ON A RESULT OF WALDSURGER II

HERVÉ JACQUET

Abstract. This is a translation of Hervé Jacquet's 1987 paper " Sur un résultat de Waldspurger", published in Compositio Mathematica. The translation was accomplished with AI. Any errors or inaccuracies are my responsibility. For typos, corrections, or suggestions, please contact yluo237@wisc.edu.

1. Introduction

1.1. Let F be a field and E a quadratic extension of F, Ω a character of the ideal class group of E, ω its restriction to the ideal class group of F. Let η denote the quadratic character of the ideal class group of F attached to E, N_0 the group of elements of $F^\times$ which are the norm of an element of E, and N_1 the group of elements of E with norm 1. For each non-zero ε in F, let G_ε be the group formed by matrices of the following type:

$$\begin{vmatrix} a & b\varepsilon \\ b^\sigma & a^\sigma \end{vmatrix}$$

where σ denotes conjugation in E with respect to F. Let G denote the group $GL(2)$, viewed as an algebraic group defined over F, Z its center, and A the subgroup of diagonal matrices. If $\varepsilon = 1$ then G_1 is isomorphic to G. In general, G_ε is an inner form of G. In particular, if ε is not a norm and π' is an automorphic representation of infinite dimension of G_ε, then the [JL85] correspondence associates to π' a cuspidal automorphic representation π of G_1. Conversely, if π is given, then there may be several ε such that the group G_ε admits an automorphic representation corresponding to π.

We now suppose that the representation π is not dihedral for the extension E. Then the lifting Π of π to the extension E is a cuspidal automorphic representation of $G(E) = \mathrm{GL}(2, E)$.

1.2. In [Wal85] Waldspurger considers two conditions relative to the representation π. Let ω be the central character of π and Ω a multiplicative character of E whose restriction to F is ω. We denote by T the subgroup of G_ε formed by matrices of the form:

$$t = \begin{vmatrix} a & 0 \\ 0 & a^\sigma \end{vmatrix};$$

Date: January 21, 2025.

it is a torus defined over F and isomorphic to the multiplicative group of E. In particular, Ω identifies with a character of the adelic group of T by the formula $\Omega(t) = \Omega(a)$. By abuse of notation, we also denote by Z the center of G_ε. The first condition is stated as follows:

(1) There exists an ε, an automorphic representation π' of G_ε corresponding to π, and an automorphic form ϕ in the space of π' such that the integral

$$\int \phi(t)\Omega^{-1}(t)dt, \quad t \in T(F_A)/T(F)Z(F_A),$$

is non-zero.

(2) The function $L(s, \Pi \otimes \Omega^{-1})$ is not zero at the point $1/2$.

Waldspurger proves that (1) implies (2) ([Wal85]). But he does not completely prove that (2) implies (1). We propose to prove this implication. For this, we will prove a "relative trace formula."

1.3. Let ω' be the lifting of ω to E, such that $\omega'(a) = \omega(aa^\sigma)$. We choose a system of representatives of the classes $F^\times/N_0$; for each ε in this system of representatives, we fix a smooth function f_ε on the group $G(E_A)$, transforming by the inverse of the character ω' under the center and with compact support modulo the center; we assume the function is zero for almost all ε. The function f_ε defines an operator in the space of cuspidal forms transforming by the character ω' of the center. This operator is represented by a kernel which we denote K_ε. We consider the expression:

$$\sum_\varepsilon \iint K_\varepsilon \left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix}, g \right] \Omega^{-1}(a) da \eta\omega(\det g) dg,$$
$$a \in E_A^\times/E^\times, \quad g \in G_\varepsilon(F)Z(F_A)\backslash G_\varepsilon(F_A). \tag{1.3.1}$$

On the other hand, for each ε we fix a smooth function f'_ε on the group $G_\varepsilon(F_A)$, transforming by the inverse of the character ω under the center and with compact support modulo the center; we assume the function is zero for almost all ε. The function defines an operator in the subspace of automorphic forms generated by cuspidal automorphic representations (i.e., of infinite dimension if G_ε is not split) which have ω as central character and are not dihedral for E. We denote by K'_ε the corresponding kernel. We also denote Ω' the character $\Omega^{-1}\omega'$. In other words, Ω' is the transform of Ω by σ. We consider the following expression:

$$\sum_\varepsilon \iint K'_\varepsilon[t_1, t_2]\Omega(t_1)\Omega'(t_2)dt_1dt_2, \quad t_1 \in T(F_A)/T(F)Z(F_A). \tag{1.3.2}$$

Then given a family of functions f_ε, there exists a family of functions f'_ε, zero for almost all ε, such that expression (1.3.1) equals expression (1.3.2).

The novelty of the formula compared to previous cases ([JL85], [Jac86]) is the presence, at least in principle, of infinitely many terms on both sides of the formula, each term involving different groups. From another point of view, the formula is somewhat intermediate between that of [JL85] and [Jac86]. It relies on an identification of the disjoint union of double classes $A(E)\backslash G(E)/G_\varepsilon(F)$ with the disjoint union of double classes $T(F)\backslash G_\varepsilon(F)/T(F)$.

The announced implication follows immediately from formula §7. In principle, one could also prove Waldspurger's arithmetic results given in [Wal85]. However, one should consider the formula itself as the main object of study. Indeed, we hope that it is the precursor of much more general formulas.

In §2, we study the properties of the above double classes. In §3 we study local orbital integrals. In §4 we calculate those of Hecke functions. In §5 we study the integral of the Eisenstein kernel and §6 that of the geometric kernel. Finally, we prove the formula in §7. In fact, we only prove the formula in the particular case where all functions f_ε except one are zero.

2. Doubles classes

In this section we keep the notations from §1 except that F is now an arbitrary field of characteristic different from 2.

2.1. We choose a non-zero ε in F and study the space of double classes $T(F)\backslash G_\varepsilon(F)/T(F)$. For g in the group $G_\varepsilon(F)$ we set $X_\varepsilon(g) = bb^\sigma\varepsilon(aa^\sigma)^{-1}$ if

$$g = \begin{vmatrix} a & b\varepsilon \\ b^\sigma & a^\sigma \end{vmatrix}.$$

It is clear that the function thus defined is constant on the double classes of the group $T(F)$ in the group $G_\varepsilon(F)$. We will say that g (or its double class) is regular if $X_\varepsilon(g)$ is neither zero nor infinite, singular in the contrary case. If g and g' are in $G_\varepsilon(F)$ and $G_{\varepsilon'}(F)$ respectively and $X_\varepsilon(g) = X_{\varepsilon'}(g')$ then $\varepsilon = \varepsilon'$ and the elements g and g' are in the same double class of the group $T(F)$. If g is regular in $G_\varepsilon(F)$ then the relation $tgt'^{-1} = g$, where t and t' are in $T(F)$, implies that t and t' are equal and in the center $Z(F)$. On the other hand, there are only two singular double classes, those of e and of the element

$$h = \begin{vmatrix} 0 & \varepsilon \\ 1 & 0 \end{vmatrix}.$$

We still consider the relation $tgt'^{-1} = g$ where t and t' are in $T(F)$, but we now take g singular. If g is e then the relation implies $t' = t$. If $g = h$, we note that h normalizes T and the relation implies $t = gt'g^{-1}$. Finally, $X_\varepsilon(g)$ is never equal to 1 and takes, as ε and g vary, all values in the set $F - 1$, augmented by a point at infinity. The verification of these assertions is elementary and left to the reader.

2.2. Let A be the subgroup of diagonal matrices in the group $G = GL(2)$. We propose to study the space of double classes $A(E)\backslash G(E)/G_\varepsilon(F)$. For this, we introduce the group P of upper triangular matrices and first study the space $P(E)\backslash G(E)/G_\varepsilon(F)$ of double classes of the groups $P(E)$ and $G_\varepsilon(F)$.

LEMMA. *If ε is not a norm then there is only one double class of the groups $P(E)$ and $G_\varepsilon(F)$. If ε is a norm let m be a matrix whose second row (r, s) satisfies $\varepsilon r r^\sigma - s s^\sigma = 0$. Then there are two double classes of the groups $P(E)$ and $G_\varepsilon(F)$, that of e and that of m.*

Proof. Let g be an element of $G(E)$ whose second row is (c, d). We first assume that $cc^\sigma\varepsilon - dd^\sigma$ is not zero, which is always the case if ε is not a norm. Then there exists an element h of $G_\varepsilon(F)$ whose second row is (c, d). The products of the row vector $(0, 1)$ by h and g are the same. Therefore the matrix $p = gh^{-1}$ fixes the vector $(0, 1)$; consequently, it is in P and $g = ph$. If ε is not a norm, the assertion of the lemma is thus proven. We now suppose that $cc^\sigma\varepsilon - dd^\sigma$ is zero and ε is a norm. Then rs^{-1} and cd^{-1} have the same norm. Therefore there exist a and z in E such that $c = azr$ and $d = a^\sigma zs$. We define

$$t = \begin{vmatrix} a & 0 \\ 0 & a^\sigma \end{vmatrix}, \quad h = m \begin{vmatrix} z & 0 \\ 0 & z \end{vmatrix} t, \quad q = gh^{-1}$$

Then h and g have the same second line. It follows that q is in P and finally $g = pmt$ with

$$p = q \begin{vmatrix} z & 0 \\ 0 & z \end{vmatrix}.$$

Since p is in P, the lemma is completely proven. □

2.3. We now introduce the involution i whose fixator is G_ε:

$$g^i = \begin{vmatrix} 0 & \varepsilon \\ 1 & 0 \end{vmatrix} g^\sigma \begin{vmatrix} 0 & \varepsilon \\ 1 & 0 \end{vmatrix}^{-1},$$

where, as above, σ denotes conjugation in E. We set $H(g) = gg^{-i}$. Then the function H is constant on the right classes of the group $G_\varepsilon(F)$; moreover $H(g) = H(g')$ implies that g and g' are in the same class of $G_\varepsilon(F)$. On the other hand, the function that associates the scalar rq/ps to the matrix h with elements p, q, r, s is constant on the double classes of the group $A(E)$. Finally, for diagonal a we have: $H(ag) = aH(g)a^{-i}$ and a^{-i} is also diagonal. We are thus led to set

$$Y_\varepsilon(g) = rq/ps$$

where p, q, r, s are the elements of the matrix $H(g)$. The function thus defined is constant on the double classes of the groups $G_\varepsilon(F)$. If g is in $P(E)mG_\varepsilon(F)$ then $s = 0$ and $Y_\varepsilon(g)$ is infinite. If g is in $P(E)G_\varepsilon(F)$ then g is in the double class of an element of the form

$$n(x) = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} \tag{2.3.1}$$

and we have:

$$Y_\varepsilon(g) = -\varepsilon^{-1} x x^\sigma (1 - \varepsilon^{-1} x x^\sigma)^{-1}. \tag{2.3.2}$$

This shows that Y_ε takes its values in the set F augmented by a point at infinity. We will say that g or its double class is regular if $Y_\varepsilon(g)$ is neither zero nor infinite.

LEMMA. *Let g and g' be regular for $G_\varepsilon(F)$ and $G_{\varepsilon'}(F)$ respectively. If $Y_\varepsilon(g) = Y_{\varepsilon'}(g')$ then $\varepsilon = \varepsilon'$ and the elements g and g' are in the same double class of the groups $A(E)$ and $G_\varepsilon(F)$.*

Proof. We can assume $g = n(x)$ and $g' = n(x')$. Formula (2.3.2) above shows that $\varepsilon^{-1}xx^\sigma = \varepsilon'^{-1}x'x'^\sigma$. It follows that x and x' have the same norm; consequently, there exists a such that $x = x'aa^{\sigma-1}$. We therefore have:

$$g = \begin{vmatrix} a & 0 \\ 0 & a^\sigma \end{vmatrix} g' \begin{vmatrix} a & 0 \\ 0 & a^\sigma \end{vmatrix}^{-1}.$$

Hence the conclusion. □

Note that $Y_\varepsilon(g)$ cannot equal 1 and takes, as ε and g vary, all values in the set $F - 1$ augmented by a point at infinity.

LEMMA 2.5. *(i) If ε is not a norm, the only singular double class is that of e.*

(ii) If ε is the norm of an element u, we set

$$m = \begin{vmatrix} 1 & u \\ 1 & -u \end{vmatrix}.$$

Then there are four singular classes, those of e, $n(u)$, m and $n(1)m$.

Proof. If ε is not a norm, it follows from the previous considerations that Y_ε does not take the infinite value. The only singular class is therefore that of e. If ε is the norm of u then $Y_\varepsilon(n(x))$ is 0 if $x = 0$ and infinite if the norm of x is ε. As in the proof of lemma (2.4) the double class of $n(x)$ depends only on the norm of x. We thus see that $P(E)G_\varepsilon(F)$ contains two singular double classes, that of e and that of $n(u)$. The other singular double classes are contained in $P(E)mG_\varepsilon(F)$ and therefore have representatives of the form $g(x) = n(x)m$. The elements $g(x)$ and $g(y)$ are in the same double class if and only if there exists a in $A(E)$ such that $g(x)$ and $ag(y)$ are in the same right classes of the group $G_\varepsilon(F)$, or, which amounts to the same thing, $H(g(x)) = H(ag(y))$. Denoting by a_1 and a_2 the diagonal elements of matrix a, we see that this last relation is equivalent to:

$$a_1 = a_1^\sigma, \quad a_2 = a_2^\sigma,$$

$$a_1 a_2^{\sigma-1} Try = Trx.$$

The first two relations imply that a_1 and a_2 are contained in F. Then we see that there exists a satisfying the last relation if and only if Trx and Try are either both zero or both non-zero. We conclude from this that there are two classes contained in $P(E)mG_\varepsilon(F)$, that of m and that of $n(1)m$. □

2.6. Given g in $G(E)$, we now study the set $Z(g)$ formed by the pairs (a, h) with a in $A(E)$ and h in $G_\varepsilon(F)$ such that $agh^{-1} = g$.

LEMMA. *(i) If g is regular or if $g = n(u)$ or if $g = mn(u)$ then $Z(g)$ is the set of pairs (z, z) with z in $Z(F)$.*

(ii) If $g = e$ then $Z(g)$ is the set of pairs (t, t) with t arbitrary in $T(F)$.

(iii) If $g = m$ then $Z(g)$ is the set of pairs $(a, g^{-1}ag)$ with a arbitrary in $A(F)$.

