{2}{3e^3}$

$$P_4(x)=1+\frac{1}{e}(x-e)-\frac{1}{2e^2}(x-e)^2+\frac{2}{3e^3}(x-e)^3$$

40. Find the Maclaurin series for $f(x)=e^x+e^{2x}$.

$e^x=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$ $\quad e^{2x}=\sum_{n=0}^{\infty}\frac{1}{n!}(2x)^n$

$$f(x)=e^x+e^{2x}=\sum_{n=0}^{\infty}\frac{1}{n!}x^n+\sum_{n=0}^{\infty}\frac{1}{n!}(2x)^n=\sum_{n=0}^{\infty}\left(\frac{1}{n!}x^n+\frac{1}{n!}(2x)^n\right)=\sum_{n=0}^{\infty}\frac{1+2^n}{n!}x^n$$