The $\omega(q)$ mock theta function and vector-valued Maass-Poincaré series

Sharon Anne Garthwaite

October 7, 2006

History

Let $p(n)$ denote the number of partitions of n.

History

Let $p(n)$ denote the number of partitions of n.

Hardy-Ramanujan-Rademacher formula (1917,1922):

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right).$$

- $I_s(z)$ is an I-Bessel function.
- $A_k(n)$ is a "Kloosterman-type" sum.

History

In 1920 Ramanujan wrote about his discovery of "very interesting functions," such as

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}$$
$$= 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 - 5q^6 + \cdots ;$$
$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2}$$
$$= 1 + 2q + 3q^2 + 4q^3 + 6q^4 + 8q^5 + 10q^6 + \cdots .$$

Here $q := e^{2\pi i z}$.

History

Define $\alpha_f(n)$ and $\alpha_\omega(n)$ by

$$f(q) = \sum_{n\geq 0} \alpha_f(n)q^n; \quad \omega(q) = \sum_{n\geq 0} \alpha_\omega(n)q^n.$$

History

Define $\alpha_f(n)$ and $\alpha_\omega(n)$ by

$$f(q) = \sum_{n\geq 0} \alpha_f(n) q^n; \quad \omega(q) = \sum_{n\geq 0} \alpha_\omega(n) q^n.$$

Andrews-Dragonette Conjecture (1952,1966,2003):

$$\alpha_f(n) = \pi(24n-1)^{-\frac{1}{4}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot I_{1/2}\left(\frac{\pi\sqrt{24n-1}}{12k}\right).$$

- $A_k(n)$ is the $p(n)$ "Kloosterman-type" sum.
- $I_{1/2}(z)$ satisfies

$$I_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sinh(z).$$

Recent work

Zwegers (Contemp. Math., 2003) :

- Vector-valued modular forms

Recent work

Zwegers (Contemp. Math., 2003) :

- Vector-valued modular forms

Bringmann and Ono (Invent. Math., 2006) :

- Weak Maass forms

Recent work

Zwegers (Contemp. Math., 2003) :

- Vector-valued modular forms

Bringmann and Ono (Invent. Math., 2006) :

- Weak Maass forms
- Andrews-Dragonette conjecture

Main Theorem

Theorem (G.)

The coefficients $\alpha_\omega(n)$ of $\omega(q)$ are

$$\frac{\pi(3n+2)^{-1/4}}{2\sqrt{2}} \sum_{\substack{k=1 \\ (k,2)=1}}^{\infty} \frac{(-1)^{\frac{k-1}{2}} A_k\left(\frac{n(k+1)}{2} - \frac{3(k^2-1)}{8}\right)}{k} I_{1/2}\left(\frac{\pi\sqrt{3n+2}}{3k}\right).$$

Main Theorem

Theorem (G.)

The coefficients $\alpha_\omega(n)$ of $\omega(q)$ are

$$\frac{\pi(3n+2)^{-1/4}}{2\sqrt{2}} \sum_{\substack{k=1 \\ (k,2)=1}}^{\infty} \frac{(-1)^{\frac{k-1}{2}} A_k\left(\frac{n(k+1)}{2} - \frac{3(k^2-1)}{8}\right)}{k} I_{1/2}\left(\frac{\pi\sqrt{3n+2}}{3k}\right).$$

Define $c(n,m)$ by formula for $\alpha_\omega(n)$ truncated at $k = 2m-1$.

n	$\alpha_\omega(n)$	$c(n,1)$	$c(n,2)$	$c(n,1000)$
1	2	1.9949	2.2428	1.9963
5	8	7.8769	8.0420	7.9958
10	29	28.6164	29.0178	29.0000
100	1995002	1994993.7262	1995001.6972	1995001.9987

Real analytic vector-valued modular forms

Define the following:

$$F(z) := \left(q^{-\frac{1}{24}} f(q),\ 2q^{\frac{1}{3}}\omega(q^{\frac{1}{2}}),\ 2q^{\frac{1}{3}}\omega(-q^{\frac{1}{2}})\right)^T.$$

$$G(z) := 2i\sqrt{3}\int_{-\bar{z}}^{i\infty} \frac{(g_1(\tau),\ g_0(\tau),\ -g_2(\tau))^T}{\sqrt{-i(\tau+z)}}\, d\tau.$$