Proof. Let p, q, r, s be the elements of matrix $H(g)$. If (a, h) is in $Z(g)$ then $H(ag) = H(g)$ and this last relation can be written, denoting by a_1 and a_2 the diagonal elements of matrix a:

$$pa_1a_2^{-\sigma} = p, \quad qa_1a_1^{-\sigma} = q, \quad ra_2a_2^{-\sigma} = r, \quad a_2a_1^{-\sigma} = s.$$

In case (i) the elements r and q are different from 0 and at least one of the elements p and s is different from 0. It follows immediately that a is in $Z(F)$ and $h = a$. In case (ii) we have $p = s = 1$ and $q = r = 0$. We deduce that a is in T and $h = a$. In case (iii) we have $p = s = 0$ and r and q are non-zero. We deduce that a is in $A(F)$. Conversely if a is in $A(F)$, then according to the previous calculation $H(ag) = H(g)$ therefore $ag = gh$ with h in $G_\varepsilon(F)$ and $h = g^{-1}ag$.

The assertions of the lemma are thus proven. □

3. Orbit integrals

We keep the notations from the previous sections but now assume that F is a local field, E a quadratic extension, η the quadratic character of F attached to E. The set of classes of N_0 in the multiplicative group of F is reduced to two elements. We denote by Ω a multiplicative character of E and by ω its restriction to F, by ω' the lifting of ω to E and by Ω' the character $\Omega^{-1}\omega'$.

3.1. We now choose a non-zero ε in F and consider a function f on the group G_ε, smooth, transforming by the inverse of character ω under the center and with compact support modulo the center. We define a function $H(x) = H(x : f)$ by the formula:

$$H(x) = \Omega(u) \iint f\left[t_1 \begin{vmatrix} 1 & u\varepsilon \\ u^\sigma & 1 \end{vmatrix} t_2\right] \Omega(t_1)\Omega'(t_2)dt_1dt_2,$$

if $x \neq 1$ and $x = uu^\sigma\varepsilon$ for at least one u;

$$H(x) = 0, \quad \text{otherwise.} \tag{3.1.1}$$

Each integral is over the compact set $T(F)/Z(F)$. A formal calculation shows that the product of the double integral by $\Omega(u)$ does not change if we replace u by $uaa^{-\sigma}$; it follows that the right-hand side depends only on the norm of u, which justifies the notation. Note that $H(1) = 0$ by definition. We propose to study the properties of function H.

PROPOSITION. *(i) The function H is zero in a neighborhood of point 1.*

(ii) It is smooth at every point of F different from 0.

(iii) There exists a function G defined in a neighborhood of 0 of F and smooth, such that $H(x) = G(x^{-1})$, for x in εN of sufficiently large absolute value. In particular:

$$G(0) = \int f \begin{bmatrix} t & 0 & \varepsilon \\ 1 & 0 & \end{bmatrix} \Omega(t)dt \cdot vol(T(F)/Z(F))$$

(iv) There exists a function I defined and smooth in a neighborhood of 0 of E, such that $H(x) = \Omega(u)I(u)$ if $x = \varepsilon uu^\sigma$ and the absolute value of x is sufficiently small.

(v) If Ω is the lift to E of a character λ of F then there exists a function J defined in a neighborhood of zero of F, such that for x in εN_0 of sufficiently small absolute value we have: $H(x) = \lambda(x)J(x)$. In particular:

$$J(0) = \int f(t)\Omega(t)dt \cdot vol(T(F)/Z(F))\lambda^{-1}(\varepsilon).$$

(vi) If H is a function satisfying properties (i) to (iv) then $H = H(f)$ for an appropriate function f.

Proof. The first assertion is evident if 1 is not in εN_0, since H is then zero on the neighborhood N_0 of 1. We now suppose that 1 is in εN_0, we have $H(1) = 0$ by definition. If x is in εN and $H(x)$ is non-zero then the matrix in the double integral must be in a fixed compact of the group $G_\varepsilon(F)/Z(F)$, thus in fact in a compact of the group $G_\varepsilon(F)$. Its determinant $1 - x$ must therefore be in a compact set of F^*, which proves the first assertion.

The second assertion is evident. To prove the third assertion, we write, after a change of variables:

$$H(x) = \iint f \begin{bmatrix} t_1 & u^{-1} & \varepsilon \\ & 1 & u^{\sigma-1} \end{bmatrix} t_2\Omega(t_1)\Omega'(t_2)dt_1dt_2 \tag{3.1.2}$$

In the p-adic case if the absolute value of u is large enough this equals:

$$\iint f_i \begin{bmatrix} t_1 & 0 & \varepsilon \\ & 1 & 0 \end{bmatrix} t_2\omega(t_1)\Omega'(t_2)dt_1dt_2$$

A change of variables gives the result in the required form. Hence the conclusion. In the real case (3.1.2) depends only on the norm of u and we conclude similarly using the following lemma:

LEMMA 3.1.3. *Let T be a smooth function defined in a neighborhood of 0 in E. We assume that $T(u)$ depends only on the norm of u. Then there exists a smooth function S in a neighborhood of 0 of F, such that $T(u) = S(uu^\sigma)$, if the absolute value of u is sufficiently small.*

Assertion (iv) is evident. To prove assertion (v) we proceed as for assertion (iii). For the value of J at point zero we obtain the integral:

$$\iint f(t_1t_2)\Omega(t_1)\Omega'(t_2)dt_1dt_2$$

A change of variables gives the result in the required form. We leave to the reader the task of proving the last assertion. □

3.2. We now consider an element ε of F, a function f on $G(E)$, transforming by the inverse of the character ω' under the center, smooth and with compact support modulo the center. We propose to examine the properties of the function $U(x) = U(x : f) = U(x : f : \varepsilon)$ defined by

$$U(x) = \Omega(u) \iint f \begin{bmatrix} a & 0 & 1 & u \\ 0 & 1 & 0 & 1 \end{bmatrix} g\Omega(a)\eta\omega(\det g)dadg$$

$$a \in E^\times, \quad g \in G_\varepsilon(F)/Z(F),$$

if $x = -\varepsilon^{-1}uu^\sigma(1-\varepsilon^{-1}uu^\sigma)^{-1}$ for at least one u;

$$U(x) = 0 \quad \text{otherwise.} \tag{3.2.1}$$

A formal calculation shows as above that the product of the double integral by $\Omega(u)$ depends only on the norm of u, which justifies the notation. On the other hand, we note that $U(1) = 0$, by definition.

3.3. To study the properties of functions U we introduce the set X formed by matrices g in $G(E)$ such that $gg^i = 1$, where i denotes the involution that fixes G_ε. (Cf. (2.3)). We denote P the application from $G(E)$ to X defined by $P(g) = gg^{-i}$.

LEMMA 3.3.1. *The application P is surjective.*

Proof. Let x be an element of X. For h in $G(E)$ we set $y(h) = h + xh^i$. We have $xy(h)^i = y(h)$ hence $x = P(y(h))$ if $y(h)$ is invertible. We will show that $y(h)$ is invertible for at least one scalar matrix h, which will prove the proposition. If h is the scalar matrix with element a and $y(h)$ is not invertible then $-aa^{-\sigma}$ is an eigenvalue of x. Since every element of norm 1 has this form and x has at most two eigenvalues, there exists at least one h such that $y(h)$ is invertible. Hence the lemma.

From the lemma we deduce that X is a closed subvariety of $G(E)$ and that P defines a diffeomorphism from $G(E)/G_\varepsilon(F)$ onto X. Let μ be a character of $E^\times$ that extends η. We set:

$$f_1(g) = f(g)\mu\Omega(\det g) \tag{3.3.2}$$

Then the integral that defines U can be written:

$$U(x) = \Omega(u) \iint f_1 \begin{bmatrix} a & 0 & 1 & u \\ 0 & 1 & 0 & 1 \end{bmatrix} g\mu^{-1}(a)dadg$$

$$a \in E^X, \quad g \in G_\varepsilon(F)/Z(F). \tag{3.3.3}$$

The integral

$$\int f_1(hg)dg$$

converges; its value depends only on the class of h modulo $G_\varepsilon(F)$, or, which amounts to the same thing, on $P(h)$. There exists therefore a function F_1 on X, smooth and with compact support, such that $F_1(P(h))$ equals the integral above. The function f_1 transforms by the character $a \to \mu\Omega(aa^{-\sigma})$ of the center. On the other hand, writing a for the scalar matrix having a for diagonal elements, we have: $P(ah) = aP(h)a^{-\sigma}$. It follows that F_1 has the following invariance property: $F_1(uy) = F_1(y)\mu\Omega(u)$ for all y in X and all u in E of norm 1. We extend it to a function still denoted F_1 on G(E) having the same invariance property, smooth and with compact support. In terms of F_1 the integral that defines $U(x)$ is written:

$$U(x) = \Omega(u) \int F_1 \begin{bmatrix} a(1-\varepsilon^{-1}uu^\sigma) & aa^{-\sigma}u \\ -\varepsilon^{-1}u^\sigma & a^{-\sigma} \end{bmatrix} \mu^{-1}(a)da \tag{3.3.4}$$

If x is finite, then $1 - \varepsilon^{-1}uu^\sigma$ is non-zero and the integrand has compact support; the original integral therefore converges for all finite x, including $x = 0$.

We now examine the behavior of the function $U(x)$. We set

$$s(u) = -\varepsilon^{-1}uu^\sigma(1-\varepsilon^{-1}uu^\sigma)^{-1} \tag{3.3.5}$$

We first examine the behavior near 1. If ε is not a norm then U is zero in the neighborhood N_0 of 1. We now suppose that ε is a norm. Let x be an element of F; if $U(x)$ is not zero then $x = s(u)$ and the integral (3.3.4) above is not zero. This implies that the matrix in the integral is in a compact set and that the absolute value of u is bounded above. It follows that that of $1 - x$ is bounded below. Therefore U is zero in the neighborhood of point 1 in all cases.

We now examine the behavior near an arbitrary point of $F^\times - 1$. It is clear that the integral defines a smooth function with compact support of $u \in E$; we conclude that U is smooth at every point of $F^\times - 1$. Combining with the previous remarks, we conclude that U is smooth at every point of $F^\times$. □

3.4. We now examine the behavior near 0. The function $U(x)$ is 0 unless $x = s(u)$, for at least one u. If x is small enough then u is as small as we want and $1 - \varepsilon^{-1}uu^\sigma$ as close as we want to 1, in particular is a norm; then x is of the form $-\varepsilon^{-1}vv^\sigma$. Conversely if x is of this form and small enough, then $1 - x$ is a norm and $x(1-x)^{-1}$ is therefore of the form $-\varepsilon^{-1}uu^\sigma$. We then have $x = s(u)$.

PROPOSITION. *(i) If x is near 0 then $U(x) = 0$ unless x is in $-\varepsilon^{-1}N_0$.*

(ii) There exists a smooth function $I(v)$ defined in a neighborhood of 0 of E, such that, for v sufficiently close to 0, we have $U(-\varepsilon^{-1}vv^\sigma) = \Omega(v)I(v)$.

(iii) We suppose that Ω is the lift of a character λ of F; then there exists a smooth function J defined near 0 on F such that $U(x) = \lambda(x)J(x)$ for x in $-\varepsilon^{-1}N_0$ and small enough. Moreover:

$$J(0) = \iint f \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g\Omega(a)da\eta\omega(\det g)dg\lambda(-\varepsilon)$$

Proof. According to the remarks preceding the proposition, if x is close enough to 0, then x is of the form $s(u)$ if and only if x is in $-\varepsilon^{-1}N_0$. The first assertion is therefore evident. Still for x sufficiently close to 0, we can write $(1-x)^{-1}$ in the form $z(x)z(x)^\sigma$, where $z(x)$ is an analytic function, defined on a neighborhood of 0 of F, with values in E. Then if x is small enough and of the form $-\varepsilon^{-1}vv^\sigma$ we can also write:

$$x = s(u) \quad \text{with} \quad u = vz(-\varepsilon^{-1}vv^\sigma)$$

Since $U(x) = \Omega(u)I_1(u)$, where I_1 is the function defined by the integral (3.3.4) in a neighborhood of 0, we obtain the second assertion with:

$$I(v) = I_1(vz(-\varepsilon^{-1}vv^\sigma)\Omega(z(-\varepsilon^{-1}vv^\sigma)) \tag{3.4.1}$$

Under the hypotheses of the third part of the lemma, $\Omega(v)$ depends only on the norm of v. The same is therefore true of $I(v)$. In the p-adic case I is constant in a neighborhood of 0 therefore certainly a smooth function of the norm of u. The same conclusion remains true in the real case according to lemma (3.1.3). Finally $J(0)$, $I(0)$ and $I_1(0)$ are equal and given by the integral of the third part of the proposition. □

3.5. We now examine the behavior of U near infinity. For this we use without proof the following lemma:

LEMMA 3.5.1. *Let ϕ be a Schwartz-Bruhat function of two variables. For all non-zero x in F, we set:*

$$I(x) = \int \phi(ax, a^{-1})\eta(a)d^\times a$$

Then there exist two Schwartz-Bruhat functions ϕ_1 and ϕ_2 such that for every non-zero x in F:

$$I(x) = \phi_1(x) + \eta(x)\phi_2(x)$$

Moreover we have:

$$\phi_1(0) = \int \phi(0, a)\eta(a)d^\times a, \quad \phi_2(0) = \int \phi(a, 0)\eta(a)d^\times a$$

The last two integrals are divergent; the last one for example is the value at point 0 of the analytic continuation of the following integral, which converges for Res strictly positive:

$$\int \phi(a, 0)|a|^s d^\times a$$

After stating the lemma we return to studying the function U. If $x = s(u)$ we have $\varepsilon^{-1}u^\sigma u = (1-x^{-1})^{-1}$. If the absolute value of x is large enough the right-hand side is a norm; we conclude that if ε is not a norm then when the absolute value of x is large enough x is not of the form $s(u)$ and $U(x)$ is zero. We now suppose that ε is indeed a norm. Then if the absolute value of x is large enough the right-hand member is a norm, thus equal to the left-hand member for an appropriate u; then $x = s(u)$. On the other hand, we can regard F_1 as a Schwartz-Bruhat function of 4 variables. We consider the integral:

$$L(y,u) = \Omega(u) \int F_1 \begin{bmatrix} ay & aa^{-\sigma}u \\ -\varepsilon^{-1}u^\sigma & a^{-\sigma} \end{bmatrix} \mu^{-1}(a)da \tag{3.5.2}$$