The $g_i(\tau)$ are the cuspidal weight 3/2 theta functions

$$H(z) := (H_0(z), H_1(z), H_2(z)) = F(z) - G(z)$$

Real analytic vector-valued modular forms

Define the following:

$$F(z) := \left(q^{-\frac{1}{24}}f(q),\ 2q^{\frac{1}{3}}\omega(q^{\frac{1}{2}}),\ 2q^{\frac{1}{3}}\omega(-q^{\frac{1}{2}})\right)^T.$$

$$G(z) := 2i\sqrt{3}\int_{-\overline{z}}^{i\infty} \frac{(g_1(\tau),\ g_0(\tau),\ -g_2(\tau))^T}{\sqrt{-i(\tau+z)}}\, d\tau.$$

The $g_i(\tau)$ are the cuspidal weight 3/2 theta functions

$$H(z) := (H_0(z), H_1(z), H_2(z)) = F(z) - G(z)$$

Real analytic vector-valued modular forms

Theorem (Zwegers)

The function $H(z)$ is a vector-valued real analytic modular form of weight $1/2$ satisfying

$$H(z+1) = \begin{pmatrix} e(-1/24) & 0 & 0 \\ 0 & 0 & e(1/3) \\ 0 & e(1/3) & 0 \end{pmatrix} H(z),$$

$$H(-1/z) = \sqrt{-iz} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(z),$$

where $e(x) := e^{2\pi i x}$.

Weak Maass forms

Theorem (Bringmann-Ono)

- $H_0(24z)$ *is a weak Maass form of weight* $1/2$ *on* $\Gamma_0(144)$ *with Nebentypus* $\left(\frac{12}{\bullet}\right)$.

Weak Maass forms

Theorem (Bringmann-Ono)

- $H_0(24z)$ *is a weak Maass form of weight* $1/2$ *on* $\Gamma_0(144)$ *with Nebentypus* $\left(\frac{12}{\bullet}\right)$.
- $H_0(24z) = P_{\frac{1}{2}}\left(\frac{3}{4}; 24z\right)$, *where*

$$P_k(s;z) := \frac{2}{\sqrt{\pi}} \sum_{M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(2)} \chi(M)^{-1}(cz+d)^{-k}\varphi_{s,k}(Mz).$$

Here

$$\varphi_{s,k}(Mz) = |y|^{-\frac{k}{2}} M_{\frac{k}{2}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|)\left(-\frac{\pi y}{6}\right) e\left(-\frac{x}{24}\right).$$

Outline of Proof

To prove the Main Theorem:

Outline of Proof

To prove the Main Theorem:

- Construct a real analytic weight 1/2 vector-valued modular form reflecting transformations of $P_{\frac{1}{2}}(\frac{3}{4}, z)$ on $\mathrm{SL}_2(\mathbb{Z})$

Outline of Proof

To prove the Main Theorem:

- Construct a real analytic weight $1/2$ vector-valued modular form reflecting transformations of $P_{\frac{1}{2}}(\frac{3}{4}, z)$ on $\mathrm{SL}_2(\mathbb{Z})$
- Express the Fourier expansions of the component functions

Outline of Proof

To prove the Main Theorem:

- Construct a real analytic weight $1/2$ vector-valued modular form reflecting transformations of $P_{\frac{1}{2}}(\frac{3}{4}, z)$ on $\mathrm{SL}_2(\mathbb{Z})$
- Express the Fourier expansions of the component functions
- Use Bringmann-Ono and the constructed vector-valued modular form to establish the coefficients of $\omega(q)$.

Constructing the modular form

Definition

If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, define,

$$\chi_0(M) := \begin{cases} i^{-1/2}(-1)^{\frac{1}{2}(c+ad+1)} e\left(\frac{3dc}{8} - \frac{(a+d)}{24c} - \frac{a}{4}\right) \omega_{-d,c}^{-1} & \text{if } c > 0,\ c \text{ even}, \\ e\left(\frac{-b}{24}\right) & \text{if } c = 0; \end{cases}$$

$$\chi_1(M) := i^{-1/2}(-1)^{\frac{c-1}{2}} e\left(\frac{3dc}{8} - \frac{(a+d)}{24c}\right) \omega_{-d,c}^{-1} \qquad \text{if } c > 0,\ d \text{ even},$$

$$\chi_2(M) := i^{-1/2}(-1)^{\frac{c-1}{2}} e\left(\frac{3dc}{8} - \frac{(a+d)}{24c}\right) \omega_{-d,c}^{-1} \qquad \text{if } c > 0,\ c, d \text{ odd},$$

Constructing the modular form

Definition

$$\mathcal{P}(z) := \left(P_0(z), P_1(z), P_2(z)\right)^T,$$

where,

$$P_0(z) := \frac{2}{\sqrt{\pi}} \sum_{M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(2)} \chi_0(M)^{-1}(cz+d)^{-1/2}\varphi_{3/4,1/2}(Mz);$$

$$P_1(z) := \frac{2}{\sqrt{\pi}} \sum_{\substack{M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = M'S \\ M' \in \Gamma_\infty \backslash \Gamma_0(2)}} \chi_1(M)^{-1}(cz+d)^{-1/2}\varphi_{3/4,1/2}(Mz);$$

$$P_2(z) := \frac{2}{\sqrt{\pi}} \sum_{\substack{M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = M'ST \\ M' \in \Gamma_\infty \backslash \Gamma_0(2)}} \chi_2(M)^{-1}(cz+d)^{-1/2}\varphi_{3/4,1/2}(Mz).$$

Connection to $H(z)$

Theorem (G.)

The function $\mathcal{P}(z)$ is a vector-valued real analytic modular form of weight $1/2$ satisfying

$$\mathcal{P}(z+1) = \begin{pmatrix} e(-1/24) & 0 & 0 \\ 0 & 0 & e(1/3) \\ 0 & e(1/3) & 0 \end{pmatrix} \mathcal{P}(z),$$

$$\mathcal{P}(-1/z) = \sqrt{-iz} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathcal{P}(z).$$

The coefficients of $\omega(q)$

- $H_1(24z) = (-i24z)^{-1/2} H_0\left(\frac{-1}{24z}\right) = (-i24z)^{-1/2} P_0\left(\frac{-1}{24z}\right) = P_1(24z).$

The coefficients of $\omega(q)$

- $H_1(24z) = (-i24z)^{-1/2} H_0\left(\frac{-1}{24z}\right) = (-i24z)^{-1/2} P_0\left(\frac{-1}{24z}\right) = P_1(24z).$
- $P_1(z) = \sum_{n\geq 0} \alpha(n) q^{\frac{n}{2}+\frac{1}{3}} + \sum_{n<0} \beta_y(n) q^{\frac{n}{2}+\frac{1}{3}}$,
  where,

$$\alpha(n) = \frac{\pi}{\sqrt{2}}(3n+2)^{-\frac{1}{4}} \sum_{\substack{k=1 \\ (k,2)=1}}^{\infty} \frac{A_k\left(\frac{n(k+1)}{2} - \frac{3(k^2-1)}{8}\right)}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{3n+2}}{3k}\right),$$

$$\beta_y(n) = \frac{\pi^{\frac{1}{2}}}{\sqrt{2}} |3n+2|^{-\frac{1}{4}} \cdot \Gamma\left(\frac{1}{2}, \frac{\pi|3n+2|\cdot y}{3}\right) \sum_{\substack{k=1 \\ (k,2)=1}}^{\infty} \frac{A_k\left(\frac{n(k+1)}{2} - \frac{3(k^2-1)}{8}\right)}{k} \cdot J_{\frac{1}{2}}\left(\frac{\pi\sqrt{|3n+2|}}{3k}\right).$$

The coefficients of $\omega(q)$

- $H_1(24z) = (-i24z)^{-1/2} H_0\left(\frac{-1}{24z}\right) = (-i24z)^{-1/2} P_0\left(\frac{-1}{24z}\right) = P_1(24z)$.
- $P_1(z) = \sum_{n \geq 0} \alpha(n) q^{\frac{n}{2}+\frac{1}{3}} + \sum_{n<0} \beta_y(n) q^{\frac{n}{2}+\frac{1}{3}}$,
  where,

$$\alpha(n) = \frac{\pi}{\sqrt{2}}(3n+2)^{-\frac{1}{4}} \sum_{\substack{k=1 \\ (k,2)=1}}^{\infty} \frac{A_k\left(\frac{n(k+1)}{2} - \frac{3(k^2-1)}{8}\right)}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{3n+2}}{3k}\right),$$

$$\beta_y(n) = \frac{\pi^{\frac{1}{2}}}{\sqrt{2}} |3n+2|^{-\frac{1}{4}} \cdot \Gamma\left(\frac{1}{2}, \frac{\pi|3n+2| \cdot y}{3}\right) \sum_{\substack{k=1 \\ (k,2)=1}}^{\infty} \frac{A_k\left(\frac{n(k+1)}{2} - \frac{3(k^2-1)}{8}\right)}{k} \cdot J_{\frac{1}{2}}\left(\frac{\pi\sqrt{|3n+2|}}{3k}\right).$$