The first variable y is in a neighborhood V of 0 of F but non-zero and the second is in a subset V' of E, reciprocal image via the norm of a neighborhood of ε. It is clear that the integral converges. To calculate it we can integrate first on $F^\times$ then on the compact set $E^\times/F^\times$. We can apply the lemma above (or rather a version of the lemma with parameters) to the interior integral. It follows that there exist two smooth functions L_1 and L_2 on $V \times V'$ such that

$$L(y,u) = L_1(y,u) + \eta(y)L_2(y,u) \tag{3.5.3}$$

Moreover we have:

$$L_1(0,u) = \Omega(u) \int F_1 \begin{bmatrix} 0 & a^{1-\sigma}u \\ -\varepsilon^{-1}u^\sigma & a^{-\sigma} \end{bmatrix} \mu^{-1}(a)da \tag{3.5.4}$$

$$L_2(0,u) = \Omega(u) \int F_1 \begin{bmatrix} a & a^{1-\sigma}u \\ -\varepsilon^{-1}u^\sigma & 0 \end{bmatrix} \mu^{-1}(a)da \tag{3.5.5}$$

The last two integrals are divergent; the second for example is the value at 0 of the analytic continuation of the following integral, which converges for Res strictly positive:

$$\Omega(u) \int F_1 \begin{bmatrix} a & a^{1-\sigma}u \\ -\varepsilon^{-1}u^\sigma & 0 \end{bmatrix} \mu^{-1}(a)|a|^s da$$

We know that F_1 has an invariance property: $F_1(hv) = F_1(h)\mu\Omega(v)$, for all v of norm 1. It follows that L has the following property: $L(y,uv) = L(y,u)$ for v of norm 1. By integrating the identity (3.4.3) over the group of elements of norm 1, we see that we can assume that L_1 and L_2 have the same invariance property as L. We can therefore write L_i as a function of the pair (y,t) with $t = -\varepsilon uu^\sigma$. Finally if the norm of x is large enough we can write $x = s(u)$. Then taking $y = 1 - \varepsilon^{-1}uu^\sigma$ and $t = -\varepsilon^{-1}uu^\sigma$, we get $U(x) = L(y,t)$. Since $t = 1/x^{-1} - 1$ and $y = -x^{-1}/x^{-1} - 1$ we can write $L_i(y,t) = M_i(x^{-1})$, where M_i is a smooth function defined in a neighborhood of 0 of F. If moreover the absolute value of x is large enough then $1 - x^{-1}$ is a norm and $\eta(y) = \eta(-x)$. Taking u of norm ε, we obtain an infinite value of x and the relation $M_i(0) = L_i(0,u)$. We thus arrive at the following proposition:

PROPOSITION. *(i) If ε is not a norm then $U(x) = 0$ if the absolute value of x is large enough.*

(ii) Suppose that ε is a norm. Then there exist two smooth functions M_i, $i = 1,2$, defined in a neighborhood of 0 of F, such that, for x large enough, we have:

$$U(x) = M_1(x^{-1}) + \eta(-x)M_2(x^{-1})$$

(iii) Under the hypotheses of (ii) if ε is the norm of v then:

$$M_1(0) = \Omega(u) \int F_1 \begin{vmatrix} 0 & a^{1-\sigma}v \\ -\varepsilon^{-1}v^\sigma & a^{-\sigma} \end{vmatrix} \mu^{-1}(a) d^\times a$$

$$M_2(0) = \Omega(u) \int F_1 \begin{vmatrix} a & a^{1-\sigma}v \\ -\varepsilon^{-1}v^\sigma & 0 \end{vmatrix} \mu^{-1}(a) d^\times a$$

The last two integrals are divergent and are defined as above by analytic continuation. They can also be interpreted as orbital integrals (divergent) attached to singular orbits. For this purpose we assume $\varepsilon = 1$ and $v = 1$. Then we have:

$$m = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}, \quad P(m) = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix},$$

$$P(n(1)) = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}, \quad P(mn(1)) = \begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} \tag{3.5.6}$$

By formal calculation we see that:

$$M_1(0) = \iint f \left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} n(1)g \right] \Omega(a)\eta\omega(\det g) d^\times a dg,$$

$$M_2(0) = \iint f \left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} n(1)mg \right] \Omega(a)\eta\omega(\det g) d^\times a dg \tag{3.5.7}$$

3.6. Let ε be given in F, a function f on $G(E)$, as above. We also give a system of representatives $\{\varepsilon_1, \varepsilon_2\}$ for the classes of N_0 in the multiplicative group of F. We assume ε_1 is in N_0. We write G_i for the group defined by ε_i.

PROPOSITION. *Given f and ε, there exist functions f_1 and f_2 on G_1 and G_2 respectively such that*

$$U(x : f : \varepsilon) = H(x, f_i) \quad \text{if} \quad x = \varepsilon_i u u^\sigma$$

The proposition is an immediate consequence of the characterization of orbital integrals H and the properties of the function U.

4. Orbit integrals: unramified case

4.1. We return to the notations of §3; we now assume that F is a non-archimedean local field and E is an unramified quadratic extension. We assume that the residual characteristic is not 2. We also assume the characters ω and Ω are unramified. Let $K = \mathrm{GL}(2, R_E)$, where R_E is the ring of integers of E. Similarly, let $K' = GL(2, R_F)$. We denote by $H(K)$ the Hecke algebra of bi-invariant functions under K with compact support. Similarly, we denote by $H(K, \omega')$ the algebra of bi-invariant functions under K, transforming by the inverse of character ω' under the center and with compact support modulo the center. We define similarly the algebras $H(K')$ and $H(K',\omega)$ of functions on the group $\mathrm{GL}(2,F)$. There exist natural homomorphisms:

$$H(K) \to H(K') \tag{4.1.1}$$

and

$$H(K,\omega') \to H(K,\omega) \tag{4.1.2}$$

The first can be defined in terms of the Satake transform. For f in $H(K)$ we set:

$$\hat{f}(X_1, X_2) = \int f\left[\begin{vmatrix}1 & u\\ 0 & 1\end{vmatrix}\begin{vmatrix}a & 0\\ 0 & b\end{vmatrix}\right] |a|^{s_1-1/2}|b|^{s_2+1/2} da\, db\, du,$$

if $X_i = q_E^{s_i}$.

The function $\hat{f}$ is a polynomial; it is the Satake transform of f. We similarly define the transform of a function f' of $H(K')$. Then if f' is the image of f under the homomorphism (4.1.1) we have:

$$\hat{f}'(X_1, X_2) = \hat{f}(X_1^2, X_2^2)$$

To define the homomorphism (4.1.2) we write an element f of $H(K,\omega')$ in the form:

$$f(g) = \int f_0(ag)\omega'(a)\, da, \quad a \in E^\times, \tag{4.1.4}$$

with f_0 in $H(K)$. Let f_0' be the image of f_0 under the homomorphism (4.1.1). Then the image f' of f under the homomorphism (4.1.2) is given by:

$$f'(g) = \int f_0'(ag)\omega(a)\, da, \quad a \in F^\times. \tag{4.1.5}$$

We are given a unit ε; thus it is a norm. We are also given a system of representatives $\{\varepsilon_1, \varepsilon_2\}$ for the classes of N_0 in the multiplicative group of F. We assume that ε_1 is a unit thus a norm. We denote by G_1 and G_2 the groups defined by ε_1 and ε_2. The group G_1 is isomorphic to the group $G(F) = \mathrm{GL}(2,F)$. In particular there exists an isomorphism of $G(F)$ that transforms the group K' into the group $K \cap G_1(F)$: such an isomorphism will be called privileged. We then take f in $H(K,\omega')$; we denote by f' its image under the homomorphism (4.1.2) and f_1 the image of f' under a privileged isomorphism. Since a privileged isomorphism is unique, up to composition with an inner automorphism of $\mathrm{GL}(2,F)$ defined by an element of K, the function f_1 is well defined. We also denote by f_2 the zero function on the group $G_2(F)$. Then we can specify proposition (3.5) as follows: PROPOSITION: With the hypotheses and notations above, we have:

$$U(x:f:\varepsilon) = H(x:f_i) \quad \text{if} \quad x = \varepsilon_i N(u)$$

The proof will occupy the remainder of §4.

4.2. It is easy to see that we can reduce to the case where $\varepsilon_1 = 1$ and $\omega = \eta$. Then $\omega' = 1$ and Ω is the unramified quadratic character of E. It will also be more convenient to formulate the above equality in terms of functions belonging to $H(K)$. Thus for such a function f:

$$U(x:f) = \Omega(y)\int f\left[\begin{vmatrix} ab & 0 \\ 0 & b\end{vmatrix}\begin{vmatrix} 1 & y \\ 0 & 1\end{vmatrix} g\right]\Omega(a)\,da\,db\,dg\text{if } x = \qquad s(y) \text{ for a } y;$$
$$= 0 \text{ otherwise.} \tag{4.2.1}$$

We can also write the above integral in the form:

$$U(x:f) = \Omega(y)\int f\left[\begin{vmatrix} a & 0 \\ 0 & b\end{vmatrix}\begin{vmatrix} 1 & y \\ 0 & 1\end{vmatrix} g\right]\Omega(a)\Omega(b)\,da\,db\,dg,$$
$$\text{if } x = s(y). \tag{4.2.2}$$

We then denote by f' the image of f in $H(K')$, then f_1 the image of f' under a privileged isomorphism of $G(F)$ onto $G_1(F)$. We write $N(u)$ for the norm of u and we set

$$H(x:f_1) = \Omega(u)\iint f\left[t_1\begin{vmatrix} 1 & u \\ u^\sigma & 1\end{vmatrix} t_2\right]\Omega(t_1)\Omega(t_2)\,dt_1\,dt_2$$
$$\text{if } x = N(u);$$
$$= 0 \text{ if } x \text{ is not a norm.} \tag{4.2.3}$$

We need to show that

$$U(x:f) = H(x:f_1) \text{ if } x \text{ is a norm;}$$
$$U(x:f) = 0 \text{ if } x \text{ is not a norm.} \tag{4.2.4}$$

By linearity, it is clear that it suffices to prove this assertion when f is the characteristic function f_m of the following set, where π denotes a uniformizer of E or F:

$$K_m = K\begin{vmatrix} \pi^m & 0 \\ 0 & 1\end{vmatrix} K$$

Still by linearity, it suffices to prove this assertion for the function

$$g_m = f_m + f_{m-1} + \cdots + f_1 + f_0$$

We denote by $U(x:m)$ the orbital integral corresponding to g_m. We will first calculate $U(x:m)$.

4.3. Let F_m be the function defined by:

$$F_m(y) = \int g_m\left[\begin{vmatrix} a & 0 \\ 0 & b\end{vmatrix}\begin{vmatrix} 1 & y \\ 0 & 1\end{vmatrix}\right]\Omega(a)\Omega(b)\,da\,db \tag{4.3.1}$$

We will calculate the function F_m then calculate the integral $U(x:m)$ in terms of F_m.

LEMMA 4.3.2. *Let Φ be the characteristic function of R_E, π a uniformizer. Then:*

$$F_m(y) = (-1)^m \Phi(\pi^m y)$$

Proof. We first calculate the function:

$$H_m(y) = \int f_m \left[\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \begin{vmatrix} 1 & y \\ 0 & 1 \end{vmatrix} \right] \Omega(a)\Omega(b)\, da\, db \tag{4.3.3}$$

This can be written as:

$$H_m(y) = \sum (-1)^{j+k} f_m \left[\begin{vmatrix} \pi^j & y\pi^j \\ 0 & \pi^k \end{vmatrix} \right]$$

Since $j + k$ equals m if the matrix above is in K_m, the sum can also be written as:

$$H_m(y) = (-1)^m \sum f_m \left[\begin{vmatrix} \pi^j & y\pi^j \\ 0 & \pi^{m-j} \end{vmatrix} \right]$$

where the sum is over all j such that:

$$j \geq 0, \quad m - j \geq 0, \quad j \geq -v(y), \quad \mathrm{Inf}[j, m - j, j + v(y)] = 0$$

If $m = 0$ the sum reduces to the single term $j = 0$ and $H_0 = \Phi$. We now assume $m > 0$. If $v(y) \geq 0$ the sum reduces to the terms $j = 0$ and $j = m$; it follows that $H_m(y) = 2(-1)^m$. If $-m < v(y) < 0$ then the sum reduces to the terms $j = -v(y)$ and $j = m$; it follows that $H_m(y) = 2(-1)^m$. If $v(y) = -m$ the sum reduces to the term $j = m = -v(y)$; then $H_m(y) = (-1)^m$. Finally if $v(y) < -m$ the sum is empty and $H_m(y)$ is zero. Therefore:

$$H_m(y) = (-1)^m [\Phi(y\pi^m) + \Phi(y\pi^{m-1}) \text{ if } m \geq 1;$$
$$H_0 = \Phi$$

By writing that F_m is the sum of H_j for $0 \leq j \leq m$ we find the desired result. ☐

4.4. The relation between $U(x : m)$ and the function F_m is the following:

LEMMA 4.4.1.

$$U(x : m) = 2\Omega(y)(1 - q^{-1})^{-1} q^{-2} F_m(y) + \Omega(y)(1 - q^{-1})^{-1} \int F_m[(y + u)(1 - N(u))^{-1}]$$
$$+ \Omega(y)(1 - q^{-1})^{-1} \int F_m[(y + u)(1 - N(u))^{-1}]$$
$$\times [1 - N(u)]_F^{-2} \Omega(1 - N(u)) du,$$
$$\text{if } x = s(y)(= -N(y)(1 - N(y))^{-1}),$$

where du denotes the Tamagawa measure and the integral is over the units of E.

Proof. We use the following integration formula on the group $G_1(F)$:

$$\int f(g)dg = (1-q^{-1})^{-1}\int f\left[t_1\begin{vmatrix}1 & u\\ u^\sigma & 1\end{vmatrix}t_2\right]dt_1dt_2[1-N(u)]_F^{-2}du,$$

where dt denotes the Tamagawa measure on the torus T and du the Tamagawa measure on the additive group of E. There exists an analogous formula for functions on $G_1(F)$ that are invariant under the center $Z(F)$. We will apply it to the restriction to $G_1(F)$ of a function f on $G(E)$ that is right-invariant under K and invariant under the center $Z(E)$ of $G(E)$. Then the integral in t_2 disappears. To evaluate the integral in u, we decompose it into three integrals corresponding to the three regions: $|u|<1$, $|u|=1$, $|u|>1$. Moreover, in the integral for $|u|>1$, we change u to u^{-1}. We obtain the product of the factor $(1-q^{-1})^{-1}$ and the following sum:

$$\int\left[t\begin{vmatrix}1 & u\\ u^\sigma & 1\end{vmatrix}\right]dt\,du + \int f\left[t\begin{vmatrix}u^{-1} & 0\\ 0 & u^{-\sigma}\end{vmatrix}\begin{vmatrix}u & 1\\ 1 & u^\sigma\end{vmatrix}\right]dt\,du$$

$$+\int f\left[t\begin{vmatrix}1-uu^\sigma & u\\ 0 & 1\end{vmatrix}\begin{vmatrix}1 & 0\\ u^\sigma & 1\end{vmatrix}\right]dt\,|1-N(u)|_F^{-2}du.$$

The first two integrals are for $|u|<1$, the last one for $|u|=1$. Taking into account the right invariance under K, this reduces to:

$$\int f(g)\,dg = q^{-2}(1-q^{-1})^{-1}2f(e)$$
$$+(1-q^{-1})^{-1}\int f\left[t\begin{vmatrix}1-N(u) & u\\ 0 & 1\end{vmatrix}\right]dt\,|1-N(u)|^{-2}du,$$

We now apply this formula to calculate $U(x:m)$. If $x=s(y)$, we obtain:

$$U(x:m) = 2\Omega(y)q^{-2}(1-q^{-1})^{-1}F_m(y)+$$
$$\Omega(y)(1-q^{-1})^{-1}\int\left[g_m\begin{vmatrix}a & 0\\ 0 & b\end{vmatrix}\begin{vmatrix}1-N(u) & y+u\\ 0 & 1\end{vmatrix}\right]|1-N(u)|_F^{-2}$$
$$\times\,\Omega(ab)\,da\,db\,du.$$

A simple change of variables finally gives the desired result. □

4.5. We must now evaluate the integral from lemma (4.4.1), taking into account the value of the function F_m given in lemma (4.3.2). In this section, we examine the case where the absolute value of y is different from 1. If it is less than 1, then we immediately obtain:

$$\Omega(y)(1-q^{-1})^{-1}(-1)^m\times\left\{2q^{-2}+\int\Phi(\pi^m(1-N(u))^{-1})\Omega(1-N(u))|1-N(u)|^{-2}\,du\right\}.$$

In the integral, we can take $z = N(u)$ as a variable. Then du must be replaced by $(1+q^{-1})\,dz$. Therefore, we have:

$$\Omega(y)(1-q^{-1})^{-1}(-1)^m \times \Big\{2q^{-2} + (1+q^{-1}) \int \Phi(\pi^m(1-z)^{-1})\Omega(1-z)|1-z|^{-2}\,dz\Big\}.$$

The integral is taken over the set of z with absolute value 1. It can be written as the difference between an integral over the set of z with absolute value at most 1 and an integral over the set of z with absolute value strictly less than 1. In this latter integral, $1-z$ is a unit and the integral equals q^{-1}. However, in the first integral, we can change z to $z+1$ and then evaluate the integral as a geometric series. Finally, we obtain the following result:

LEMMA 4.5.1. *If $|y| < 1$ and $x = s(y)$, we have:*

$$U(x:m) = \Omega(y)q^m.$$

We now move to the case where the absolute value of y is strictly greater than 1. We first obtain:

$$(-1)^m(1-q^{-1})^{-1}\Omega(y)\times\Big\{2q^{-2}\Phi(y\pi^m)+\int \Phi[y\pi^m(1-N(u))^{-1}]|1-N(u)|^{-2}\Omega(1-N(u))\,du\Big\}.$$

Again, in the integral we can take $z = N(u)$ and replace du by $(1+q^{-1})\,dz$. We obtain an integral over the set of z with absolute value 1 which we treat as above. Finally, we obtain:

LEMMA 4.5.2. *If $w|y| > 1$ and $x = s(y)$ then:*

$$\begin{aligned} U(x:m) &= q^{m+v(y)}, && \text{if } 1 < |y| \le |\pi^{-m}|,\\ U(x:m) &= 0 && \text{if } |\pi^{-m}| < |y| \end{aligned}$$

4.6. We now move to the case where y is of absolute value 1. We will prove the following result:

LEMMA 4.6.1. *Suppose $x = s(y)$ with y of absolute value 1.*
(i) If $N(y) - 1$ is also of absolute value 1 then:

$$U(x:m) = q^m$$

(ii) If $N(y) - 1$ is not of absolute value 1 we set $z = N(y) - 1$. Then:

$$U(x:m) = q^m[(-1)^{v(z)} + 1]/2.$$

We return again to the integral of lemma 4.4.1. Since y is a unit we can change u to uy. On the other hand, the functions that we consider are invariant under a homothety of unit ratio. We can see that we can write:

$$U(x:m) = \Omega(y)(-1)^m(1-q^{-1})^{-1} \times \Big\{2q^{-2} + \int \Phi[(1+u)A^{-1}\pi^m]\Omega(A)|A|_F^{-2}du\Big\}$$

with $A = N(y)^{-1} - N(u)$.

We now choose a system of representatives for the classes of $1+P_E$ in the unit group of E. It will be convenient to take the q^2-1 roots of unity in E for the system of representatives. Then we can set $u=t(1+v)$, where t runs through this set of representatives and v through the set P_E. The integral above becomes:

$$U(x:m)=(-1)^m(1-q^{-1})^{-1}\times\Big\{2q^{-2}+\sum_t\int\Phi[(1+t+tv)B^{-1}\pi^m]\Omega(B)|B|_F^{-2}dv\Big\},$$

with $B=N(y)^{-1}N(t)^{-1}-1-Q(v)$, $Q(v)=Tr(v)+N(v)$.

In the sum over t we will distinguish the t for which $N(y)N(t)$ is not congruent to 1 modulo P_F. There are q^2-q-2 such t. Moreover for such a t, B is a unit, $1+t+tv$ an integer and the integral above is independent of t with value q^{-2}. The contribution of these t is therefore $1-q^{-1}-2q^{-2}$. Adding this to the first term in the expression above, we see that:

$$U(x:m)=(-1)^m\times\Big\{1+(1-q^{-1})^{-1}\sum_t\int\Phi[(1+t+tv)B^{-1}\pi^m]\Omega(B)|B|_F^{-2}dv\Big\},$$

with $B=N(y)^{-1}N(t)^{-1}-1-Q(v)$, $Q(v)=Tr(v)+N(v)$, where the sum now runs over the set $X(y)$ formed by the t such that $N(y)N(t)$ is congruent to 1 modulo P_F. Note that if $N(y)$ is itself congruent to 1 modulo P_F then $t=-1$ is in $X(y)$ because $N(-1)=1$. If on the contrary $N(y)$ is not congruent to 1 modulo P_F then -1 is not in $X(y)$.

We first assume that $N(y)$ is not congruent to 1 mod P_F. Then $1+t$ is a unit since -1 is not in $X(y)$. The same is true for $1+t+tv$ which therefore "disappears" from the integral. We can then take $Q(v)$ as a variable, that is to say use the integration formula:

$$\int f[Q(v)]dv=q^{-1}\int f[w]dw,\quad v\in P_E,\quad w\in P_F.$$

Therefore we have:

$$U(x:m)=(-1)^m\times\Big\{1+(1-q^{-1})^{-1}q^{-1}\sum_t\int\Phi[B^{-1}\pi^m]\Omega(B)|B|_F^{-2}dw\Big\},$$

with $B=N(yt)^{-1}-1-w$, $w\in P_F$.

Since $N(yt)^{-1}-1$ is in P_F, we can make it disappear from the integral by a translation on w. The integral thus has a value independent of t which is moreover easy to calculate. As for the number of elements of $X(y)$, it is the number of elements in the finite field with q^2 elements having norm 1 in the field with q elements. This is therefore $q+1$. In total, we find for $U(x:m)$ the value announced in (4.6.1) (i).

We now assume $N(y)$ congruent to 1 mod P. Then -1 is in the set $X(y)$. In the sum over t we therefore separate the terms with $t \neq -1$ from the term $t = -1$. We thus obtain:

$$U(x:m) = (-1)^m \times \Big\{1 + (1-q^{-1})^{-1} \sum_t \int \Phi[(1+t+tv)B^{-1}\pi^m]\Omega(B)|B|_F^{-2} dv + (1-q^{-1})^{-1} \int \Phi[vC^{-1}\pi^m]\Omega(C)|C|_F^{-2} dv\Big\}$$

with $B = N(y)^{-1}N(t)^{-1} - 1 - Q(v)$, $Q(v) = Tr(v) + N(v)$, $C = N(y)^{-1} - 1 - Q(v)$. The first expression can be calculated as above, except that there are only q terms in the sum. We find that its value is

$$(q+1)^{-1}[(-q) - (-q)^{m+1}]$$

To calculate the second integral we use the following lemma:

LEMMA 4.6.2. *Let z be an element of F of absolute value less than 1. Then we have:*

$$\int \Phi[vD^{-1}\pi^m]\Omega(D)|D|^{-2} dv = (1-q^{-1})[-(1+q)^{-1} + q^m(-1)^m((q+1)^{-1} + 2^{-1}((-1)^{v(z)}-1))],$$

where we have set $D = z - Q(v)$ and $v \in P_E$.

Applying the lemma to $z = N(y)^{-1} - 1$ and adding the result to the value of the first integral, we see that $U(x:m)$ indeed has the given value; note that z and $N(y) - 1$ have the same valuation since $N(y)$ is a unit.

4.7. We will now prove lemma (4.6.2) which will complete the proof of lemma (4.6.1). We set $P = P_F$, $P' = P_E$, $G_i = 1 + P^i$ and $G'_i = 1 + P'^i$. We denote Tr the trace and N the norm. Then $Tr(P') = P$ and $N(G'_i) = G_i$. Let K be the kernel of N in G'_1. We will use without proof the following result:

LEMMA 4.7.1. *The index of $K \cap G'_j$ in K is q^{j-1}.*

To calculate the integral of lemma 4.6.2 we use the following integration formula:

$$\int_{P'} F(u+1) du = q^{-1} \int_P dv \int_K F[(1+u_0)k] dk,$$

where in the inner integral we choose a u_0 such that $N(1+u_0) = 1+v$ and dk denotes the Haar measure of volume 1 on K. Noting that $Q(u) = v$ if $N(1+u) = 1+v$, we see that the integral of lemma 4.6.2 can therefore be written as:

$$q^{-1} \int_P \Omega(z-v)|z-v|^{-2} dv \int \Phi[u(z-v)^{-1}\pi^m] dk,$$

where we write $1 + u = (1+u_0)k$ with $N(1+u_0) = 1+v$. We write the outer integral as the sum of an integral over the set of v such that $1 \leq |(z-v)\pi^{-m}|$ and an integral over the

v such that $|(z-v)\pi^{-m}| < 1$:

$$q^{-1}\int \Omega(z-v)|z-v|^{-2}dv \int \Phi[u(z-v)^{-1}\pi^m]dk, \quad |z-v| \geq q^{-m};$$

$$q^{-1}\int \Omega(z-v)|z-v|^{-2}dv \int \Phi[u(z-v)^{-1}\pi^m]dk, \quad |z-v| < q^{-m}. \tag{4.7.2}$$

In the first integral, we note that in the domain of integration the function Φ is evaluated on an integer; the inner integral is therefore independent of v with a value equal to 1. By changing v to $v+z$, we finally obtain:

$$q^{-1}\int \Omega(v)|v|^{-2}dv, \quad q^{-m} \leq |v| < 1. \tag{4.7.3}$$

The value of this integral is:

$$(1-q^{-1})(q+1)^{-1}[-1+(-1)^m q^m].$$

We now consider the second integral (4.7.2). For u in P' we have $|u|_E \geq |Q(u)|_E$. In the inner integral we thus have:

$$|(z-v)\pi^{-m}|_E \geq |u|_E \geq |Q(u)|_E = |v|_E.$$

We can therefore consider the outer integral as being over the set of v such that

$$q^{-m}|v| \leq |z-v| < q^{-m}.$$

We now consider such a v. There exists a u_0 with $|u_0|_E = |v|_E$ such that $N(1+u_0) = 1+v$. We then set $1+u = k(1+u_0)$ and the inner integral is over the set of k such that $|u| \leq |(z-v)\pi^{-m}|$. Setting $j = v[(z-v)\pi^{-m}]$ this means that k is in $G'_j \cap K$. The inner integral is therefore the volume of this intersection, which is q^{1-j} according to lemma (4.7.1), or equivalently $q|(z-v)\pi^{-m}|$. In total, the second integral (**??**) becomes

$$q^m \int \Omega(z-v)|z-v|^{-1}dv, \quad q^{-m}|v| \leq |z-v| < q^{-m}.$$

By changing v to $v+z$ we arrive at:

$$q^m \int \Omega(v)|v|^{-1}dv, \quad q^{-m}|v+z| \leq |v| < q^{-m}. \tag{4.7.4}$$

The first inequality is automatically satisfied if $|z| \leq |v|$. If $|z| > |v|$ it reduces to the inequality $|v| \geq q^{-m}|z|$. We thus see that the domain of integration is in fact defined by the inequalities $q^{-m}|z| \leq |v| < q^{-m}$. The calculation of the integral is then immediate. We find:

$$(1-q^{-1})q^m(-1)^m[-1+(-1)^{v(z)}]/2.$$

Adding the results of calculating (4.7.3) and (4.7.4), we indeed arrive at lemma 4.6.2.