Maass-Poincaré series of all weights

Define

$$P(N,\chi,m,k,s;z) := \sum_{M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_0(N)} \chi(M)^{-1}(cz+d)^{-k}\varphi_{s,k,m}(Mz).$$

- $k \in \frac{1}{2}\mathbb{Z}$, $N \in \mathbb{N}$, $0 > m \in \mathbb{Q}$, $s \in \mathbb{C}$, and χ is a multiplier system for $\Gamma_0(N)$.
- $\varphi_{s,k,m}(z) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|)\,(4m\pi y)\, e\,(mx)\,.$

Properties of $P(N, \chi, m, k, s; z)$

- Absolutely convergent for $\Re(s) > 1$.

Properties of $P(N, \chi, m, k, s; z)$

- Absolutely convergent for $\Re(s) > 1$.
- If $\Re(s) > 1$ and $V = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0(N)$ then

$$P(N, \chi, m, k, s; Vz) = \chi(V)(\gamma z + \delta)^k P(N, \chi, m, k, s; z).$$

Properties of $P(N, \chi, m, k, s; z)$

- Absolutely convergent for $\Re(s) > 1$.
- If $\Re(s) > 1$ and $V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N)$ then

$$P(N, \chi, m, k, s; Vz) = \chi(V)(\gamma z + \delta)^k P(N, \chi, m, k, s; z).$$

- If $k < 0$ and $s = 1 - k/2$ or $k > 2$ and $s = k/2$, then $P(N, \chi, m, k, s; z)$ is a weak Maass form of weight k on $\Gamma_0(N)$ with Nebentypus if χ is of the correct form.

Properties of $P(N, \chi, m, k, s; z)$

- Absolutely convergent for $\Re(s) > 1$.
- If $\Re(s) > 1$ and $V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N)$ then

$$P(N, \chi, m, k, s; Vz) = \chi(V)(\gamma z + \delta)^k P(N, \chi, m, k, s; z).$$

- If $k < 0$ and $s = 1 - k/2$ or $k > 2$ and $s = k/2$, then $P(N, \chi, m, k, s; z)$ is a weak Maass form of weight k on $\Gamma_0(N)$ with Nebentypus if χ is of the correct form.
- We can express the Fourier expansion for $P(N, \chi, m, k, s; Vz)$, $V \in \mathrm{SL}_2(\mathbb{Z})$.

Summary

- Mock theta functions are the holomorphic projection of weight $1/2$ weak Maass forms

Summary

- Mock theta functions are the holomorphic projection of weight $1/2$ weak Maass forms
- These Maass forms are weight $1/2$ vector-valued modular forms

Summary

- Mock theta functions are the holomorphic projection of weight $1/2$ weak Maass forms
- These Maass forms are weight $1/2$ vector-valued modular forms
- For $f(q)$ we can construct a Maass-Poincaré series whose Fourier expansion yields $\alpha_f(n)$.

Summary

- Mock theta functions are the holomorphic projection of weight $1/2$ weak Maass forms
- These Maass forms are weight $1/2$ vector-valued modular forms
- For $f(q)$ we can construct a Maass-Poincaré series whose Fourier expansion yields $\alpha_f(n)$.
- We can use the transformation properties of the Maass form and Maass-Poincaré series to find $\alpha_\omega(n)$.

Summary

- Mock theta functions are the holomorphic projection of weight 1/2 weak Maass forms
- These Maass forms are weight 1/2 vector-valued modular forms
- For $f(q)$ we can construct a Maass-Poincaré series whose Fourier expansion yields $\alpha_f(n)$.
- We can use the transformation properties of the Maass form and Maass-Poincaré series to find $\alpha_\omega(n)$.
- We can do this construction and express the Fourier coefficients for the general $P(N, \chi, m, k, s; Vz)$, $V \in \mathrm{SL}_2(\mathbb{Z})$.