4.8. In summary, for x of the form $s(y)$, $U(x : m)$ is given by the following formulas:

$$\begin{array}{rll} & \Omega(y)q^m & \text{if} \quad v(y) > 0, \\ & 0 & \text{if} \quad v(y) < -m, \\ & q^{m+v(y)} & \text{if} \quad -m \le v(y) < 0, \\ q^m 2^{-1}[1 + (-1)^{v(z)}] \quad \text{with} & z = N(y) - 1 & \text{if} \quad v(y) = 0. \end{array}$$

It remains to translate these formulas in terms of the variable x. We thus arrive at the following result:

PROPOSITION. *(i)* $U(x : m) = 0$ *unless the valuation* $v(x)$ *of* x *is even.*

(ii) We now assume $v(x)$ *even. Then* $U(x : m)$ *is given by the following formulas:*

$$\begin{array}{ll} q^m(-1)^{v(x)/2} & \textit{if} \quad v(x) > 0; \\ q^m & \textit{if} \quad v(x) < 0; \\ q^m q^{-v(1-x)/2} & \textit{if} \quad v(x) = 0, \quad v(1-x) \textit{ is odd and } v(1-x) \le 2m; \\ 0 & \textit{if} \quad v(x) = 0 \quad \textit{and} \quad v(1-x) \textit{ is odd} \\ & \textit{or if } v(x) = 0 \quad \textit{and} \quad v(1-x) > 2m. \end{array}$$

4.9. Let f' be a function on $G(F)$ that is K'-invariant. This function can be extended to a function on $G(E)$ invariant under K. We can then restrict this function to $G_1(F)$; this is nothing other than the image of f' under a privileged isomorphism from $G(F)$ to $G_1(F)$. We will still denote by f' the extension and restriction of this extension. Let g'_m be the image of g_m under the homomorphism (4.1.1). We now calculate the orbital integral $H(x : g'_m)$. Let $f'_{a,i}$ first be the characteristic function of the set:

$$K' \begin{vmatrix} \pi^{a+i} & 0 \\ 0 & \pi^i \end{vmatrix} K'.$$

We write simply f'_a for $f'_{a,0}$. We first have the relation:

$$H(x : f'_{a,i}) = (-1)^i H(x : f'_a). \tag{4.9.1}$$

Since $T(F)$ is contained in $Z(F)K'$ we immediately have:

$$H(x : f'_a) = \Omega(u) \sum_i (-1)^i f'_a[\pi^i \begin{pmatrix} 1 & u \\ u^\sigma & 1 \end{pmatrix}] \quad \text{if} \quad x = N(u).$$

If $|x| < 1$ then the matrix above is in K; this expression is therefore zero unless $a = 0$, in which case it is $(-1)^{v(x)/2}$. Similarly if $|x| > 1$ then the matrix above can be written:

$$\begin{vmatrix} u & 0 \\ 0 & u^\sigma \end{vmatrix} \begin{vmatrix} u^{-1} & 1 \\ 1 & u^{\sigma-1} \end{vmatrix}$$

The first matrix is in $T(F)$ thus in $Z(F)K$. The second is in K. The value of the expression above is therefore zero, unless $a = 0$ in which case it is 1. Finally we assume $|x| = 1$. Then the matrix above can be written:

$$\begin{vmatrix} 1 & u \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1-N(u) & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ u^\sigma & 1 \end{vmatrix}.$$

The first and third matrices are in K. The expression above is therefore zero unless $a = v(1-N(u))$, in which case it has the value 1. In summary, we have proved the following lemma:

LEMMA 4.9.2. *The function $H = H(x : f'_a)$ is given by the following formulas:*

(i) $H = 0$ if the valuation of x is odd;

Now suppose the valuation of x is even.

(ii) if $v(x) > 0$, $H = 0$ unless $a = 0$, in which case $H = (-1)^{v(x)/2}$;

(iii) if $v(x) < 0$, $H = 0$ unless $a = 0$, in which case $H = 1$;

(iv) $v(x) = 0$, $H = 0$ unless $a = v(1-x)$ in which case $H = 1$. Now we need to calculate g'_m in terms of functions $f_{a,i}$. This is a classical calculation that we leave to the reader (Cf. [Lan80]*).*

LEMMA 4.9.3. *We have:*

$$g'_m = \sum_{0 \le a \le m} f'_{2a} + \sum_{0 \le a \le m} \sum_{1 \le i \le m-a} (-1)^i q^i (1-q^{-1}) f'_{2a,i}$$

By applying relation (4.9.1) we immediately obtain

$$H(x : g'_m) = \sum_{0 \le a \le m} q^{m-a} H(x : f'_{2a})$$

If $v(x)$ is odd $H(x : g'_m) = 0$ by definition. Now suppose $v(x)$ is even. If $v(x) > 0$ this sum reduces to the term $a = 0$ and thus has the value $q^m(-1)^{v(x)/2}$. If $v(x) < 0$ the sum still reduces to the term $a = 0$ and takes the value q^m. Finally if $v(x) = 0$ the sum reduces to the term $a = v(1-x)/2$. It is therefore zero unless $v(1-x)$ is even and $v(1-x) \le 2m$. Its value is then $q^{m-v(1-x)/2}$.

By comparing with proposition 4.8 we obtain the identity:

$$H(x : g'_m) = U(x : m)$$

This is identity (4.2.4) for the function g_m and proposition 4.1 is thus established.

5. Integral of Eisenstein series

5.1. We now take for F a number field and for E a quadratic extension of F. Let ω be a character of the ideal class group of F. We assume it is trivial on the subgroup of ideals of E whose finite components are 1 and whose infinite components are all equal and positive. Unless explicitly stated otherwise, we make the same assumption for all characters of the ideal class group of F or E. We denote by ω' the lift of ω to the field E. We consider the Eisenstein series integrals of the group $G(E)$ which we will need for the relative trace formula. We consider a function ϕ on $G(E_A)$ such that:

$$\phi \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} g = \chi(a)\chi'(b)|a|^{u+1/2}|b|^{u-1/2}, \tag{5.1.1}$$

where χ and χ' are characters of the ideal class group of E whose product is ω'. Despite the adopted notations, the function ϕ depends on u. It is determined by its restriction to the usual maximal compact subgroup K. In general, we assume this restriction is independent of u. We will denote by $\varrho(\chi, \chi', u)$ the representation of $G(E_A)$ in the space of functions transforming as above. We write $f \cdot \phi$ instead of $\varrho(\chi, \chi', u)(f)\phi$ when this does not lead to confusion. We denote by B the subgroup of upper triangular matrices and we set:

$$E(g, \phi, \chi, \chi', u) = \sum_{B(E)\backslash G(E)} \phi(\gamma g),$$

where the sum is defined by analytic continuation.

5.2. We will need a formula for the Mellin transform of E. Let ψ be a non-trivial character of the ideal class group of E. We set:

$$W(g) = \int E \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\psi(x)dx.$$

We also set, for Ω a character of the ideal class group of E:

$$L(\Omega^{-1}, \phi, \chi, \chi', u) = \int W \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} |a|^{s-1/2}\Omega^{-1}(a)d^\times a|_{s=1/2}. \tag{5.2.1}$$

More precisely, the integral converges when the real part of s is large enough and extends to a meromorphic function of s, holomorphic at point $1/2$. We now integrate E over the set of diagonal matrices $\mathrm{diag}(a, 1)$ with $c^{-1} < |a| < c$; the integral is taken modulo rational elements. The Haar measure is the product of local Tamagawa measures by the residue at point 1 of the function $L(s, 1_E)$. Moreover, given the result we have in mind (cf. (5.6)), the choice of measure matters little. We obtain the following relation:

$$\begin{aligned}
&\int_{c^{-1}}^{c} E\left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix}, \phi, \chi, \chi', u\right] \Omega^{-1}(a)da = L(\Omega^{-1}, \phi, \chi, \chi', u) \\
&+\delta(\chi\Omega^{-1})c^{u+1/2}(u+1/2)^{-1}\phi(e) \\
&+\delta(\chi'\Omega^{-1})c^{-u+1/2}(-u+1/2)^{-1}(M(u, \chi, \chi')\phi(e)) \\
&+\delta(\chi'\Omega^{-1})c^{u+1/2}(u+1/2)^{-1}\phi(w) \\
&+\delta(\chi\Omega^{-1})c^{-u+1/2}(-u+1/2)^{-1}(M(u, \chi, \chi')\phi)(w) \\
&+R(c, u, f \cdot \phi)
\end{aligned} \tag{5.2.2}$$

where we set $\delta(\chi) = 1$ if χ is not trivial, $\delta(\chi) = 0$ if χ is trivial, M denotes the intertwining operator and w is the matrix:

$$w = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

The "remainder" $R(c, u, \Phi)$ tends to 0 when c tends to infinity. For the proof see [Jac86] §8. From now on we assume that the restriction of Ω to F is ω.

We will need precise estimates on the introduced quantities:

LEMMA 5.2.3. *The function $L(\cdot, \cdot, u)$ and its derivatives are of slow growth on the line $Re(u) = 0$.*

Proof. We can assume that the function ϕ is a product of local functions ϕ_v. Let S be a finite set of places of E containing all infinite places and such that, for all v not in S, ϕ_v is K_v-invariant and the characters χ_v and χ'_v are unramified. Then, at places v not in S, we can write

$$\phi_v(g) = L(2u+1, \chi_v \chi_v'^{-1})^{-1} \int \Phi_v[(0,1)g]|t|^{2u+1} \chi_v \chi_v'^{-1}(t) d^\times t \\ \times \chi_v(\det g) |\det g|^{u+1/2} \tag{5.2.4}$$

where Φ_v is the characteristic function of integers. At places v in S we can write

$$\phi_v(g) = \int \Phi_v[(0,1)g]|t|^{2u+1} \chi_v \chi_v'^{-1}(t) d^\times t \chi_v(\det g) |\det g|^{u+1/2}, \tag{5.2.5}$$

where Φ_v is a Schwartz-Bruhat function such that the function $\Phi_v[(0,1)g]$ on $SL(2, F_v)$ has compact support, modulo the group of strictly upper triangular matrices. This being so, denoting by Φ the product of functions Φ_v, we obtain for $L(\cdot, \cdot, u)$ the expression

$$L(\Omega^{-1}, \phi, \chi, \chi', u) = L(2u+1, \chi \chi'^{-1S})^{-1} \times \iint \Phi'(a,b) \Omega^{-1} \chi(a) |a|^{u+1/2} da \Omega^{-1} \chi'(b) |b|^{-u+1/2} db \tag{5.2.6}$$

where Φ' denotes a partial Fourier transform of Φ. The factor L in this formula denotes the product of factors $L(2u+1, \chi_v \chi_v'^{-1})$ for v not in S. On the line $Re(u) = 0$ it is of slow growth as well as all its derivatives, in fact of logarithmic growth. Its inverse is of slow growth; its derivatives are therefore also of slow growth. The second factor is a Tate integral without poles on the same line. It is therefore bounded as well as all its derivatives. Hence the lemma. □

LEMMA 5.2.8. *On the line $Re(u) = 0$ the function $R(c, u, \phi)$ and its derivatives are of slow growth. Moreover, when c tends to infinity, the function and its derivatives tend to 0 in the space of functions of slow growth.*

Proof. We again use the integral expression above for the function ϕ. We then obtain for $R(c, u, \phi)$ the expression:

$$L(2u+1, \chi \chi'^{-1S})^{-1} \int \Big[\int \Phi'(ta, t^{-1}) \chi \chi'^{-1}(t) |t|^{2u} dt \Big] \Omega^{-1} \chi(a) |a|^{u+1/2} d^\times a,$$

where the exterior integral is over ideals of norm greater than c or less than c^{-1}. Let $R_+(c,u)$ be the integral over a such that $|a| > c$. There exists a Schwartz-Bruhat function $\phi \geq 0$ such that:

$$\int |\Phi'|(ta, t^{-1}) d^\times t \leq \phi(a)|a|^{-1}$$

We thus obtain a majorization of $R_+(c,u)$ by the integral

$$\int \phi(a)|a|^{1/2} d^\times a, \quad |a| > c$$

which decreases rapidly for large c. We thus have the desired majorization for $R_+(u,c)$. For $R_-(c,u)$ we have, after a change of variables, the following integral representation, where we integrate over a of norm greater than c:

$$L(2u+1, \chi\chi'^{-1S})^{-1} \times \int \Big[\int \Phi'(t^{-1}, at)\chi'\chi^{-1}(t)|t|^{-2u} dt \Big] \chi^{-1}(a)|a|^{-u+1/2} d^\times a$$

We thus obtain an estimate for $R_-(c,u)$ and then $R(c,u)$. Finally, we obtain similar estimates for the derivatives of $R(c,u)$ by replacing the factors $|t|^{2u}$ and $|a|^u$ with their derivatives. □

LEMMA 5.2.9. *On the line $Re(u) = 0$ the function:*

$$\int_{c^{-1}}^{c} E\left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix}, f \cdot \phi, \chi, \chi', u \right] da$$

and all its derivatives are rapidly decreasing. The function $M(u,\chi,\chi')\phi(e)$ and its derivatives are of slow growth.

Proof. We can write

$$f \cdot \phi = \sum_i m_i(u)\phi_i$$

where the functions m_i are rapidly decreasing. We thus see that for the first assertion it suffices to prove that the integral obtained by replacing $f \cdot \phi$ with ϕ is of slow growth as well as all its derivatives. Returning to expression (5.2.2) and using the preceding lemmas, we see that it suffices to verify that $M(u,\chi,\chi')\phi(e)$ is of slow growth as well as all its derivatives on the line $Re(u) = 0$. For this purpose we write:

$$M(u,\chi,\chi')\phi(e) = L(2u, \chi\chi'^{-1})L(2u+1, \chi\chi'^{-1})^{-1} \prod_v R(u, \chi_v, \chi'_v)\phi_v(e)$$

where $R(u,\chi_v,\chi'_v)$ is the normalized intertwining operator. In the infinite product almost all factors are 1. Moreover, if v is a finite place the corresponding factor is a rational function in q_v^{-u}. If v is an infinite place the corresponding factor is a rational function in u. Using the functional equation of the L functions, the ratio, up to an exponential factor, can be written in the form:

$$L(1-2u, \chi'\chi_S^{-1})L(1+2u, \chi\chi_S'^{-1})^{-1} \times L(1-2u, \chi'\chi^{-1S})L(1+2u, \chi\chi'^{-1S})^{-1},$$

where S now denotes the set of places at infinity and the first factor is the product of local L factors for all places in S. As in the proof of lemma 5.2.3 the second ratio and its derivatives grow slowly. The first ratio has absolute value 1; moreover according to the classical properties of the gamma function, the quotient of a derivative of the function $L(1+2u,\cdot,\cdot,s)$ by itself is of slow growth. It follows that all derivatives of the first quotient are of slow growth. Hence the conclusion. □

We will also need the following result:

LEMMA 5.2.10. *Suppose that χ is the lift of a character of F and that the restriction of χ to F equals $\omega\eta$. Then the function $L(\Omega^{-1}, f\cdot\phi,\chi,\chi',u)$ is zero at 0.*

Proof. Under the hypotheses of the lemma, we have $\chi=\chi'$. As above, denoting by S the set of infinite places, we have:

$$L(\Omega^{-1},\phi,\chi,\chi',u)=L(2u+1,1^S)^{-1}\times\iint\Phi'(a,b)\Omega^{-1}(a)\chi(a)|a|^{u+1/2}da\Omega^{-1}(b)\chi'(b)|b|^{-u+1/2}db$$

The first factor has a zero at point 0 and the other two have no pole. The lemma follows. □

5.3. Let χ still be a character of the ideal class group of E and χ' the character $\omega'\chi^{-1}$. If ϕ is as above we will set:

$$I(\phi,\chi,\chi',u)=\int\phi(g)\eta\omega^{-1}(\det g)dg,\quad g\in Z(F_A)T(F)\backslash G_6(F_A). \tag{5.3.1}$$

To calculate the integral, we can first integrate over the torus T. The integral over the torus reduces to the integral of $\chi(a/a^\sigma)$ over the quotient of $E_A^\times$ by the product $E^\times F_A^\times$. It is therefore zero unless χ is invariant under conjugation of E with respect to F. Under this hypothesis it equals the volume of this quotient and the total integral equals:

$$I(\phi,\chi,\chi',u)=\int\phi(g)\eta\omega^{-1}(\det g)dg\ \mathrm{vol}(E_A^\times/F_A^\times E^\times),$$

$$g\in Z(F_A)T(F_A)\backslash G_6(F_A). \tag{5.3.2}$$

LEMMA 5.3.3. *If ε is not a norm and the real part of u is large enough then:*

$$\int_{Z(F_A)G_6(F)\backslash G_6(F_A)}E(g,\phi,\chi,\chi'u)dg=I(\phi,\chi,\chi',u).$$

Proof. Indeed we have $G(E)=P(E)G_6(F)$ and the intersection of these two subgroups is $T(F)$. The series that defines the Eisenstein series can therefore be written:

$$E(g)=\sum\phi(\gamma g),\quad \gamma\in T(F)\backslash G_6(F).$$

and the assertion of the lemma is therefore immediate. □

If ε is a norm, the above integral diverges and we must use a truncation operator. For this purpose we assume $\varepsilon = 1$, we choose an element of E with zero trace s and we set:

$$m = \begin{vmatrix} s & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{vmatrix}.$$

Then we verify that $mG_1(F)m^{-1} = G(F)$. Moreover, according to lemma 2.2 we have:

$$G(E) = P(E)G_1(F) \cup P(E)mG_1(F).$$

We therefore also have:

$$G(E) = P(E)G_1(F) \cup P(E)G(F)m$$

If f is a function on $G(E)\backslash G(E_A)$ the function truncated "at height c" is the function $T^c f$ on $G_1(F)\backslash G_1(F_A)$ given by the sum:

$$T^c f(g) = f(g) - \sum f_N(\gamma m g) H_c(\gamma m g m^{-1}), \quad \gamma \in P(F)\backslash G(F),$$

where f_N is the constant term of f, that is the integral:

$$f_N(g) = \int f \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g dx, \quad x \in E_A/E.$$

and we have set:

$$H(g) = |ab^{-1}|_E \quad \text{if} \quad g = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} k \quad \text{with} \quad k \in K,$$

$$H_c(g) = H(g) \quad \text{if} \quad H(g) \geq c, \quad = 0 \quad \text{otherwise.}$$

LEMMA 5.3.4. *Let $\varepsilon = 1$. Then, for a suitable choice of Haar measures we have:*

$$\int T^c E(g, \phi, \chi, \chi', u) \eta\omega^{-1}(\det g) \mathrm{d}g = I(\phi, \chi, \chi', u)$$

$$+\delta(\chi\omega^{-1}\eta) c^{2u} (2u)^{-1} \int \phi(km) \mathrm{d}k$$

$$-\delta(\chi'\omega^{-1}\eta) c^{-2u} (2u)^{-1} \int [M(u, \chi, \chi')\phi].(km) \mathrm{d}k.$$

In this formula k is in the usual maximal compact subgroup K_F of $G(F_A)$; moreover the first δ for example is 1 if the restriction of χ to F equals $\omega\eta^{-1}$, 0 otherwise. Finally $I = 0$ unless χ is a lifting.

Proof. By combining the series that define E and its truncation, we obtain the expression:

$$T^c E = \sum_{T(F)G_1(F)} \phi(\gamma g) + \sum_{P(F)G(F)} [\phi(\gamma m g) - E_N(\gamma m g) H_c(\gamma m g m^{-1})].$$

We now integrate this expression against $\eta\omega^{-1}$ on the quotient of $G_1(F_A)$ by $Z(F_A)G_1(F)$. The first sum gives the integral I. The second becomes the integral:

$$\int_{P(E)Z(F_A)G(F_A)} [\phi(gm) - E_N(gm) H_c(g)] \eta\omega^{-1}(\det g) \mathrm{d}g,$$

or alternatively:

$$\int \phi(gm)[1 - H_c(g)]\eta\omega^{-1}(\det g)\mathrm{d}g - \int M(u,\chi,\chi')\phi(gm)H_c(g)\eta\omega^{-1}(\det g)\mathrm{d}g.$$

The integrals are taken over $P(E)Z(F_A)\mathcal{N}_{n+1}(F_A)$ where N is the group of upper unipotent matrices. We arrive at the desired result using the Iwasawa decomposition. □

5.4. We will need the analytic properties of the functions $I(\dots u)$.

LEMMA 5.4.1. *Suppose that χ is the lifting of a character of F. Then $I(\dots u)$ is holomorphic and rapidly decreasing in the strip $0 \le \mathrm{Re} u \le 1/2$, except for a simple pole at $u = 0$ and a simple pole at $u = 1$ if the restriction of χ to F equals $\omega\eta$. Its derivatives have slow growth on the line* $\mathrm{Re}(u) = 1/2$.

Proof. If χ is the lifting of the character μ, then the restriction of χ to F is the square of μ. We use the same integral representation as above for the function ϕ, but we write it in the form:

$$\phi(g) = L(2u+1, \chi\chi'^{-1S})^{-1} \int \Phi[(0,1)|\begin{pmatrix} t & 0 \\ 0 & t^\sigma \end{pmatrix} g]\mu^2\omega^{-1}(tt^\sigma)|tt^\sigma|_E^{|u+1/2}$$

$$\mathrm{d}t\, \chi(\det g)|\det g|_E^{u+1/2},$$

the integral being over the torus T. By integrating this over the quotient $T(F_A)G_\varepsilon(F_A)$ we obtain an integral over $G_\varepsilon(F_A)$:

$$L(2u+1, \chi\chi'^{-1S})^{-1} \int \Phi[(0,1)g]\mu^2\omega^{-1}\eta(\det g)|\det g|^{2u+1}\mathrm{d}g.$$

The function L is not zero in the band in question. Its inverse is therefore holomorphic and has slow growth in the band. The second factor is a Tate integral for the division algebra whose multiplicative group is G_ε. According to the theory of these integrals, it is the product of an entire function, the function $L(2u+1, \mu^2\omega^{-1}\eta)$ and the function $L(2u, \mu^2\omega^{-1}\eta)$. Moreover, in the first factor, the character is composed of $\mu^2\omega^{-1}$ and the norm. The holomorphic properties are thus manifest. A Tate integral is bounded at infinity in any vertical band. Its derivatives similarly have slow growth in any vertical band. It is therefore clear that the function has slow growth in the band. We move on to the behavior on the line $\mathrm{Re} u = 1/2$. Since $L(2u+1, \mu^2\omega^{-1}\eta^S)$ is defined by a convergent product (or a convergent Dirichlet series) its derivatives are bounded on the line in question. We conclude that the derivatives of the product have slow growth. Hence the conclusion. □

LEMMA 5.4.2. *Suppose that the character χ is a lifting and its restriction to F is ω. Let z^* denote the complex conjugate of the complex number z. Then the product*

$$M(u,\chi,\chi')f \cdot \phi(e)I(\phi,\chi,\chi',-u^*)^*$$

is holomorphic and rapidly decreasing in the strip $0 \le \mathrm{Re} u \le 1/2$. Its derivatives are rapidly decreasing on the line $\mathrm{Re} u = 1/2$.

Proof. By hypothesis we have $\chi = \chi'$. By linearity we can assume that we are in the following situation: the function ϕ is the product of local functions ϕ_v and S is a finite set of places of E. For v not in S the function ϕ_v transforms under K_v according to the character $\chi_v(\det k)^{-1}$. For v in S we have:

$$\int_{K_v} \phi_v(gk)\chi_v(\det k)^{-1}\mathrm{d}k = 0$$

Note that S may not contain certain archimedean places. We then have an integral representation analogous to the one we have used until now. In particular, the quantity in question can be written:

$$L(2u,1)L(2u+1,1)^{-1}R(u,\chi,\chi')f\cdot\phi(e)L(-2u+1,1^S)^{-1}\times\int \Phi^*[(0,1)g]\eta(\det g)|\det g|^{1-2u}\mathrm{d}g.$$

The last integral is a Tate integral over the group G_ε. We write $L(2u,1) = L(1-2u,1)$, up to an exponential factor. We thus obtain the expression in the form of a product of factors:

$$L(2u+1,1^S)^{-1}[\prod_{v\in S} L(1-2u,1_v)L(1+2u,1_v)^{-1}f_v\cdot R(u,\chi_v,\chi'_v)\phi_v(e)]$$
$$\times\int \Phi^*[(0,1)g]\eta(\det g)|\det g|^{-2u+1}\mathrm{d}g.$$

This formula already gives holomorphicity. Indeed, the last factor is a holomorphic multiple of $L(1-2u,\eta)L(-2u,\eta)$ and is thus holomorphic. The first factor and the normalized intertwining operator are holomorphic in the band in question. The factor $L(-2u+1,1_v)$ where v is in archimedean S has a pole at point $1/2$; however, according to our conventions $R(u,\chi,\chi')\phi_v = 0$ at point $1/2$. Thus the pole of the factor $L(-2u+1,1_v)$ is compensated by a zero. A similar argument applies to the poles of the factor $L(-2u+1,1_v)$ for finite v in S. To obtain the required estimate we can further modify S as follows: we enlarge S by adding all places at infinity. Then the first factor has slow growth in the band; its derivatives have slow growth on the line $\mathrm{Re}u = 1/2$. The last factor is bounded as are all its derivatives in the band. The factor containing the intertwining operator is rapidly decreasing for archimedean v. According to the properties of the gamma function, the factor corresponding to an archimedean place v in S has slow growth in the band as do all its derivatives. Finally, the remaining factors are rational functions of q_v^{-s} without poles in the band. Our assertion follows. □

5.5. Let us now consider the kernel K_{eis}. Recall that z^* denotes the complex conjugate of z. We have:

$$K_{eis}(g,h) = 1/4i\pi\sum_{\chi,\phi}\int E(g,f,\phi,\chi,\chi'u)\cdot E(h,\chi,\chi',u)^* du;$$

the sum is over all characters χ and, for each χ, over an orthonormal basis of the representation space $\rho(\chi,\chi',u)$, considered as acting on a function space on K. We have written and will often write in the sequel $f\cdot\phi$ for the action of f on ϕ in the representation $\rho(\chi,\chi',u)$.

We choose $C > 0$ and $c > 0$ and apply to the kernel the truncation operator with respect to the second variable "at height c". It is clear that we can exchange the integration and truncation. We consider the following integral:

$$\int_{C^{-1}}^{C} \int T_2^c K_{eis} \left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix}, h \right] \Omega^{-1}(a) \eta\omega(\det h) dh d^{\times} a. \tag{5.5.1}$$

The integral in h is over the quotient $G_v(F)Z(F_\lambda)\backslash G_v(F_\lambda)$; the integral in a is over the elements a of the group of ideal classes of E such that $C^{-1} < |a| < C$. We will assume that ε is a norm and even that $\varepsilon = 1$. We will indicate in passing the modifications to make if ε is not a norm.

LEMMA 5.5.2. *There exists a polynomial $P(t)$ such that $|T^c E(h, \phi, \chi, \chi', it)| \leq P(t)$ for all t and all h. Moreover, given f, χ, a compact subset M and an integer N, there exists a constant C such that $|E(h, f, \phi, \chi, \chi', it)| \leq C|t|^{-N}$ for h in M.*

Proof. The second assertion is standard. The first one is [JL85, Lemma 8.2.1]. □

The lemma shows that the integral (5.5.1) is in fact equal to a finite sum of integrals:

$$\int \left[\int_{C^{-1}}^{C} E \left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} \right], f \cdot \phi, \chi, \chi', u)^* \Omega^{-1}(a) d^{\times} a \times \int T^c E(h, \phi, \chi, \chi', u)^* \eta\omega(\det h) dh \right] du.$$

According to (5.3.4), setting:

$$\int_{C^{-1}}^{C} E \begin{pmatrix} |a & 0| \\ 0 & 1 \end{pmatrix}, f \cdot \phi, \chi, \chi', u \Omega^{-1}(a) d^{\times} a = J(C, u),$$

we can write this in the form:

$$\begin{aligned} &\int J(C,u) I(\phi, \chi, \chi', u)^* du \\ &+\delta(\chi\omega^{-1}\eta) \int c^{-2u}(-2u)^{-1} J(C,u) \int \phi(km) dk^* du \\ &-\delta(\chi'\omega^{-1}\eta) \int c^{2u}(-2u)^{-1} J(C,u) \int M(u, \chi, \chi') \phi(km) dk^* du. \end{aligned} \tag{5.5.3}$$

The first term is zero unless χ is a lift. The last two terms are zero unless the restriction of χ to F is $\omega\eta$. Moreover, according to lemma (5.2.9), $J(C,u)$ and its derivatives decrease rapidly on the line $Reu = 0$. The last two integrals are therefore oscillating integrals with a limit as c tends to infinity, namely:

$$i\pi/2 \int_{C^{-1}}^{C} E \begin{bmatrix} |a & 0| \\ 0 & 1 \end{bmatrix}, f \cdot \phi, \chi, \chi', 0 \Omega^{-1}(a) d^{\times} a \int \phi(km) dk^*,$$

$$i\pi/2 \int_{C^{-1}}^{C} E \begin{bmatrix} |a & 0| \\ 0 & 1 \end{bmatrix}, f \cdot \phi, \chi, \chi', 0 \Omega^{-1}(a) d^{\times} a \times \int M(0, \phi, \chi, \chi') \phi(km) dk^*.$$

First, suppose that χ is the lift to E of a character μ of F. Since the restriction of χ to F is μ^2, we must have $\mu^2 = \omega\eta$. It follows that χ' equals χ. According to a well-known result, we then have $M(0,\chi,\chi') = -1$ and the preceding terms cancel out. Now suppose that χ is not a lift. We now use expression (5.2.4) to calculate the above integrals. The terms $\delta(\chi\Omega^{-1})$ and $\delta(\chi'\Omega^{-1})$ are zero because the restriction of Ω to F is ω while the restriction of χ and χ' to F is $\omega\eta$. The term $R(C,u,f,\phi)$ tends to 0 as C tends to infinity. We thus finally obtain the following contribution for the double limit of the last two terms of (5.5.3):

$$\sum_{\phi,\chi} L(\Omega^{-1}, f\cdot\phi,\chi,\chi',0)(i\pi/2)\times\left[\int\phi[km]dk^* + \int[M(0,\chi,\chi')\phi](km)dk^*\right]. \qquad (5.5.4)$$

The sum is over all χ whose restriction to F is $\omega\eta$ and which are not lifts. These terms are not present if ε is not a norm.

We now examine the first term of (5.5.3). We replace the integral in a by its expression derived from (5.2.2). We obtain the following terms:

$$\int L(\chi^{-1}f\cdot\phi,\chi,\chi',u)I(\phi,\chi,\chi',u)^*du$$
$$+\delta(\chi\Omega^{-1})\int C^{u+1/2}(u+1/2)^{-1}f\cdot\phi(e)I(\phi,\chi,\chi',u)^*du$$
$$+\delta(\chi'\Omega^{-1})\int C^{-u+1/2}(-u+1/2)^{-1}f\cdot M(u,\chi,\chi')\phi(e)I(\phi,\chi,\chi',u)^*du$$
$$+\delta(\chi'\Omega^{-1})\int C^{u+1/2}(u+1/2)^{-1}f\cdot\phi(e)I(\phi,\chi,\chi',u)^*du$$
$$+\delta(\chi\Omega^{-1})\int C^{-u+1/2}(-u+1/2)^{-1}f\cdot M(u,\chi,\chi')\phi(w)I(\phi,\chi,\chi',u)^*du$$
$$+\int R(C,u,f\cdot\phi)I(\phi,\chi,\chi',u)^*du.$$

Each term is zero unless χ is a lift. We can therefore assume this is the case. If moreover the restriction of χ to F is the character $\omega\eta$ then I has a pole at point $u=0$ (lemma (5.4.1)); the integrals above are then improper. When C tends to infinity, the last term tends to 0 as follows from lemma (5.2.8). We now examine the limit as C tends to infinity of integrals 2 to 5. They are zero unless $\chi=\chi'=\Omega$. Since the restriction of Ω to F is ω, that of χ is also ω and in particular different from $\omega\eta$. According to lemma (5.4.1) the function $f\cdot\phi(e)I(\phi,\chi,\chi',-u^*)^*$ is holomorphic in the band $-1/2\le Re u\le 0$, with rapid decay; its derivatives have rapid decay on the line $Re u=-1/2$. We can therefore deform the integration contour and write integral 2 in the form:

$$\int C^v v^{-1} f\cdot\phi(e)I(\phi,\chi,\chi',1/2-v^*)^*dv,$$

the integral being extended to the imaginary axis, except that the arc from $-i\varepsilon$ to $i\varepsilon$ is replaced by the semicircle passing through $-i\varepsilon$, ε, $i\varepsilon$; letting ε tend to 0 we see that the above expression equals the (improper) integral over the entire imaginary axis plus $i\pi$ times the value of the integrand at point $v=0$. Now letting C tend to infinity, we finally obtain:

$$2i\pi f\cdot\phi(e)I(\phi,\Omega,\Omega,1/2)^*.$$

We proceed similarly with integral 4 and obtain an analogous result. For integrals 3 and 5, we proceed as above except that the integration contour becomes the line $Reu = 1/2$ suitably deformed and we use lemma (5.4.2). In total, we obtain the following expression for the double limit of the first term of (5.5.3):

$$\begin{aligned}&\int L(\Omega^{-1}, f\cdot\phi, \chi, \chi', u)I(\phi, \chi, \chi', u)^* du\\&+\{2i\pi[\varrho(\Omega,\Omega,-1/2)(f)\phi(e)+\varrho(\Omega,\Omega,-1/2)(f)\phi(w)]I(\phi,\Omega,\Omega,1/2)^*\\&+2i\pi[\varrho(\Omega,\Omega,-1/2)(f)M(1/2,\Omega,\Omega)\phi(e)\\&+\varrho(\Omega,\Omega,-1/2)(f)M(1/2,\Omega,\Omega)\phi(w)]\}I(\phi,\Omega,\Omega,-1/2)^*,\end{aligned}$$

the last two terms being present only if Ω is a lift. In summary, we have the following formula:

PROPOSITION. *Consider the integral (without T_2^c if ε is not in N_0)*

$$\int_{C^{-1}}^{C} T_2^c E_{eis}\begin{pmatrix}|a & 0|\\ 0 & 1\end{pmatrix}, h\eta\omega(\det h)dhd^\times a.$$

The double limit where c then C tends to infinity exists. It equals $4i\pi^{-1}$ times:

$$\begin{aligned}&\sum_{\chi,\phi}\int L(\Omega^{-1}, f\cdot\phi,\chi,\chi',u)I(\phi,\chi,\chi',u)^*du\\&+\sum_{\phi}2i\pi\varrho(\Omega,\Omega,-1/2)(f)\phi(e)I(\phi,\Omega,\Omega,1/2)^*\\&+\sum_{\phi}2i\pi\varrho(\Omega,\Omega,-1/2)(f)\phi(w)I(\phi,\Omega,\Omega,1/2)^*\\&+\sum_{\phi}2i\pi\varrho(\Omega,\Omega,-1/2)(f)M(1/2,\Omega,\Omega)\phi(e)I(\phi,\Omega,\Omega,-1/2)^*\\&+\sum_{\phi}2i\pi\varrho(\Omega,\Omega,-1/2)(f)M(1/2,\Omega,\Omega)\phi(w)I(\phi,\Omega,\Omega,-1/2)^*\\&+\sum_{\chi,\phi}L(\Omega^{-1},f\cdot\phi,\chi,\chi',0)(i\pi/2)\\&\times\left[\int\phi[km]dk^*+\int[M(0,\chi,\chi')\phi](km)dk^*\right]\end{aligned}$$

The first sum is over all χ which are lifts. The following terms are zero unless Ω is the lift of a character of F. The last sum is over all characters χ which are not lifts and whose restriction to F is $\omega\eta$. It is zero if ε is not a norm.

5.6. For the application we have in mind, we can reformulate this identity as follows. In the first integral, the factor I may have a pole at point $u = 0$ if the restriction of χ to F is $\omega\eta$. However, according to lemma (5.2.8) the first factor then has a zero. This integral is therefore always an ordinary integral. Let S be a finite set of places of E containing all places at infinity; if ε is not a norm we assume that S contains all places where ε is not a norm as well as all places where ε is not a unit. For v not in S we take for f_v a K_v-invariant function. We write f^S (resp. f_S) for the product of f_v, v not in S (resp. in S). Then the Satake transform $f^S(\chi, \chi', u)$ is defined by the formula:

$$f^S(\chi, \chi', u)\phi = \varrho(\chi, \chi', u)(f^S)\phi,$$

if ϕ is K^S-invariant. Then the preceding expression has the form:

$$\sum \int L(\Omega^{-1}, f_S, \phi, \chi, \chi', u) I(\phi, \chi, \chi', u)^* f^S(\chi, \chi', u) du$$
$$+ \sum_{\chi} A(f_S, \phi, \chi) f^S(\chi, \chi', 0) + B(f_S, \phi, \Omega) f^S(\Omega, \Omega, 1/2),$$

where A and B depend linearly on their first argument. Moreover, the characters that appear in the sums are the unramified characters outside of S. The first sum is over all characters χ which are lifts, the second over all characters which are not lifts and whose restriction to F is $\omega\eta$. Furthermore, the second sum is zero if ε is not a norm. Finally, the last term is zero unless Ω is a lift.

5.7. We now study the integral from proposition (5.6) for the special kernel. This is given by the sum:

$$K_{spe}(x, y) = 1/vol \sum_{\chi} \int f(g) \chi(\det g) dg \chi(x) \chi(y)^*,$$

the sum being over all characters χ whose square is ω'. We first assume $\varepsilon = 1$. The integral in question is written:

$$\int_{C^{-1}}^{C} T_2^c K_{spe} \begin{pmatrix} |a & 0| \\ 0 & 1 \end{pmatrix}, h\eta\omega(\det h) dh \Omega^{-1}(a) d^\times a.$$

The integral in a is zero unless χ equals Ω. The integral in h gives a difference:

$$\int_{Z(F_\lambda)G_1(F)\backslash G_1(F_\lambda)} \chi^*(g)\omega\eta(\det g) dg - \int_{Z(F_\lambda)P(F)\backslash G(F_\lambda)} \chi^* \omega\eta(\det gm) H_c(gm) dg.$$

Both integrals are zero unless the restriction of χ to F equals $\eta\omega$. We conclude that the total integral is zero. We have an analogous conclusion for the case where ε is not a norm. Hence:

PROPOSITION. *If $\varepsilon = 1$ then the integral*

$$\int_{C^{-1}}^{C} T_2^c E_{spe} \begin{pmatrix} |a & 0| \\ 0 & 1 \end{pmatrix}, h\eta\omega(\det h) dh \Omega^{-1}(a) d^\times a.$$

is convergent and zero. If ε is not a norm the integral

$$\int_{C^{-1}}^{C} E_{spe} \begin{pmatrix} |a & 0| \\ 0 & 1 \end{pmatrix}, h\eta\omega(\det h) dh \Omega^{-1}(a) d^\times a.$$

is convergent and zero.

6. Integration of kernel K

6.1. We keep the notations from §5. We set

$$K(x,y)=\sum f(x^{-1}\xi y),\quad \xi\in G(E)/Z(E).$$

If $\varepsilon=1$ we study the integral

$$\int_{C^{-1}}^{C}\int T_2^cK\left[\begin{vmatrix}a&0\\0&1\end{vmatrix},h\right]\eta\omega(\det h)\Omega^{-1}(a)dhda,$$

$$a\in E_\Lambda^X/E^X,\quad h\in Z(F_\Lambda)G_1(F)\backslash G_1(F_\Lambda)\tag{6.1.1}$$

If ε is not a norm we study the integral

$$\int_{C^{-1}}^{C}\int K\left[\begin{vmatrix}a&0\\0&1\end{vmatrix},h\right]\eta\omega(\det h)\Omega^{-1}(a)dhda,$$

$$a\in E_\Lambda^X/E^X,\quad h\in Z(F_\Lambda)G_\varepsilon(F)\backslash G_\varepsilon(F_\Lambda).\tag{6.1.2}$$

6.2. We will first show that the truncation operator in (6.1.1) is superfluous. For this purpose, we set:

$$S^c\phi(g)=(T^c-1)\phi(g)=\sum\phi_N(\gamma mg)H_c(\gamma mgm^{-1}).$$

We then have the following proposition:

PROPOSITION. *Given f and $C>0$, there exists $D>0$ such that the relations $c>D$ and $C^{-1}<|a|<C$ imply*

$$S_2^cK\left[\begin{vmatrix}a&0\\0&1\end{vmatrix},h\right]=0.$$

Proof. We use without proof the following lemma:

LEMMA 6.2.1. *Let U be a compact subset of $SL(2,F_\Lambda)$. Then there exists $D>0$ such that the relations*

$$g\in U,\quad \gamma\in SL(2,F)\quad \textit{and}\quad H(\gamma g)>D$$

imply that γ is triangular.

□

This being the case, the constant term of K with respect to the second variable is written:

$$K_{2N}\left[\begin{vmatrix}a&0\\0&1\end{vmatrix},y\right]=\int_{N(E_\Lambda)}\sum_{\gamma\in Z(E)\backslash G(E)/N(E)}f\left[\begin{vmatrix}a^{-1}&0\\0&1\end{vmatrix}\gamma ny\right]dn.$$

It follows therefore:

$$S_2^cK\left[\begin{vmatrix}a&0\\0&a\end{vmatrix},g\right]=\sum_{\theta\in P(F)\backslash G(F)}\int_{N(E_\Lambda)}\sum_{\gamma}f\left[\begin{vmatrix}a^{-1}&0\\0&1\end{vmatrix}\gamma n\theta mg\right]H_c(\theta mgm^{-1})dn.$$

In this expression we can assume that a is in a compact set. Under this hypothesis, if the integrand is not zero then $h = \gamma n\theta mgm^{-1}$ is in a compact set modulo the center and $H(\gamma^{-1}h) > c$. According to the lemma, we therefore have $\gamma \in P(E)$ if c is large enough. Thus the preceding expression, for c large enough, reduces to:

$$\sum_{\theta \in P(F)\backslash G(F)} \int_{N(E_\Lambda)} \sum_\alpha f\left[\begin{vmatrix}\alpha a^{-1} & 0\\ 0 & 1\end{vmatrix} n\theta mg\right] H_c(\theta mgm^{-1})dn.$$

We now write:

$$\theta mgm^{-1} = \begin{vmatrix}b & 0\\ 0 & 1\end{vmatrix} n'k, \quad |b| \geq c, \quad k \in K.$$

For the preceding expression, it follows:

$$\sum_\theta \int \sum_\alpha f\left[\begin{vmatrix}a^{-1}\alpha b & 0\\ 0 & 1\end{vmatrix} nkm\right] dn|b|.$$

If this expression is not zero then $|\alpha a^{-1}b| = |a^{-1}b|$ is in a compact set of $\mathbb{R}^\times$. The same is therefore true of $|b|$. This expression is thus zero if c is large enough. Hence the conclusion.

6.3. We are now reduced in all cases to considering the integral (6.1.2). We will prove its convergence. For this we write K as the sum of K_{reg} and K_{sin} with:

$$K_{reg}(x,y) = \sum f(x^{-1}\xi y), \quad \xi \quad \text{regular} \tag{6.3.1}$$

$$K_{sin}(x,y) = \sum f(x^{-1}\xi y), \quad \xi \quad \text{singular} \tag{6.3.2}$$

In this section we study the integral of K_{reg}. Every regular element can be written as:

$$\gamma = \begin{vmatrix}\alpha & 0\\ 0 & 1\end{vmatrix} n(\xi)\mu, \quad n(\xi) = \begin{vmatrix}1 & \xi\\ 0 & 1\end{vmatrix},$$

with

$$\alpha \in E^X, \quad \mu \in G_\varepsilon(F)/Z(F), \quad N(\xi) \neq 0, \quad \varepsilon.$$

Moreover α, μ and $N(\xi)$ are uniquely determined. It immediately follows that:

$$\int_{C^{-1}}^{C} \int K_{reg}\left[\begin{vmatrix}a & 0\\ 0 & 1\end{vmatrix}, h\right] \eta\omega(\det h)dh\Omega^{-1}(a)da$$

$$= \sum_{N(\xi)} \int_{C^{-1}}^{C} \int f\left[\begin{vmatrix}a & 0\\ 0 & 1\end{vmatrix} n(\xi)h\right] \eta\omega(\det h)dh\Omega(a)da \tag{6.3.3}$$

where in the right-hand side a is in E_Λ^X, h in $G_\varepsilon(F_\Lambda)/Z(F_\Lambda)$ and the sum is over the norm of ξ, assumed different from 0 and ε. The support of function f is contained in a compact set modulo the center; the image of this set by the application $P(g) = gg^{-1}$ associated to G_ε is thus contained in a union

$$\bigcup \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} M, \quad N(u) = 1,$$

where M is compact. If the integrand of the right-hand side of (6.3.3) is not zero then the image of the inner matrix by P is in the preceding set, which gives:

$$\begin{vmatrix} a^{-1}u & \xi u \\ -\varepsilon^{-1}\xi^\sigma a^{\sigma^{-1}} u & (1-\varepsilon^{-1}N(\xi))a^\sigma u \end{vmatrix} \in M,$$

for at least one u of norm 1. The element ξu is in a compact set of adèles. The same is therefore true of its norm $N(\xi)$ which thus takes only a finite number of values. According to the choice of ξ, the same applies to ξ. Therefore u is in a compact set of adèles. Since the quotient $N_1(F_\Lambda)/N_1(F)$ is compact, u is thus in fact in a compact set of $N_1(F_\Lambda)$, or equivalently a compact set of ideals; thus a^{-1} is in a compact set of adéles. Since $1-\varepsilon^{-1}N(\xi)$ takes only a finite number of values, a^σ is also in a compact set of adèles. It follows that a is in a compact set of ideals; this implies that h is itself in a compact set. This shows that the right-hand side of (6.3.3) converges and that its value is independent of C, provided that C is large enough. Moreover, this value is nothing other than the value of the analogous integral without restriction on a. A similar conclusion applies to the left-hand side and the equality is verified. We thus obtain:

$$\lim_{C\longrightarrow\infty} \int_{C^{-1}}^{C} \int K_{reg}\left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix}, h\right] \eta\omega(\det h) dh \Omega^{-1}(a) da$$
$$= \sum_{N(\xi)} \Omega(\xi) \iint f\left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} n(\xi) h\right] \eta\omega(\det h) dh \Omega(a) da \tag{6.3.4}$$

the factor $\Omega(\xi)$ being in fact equal to one.

Now we write the integrals of (6.3.4) as a product of local integrals. If v is a place of F inert in E and w the unique place of E above v, we set:

$$U_v(x) = \Omega_v(u) \iint f_w \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} h \Omega_w(a) \eta_v \omega_v(\det h) dh da \tag{6.3.5}$$

if $x = -\varepsilon^{-1}N(u)(1-\varepsilon^{-1}N(u))^{-1}$ for at least one u, $= 0$ otherwise.

Then the local factor at place v in expression (6.3.4) is nothing other than:

$$U_v(\mu), \quad \text{where} \quad \mu = -\varepsilon^{-1}N(\xi)(1-\varepsilon^{-1}N(\xi))^{-1}. \tag{6.3.6}$$

We now consider a place v that decomposes into v_1 and v_2. We thus have isomorphisms:

$$E_{v_1} \longrightarrow F_v, \quad E_{v_2} \longrightarrow F_v$$

and we denote by ξ_1 and ξ_2 the images of ξ in F_v. The group $G_{\varepsilon v}$ is isomorphic to the group of pairs (h_1, h_2) such that

$$h_2 = \begin{vmatrix} 0 & \varepsilon \\ 1 & 0 \end{vmatrix} h_1 \begin{vmatrix} 0 & \varepsilon \\ 1 & 0 \end{vmatrix}^{-1}, \quad h_1 \in GL(2, F_v).$$

Then the local factor at place v in expression (6.3.4) is written:

$$\int f_{v_1} \left[\begin{vmatrix} a_1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & \xi_1 \\ 0 & 1 \end{vmatrix} h_1 \right] \int f_{v_2} \left[\begin{vmatrix} a_2 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & \xi_2 \\ 0 & 1 \end{vmatrix} h_2 \right] \Omega_{v_1}(a_1)\Omega_{v_2}(a_2)\omega_v(\det h_1)$$

$$da_1 da_2 dh_1 \Omega_{v_1}(\xi_1)\Omega_{v_2}(\xi_2).$$

We now introduce the function f_v on $GL(2, F_v)$ defined by:

$$f_v(g) = \int f_{v_1}(gh_1) f_{v_2}(h_2)\omega_v(\det h_1) dh_1. \tag{6.3.7}$$

Then f_v transforms according to the inverse of ω_v under the center. We define a function U_v on the multiplicative group of F_v by the formula:

$$f_v(g) = \int f_{v_1}(gh_1) f_{v_2}(h_2)\omega_v(\det h_1) dh_1. \tag{6.3.8}$$

Then f_v transforms according to the inverse of ω_v under the center. We define a function U_v on the multiplicative group of F_v by the formula:

$$U_v(x) = \Omega_{v_1}(x) \iint f_v \left[\begin{vmatrix} a_1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & x \\ 1 & 1 \end{vmatrix} \begin{vmatrix} a_2 & 0 \\ 0 & 1 \end{vmatrix} \right] \Omega_{v_1}(a_1)\Omega_{v_2}(a_2) da_1 da_2 \tag{6.3.9}$$

if $x \neq 1$, $= 0$ if $x = 1$. One can easily see that this integral converges. After a change of variables, the factor corresponding to place v in integral (6.3.4) becomes:

$$\Omega_{v_2}(-\varepsilon)U_v(\mu), \quad \mu = -\varepsilon^{-1}\xi_1\xi_2(1 - \varepsilon^{-1}\xi_1\xi_2)^{-1}. \tag{6.3.10}$$

In total, we see that the integral of K_{reg} can be written:

$$\iint K_{reg} \cdots = \sum_{\mu} \prod_{v} U_v(\mu) \prod \Omega_{v_2}(-\varepsilon); \tag{6.3.11}$$

the second product is over the set of places v decomposed in E; for such a place we choose, once and for all, a numbering v_1, v_2 of the places of E above v; the sum is over all elements μ of F^X of the form

$$-\varepsilon^{-1}N(\xi)(1 - \varepsilon^{-1}N(\xi))^{-1}, \quad \text{with} \quad N(\xi) \neq \varepsilon.$$

If μ is not of this form there exists at least one place v inert in E such that μ is not of the form

$$-\varepsilon^{-1}N(u)(1 - \varepsilon^{-1}N(u))^{-1} \quad \text{with} \quad u \in F_v^{\times}.$$

Then $U_v(\mu) = 0$. We can therefore regard the sum in (6.3.11) as being over all $\mu \neq 0$ in F.

6.4. We move on to the singular terms. If $\varepsilon = 1$, then there are four double singular classes, those of n_1, $n_1 r$, e, r, where we have set:

$$r = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}, \quad n_1 = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \tag{6.4.1}$$

We denote by K_i, $1 \le i \le 4$, the corresponding kernels. According to lemma 2.6 they are given by:

$$\begin{aligned}
K_1 &= \sum f \left[x^{-1} \begin{vmatrix} \alpha & 0 \\ 0 & 1 \end{vmatrix} n_1 \gamma y \right], \quad \alpha \in E^\times, \quad \gamma \in G_1(F)/Z(F), \\
K_2 &= \sum f \left[x^{-1} \begin{vmatrix} \alpha & 0 \\ 0 & 1 \end{vmatrix} n_1 r \gamma y \right], \quad \alpha \in E^\times, \quad \gamma \in G_1(F)/Z(F), \\
K_3 &= \sum f \left[x^{-1} \begin{vmatrix} \alpha & 0 \\ 0 & 1 \end{vmatrix} \gamma y \right], \quad \alpha \in E^\times/N_1(F), \quad \gamma \in G_1(F)/Z(F), \\
K_4 &= \sum f \left[x^{-1} \begin{vmatrix} \alpha & 0 \\ 0 & 1 \end{vmatrix} r \gamma y \right], \quad \alpha \in E^\times/F^\times, \quad \gamma \in G_1(F)/Z(F).
\end{aligned} \tag{6.4.2}$$

If ε is not a norm then the only singular class is that of e and K_{sin} reduces to K_3.

6.5. We study the integral of K_3. We obtain:

$$\iint f \left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} \right] h \, \Omega(a) \eta\omega \, (\det h) da \, dh, \tag{6.5.1}$$

where a is integrated over the set of elements of $E_A^\times/N_1(F)$ satisfying the inequality $C^{-1} < |a| < C$. As in (6.3) there exists a compact set M of $\mathrm{GL}(2, E_A)$ such that if the integrand is not zero then

$$uP \begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} \in M$$

for a u of norm 1. This gives

$$\begin{vmatrix} au & 0 \\ 0 & a^{-\sigma} u \end{vmatrix} \in M.$$

It follows that the norm of a is in a compact set and that a itself is in a compact subset modulo $N_1(F_A)$. Since $N_1(F_A)/N_1(F)$ is compact, we see that a is actually in a compact set. It follows that the integral converges and its value is independent of C, provided that C is large enough. This value is moreover that of the integral obtained by integrating over the whole quotient $E_A^\times/N_1(F)$.

We now integrate over $N_1(F_A)/N_1(F)$ (variable b) then over $E_A^\times/N_1(F_A)$. We get:

$$\int f \left[\begin{vmatrix} ab & 0 \\ 0 & 1 \end{vmatrix} \right] h \, \Omega(a)\Omega(b) \eta\omega \, (\det h) \, dh \, da \, db.$$

If we write $b = u^{1-\sigma}$ this becomes:

$$\iint f\left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix}\begin{vmatrix} u & 0 \\ 0 & u^\sigma \end{vmatrix}\right] h\ \Omega(a)\Omega(b)\omega(N(u))\eta\omega(\det h)\ dh.$$

After a translation on h we obtain as a factor the integral of Ω over the quotient $N_1(F_A)/N_1(F)$. This factor is therefore zero, unless Ω is trivial on this subgroup, that is, the lifting of a character λ of F. If this is the case, the integral becomes:

$$\mathrm{vol}(N_1(F_A)/N_1(F)) \iint f\left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} h\right] \Omega(a)\eta\omega(\det h)\ dh\ da,$$

$$a \in E_A^\times/N_1(F_A), \quad h \in G_1(F_A)/Z(F_A). \qquad (6.5.2)$$

We now relate this integral to the functions U_v and f_v (Cf. (6.3.5), (6.3.8), (6.3.9)). Let v be a place of F inert in E and w the corresponding place of E. Then we know (Prop. 3.4) that

$$\lim_{x\longrightarrow 0, x\in -\varepsilon N_{0v}} U_v\lambda_v^{-1}(x) = \iint f_w \left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} h\right] \Omega_w(a)\eta_v\omega_v(\det h)\ da\ dh\ \lambda(-\varepsilon). \qquad (6.5.3)$$

In the integral of (6.5.2) the factor corresponding to the place v has the same form as above except that the integral is over the quotient $E_v^\times/N_{1v}$; this factor is therefore

$$\lim_{x\longrightarrow 0, x\in -\varepsilon N_{0v}} U_v\lambda_v^{-1}(x)\ \mathrm{vol}(N_{1v})^{-1}\ \lambda(-\varepsilon). \qquad (6.5.4)$$

If v decomposes into v_1 and v_2 then the local factor corresponding to v in (6.5.2) is nothing but:

$$\iint f_{v_1}\left[\begin{vmatrix} a_1 & 0 \\ 0 & 1 \end{vmatrix} h_1\right] f_{v_2}\left[\begin{vmatrix} a_2 & 0 \\ 0 & 1 \end{vmatrix} h_2\right] \Omega_{v_1}(a_1)\Omega_{v_2}(a_2)\omega_v(\det h_1)\ dh_1, \qquad (6.5.5)$$

the integral being over the group of pairs (a_1, a_2), modulo the subgroup of pairs of the form (b, b^{-1}). After a change of variables this can be written:

$$\int f_v(a)\lambda_v(a)\ da, \quad a \in A_v/Z_v.$$

where f_v is the function (6.3.8). In summary:

PROPOSITION. *We have:*

$$\lim_{C\longrightarrow\infty} \int_{C^{-1}}^{C} \int K_3 = \iint K_3.$$

These integrals are zero unless Ω is of the form $\lambda \circ N$. If this is the case, the value of this expression is the product of the following quantities:

(i) vol($E_A^\times/N_1(F_A)$);

(ii) for each inert place v:

$$\lim_{x\longrightarrow 0, x\in -\varepsilon N_{0v}} U_v\lambda_v^{-1}(x)\ vol(N_{1v})^{-1}\ \lambda(-\varepsilon);$$

(iii) for each decomposed place v:

$$\int f_v(a)\lambda_v(a)\, da, \quad a \in A_v/Z_v.$$

6.6. We now study the integral of K_4. According to lemma (2.6) it can be written as

$$\iint f\left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} rh\right] \Omega(a)\omega(\det h)\, da\, dh,$$

$$a \in E_A^\times/F^\times, \quad C^{-1} < |a| < C, \quad h \in G_1(F_A)/Z(F_A).$$

As above there exists a compact set M such that if the integrand is not zero then

$$uP\left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} rh\right] \in M,$$

where u is of norm 1. This can also be written:

$$\begin{vmatrix} 0 & a^{1-\sigma}u \\ -u & 0 \end{vmatrix} \in M.$$

It follows first that u is in a compact set, then that $a^{1-\sigma}$ is in a compact set. Thus a is of the form btc where b is in a compact set, c in the group of ideles of F of norm 1 and t in a subgroup of the ideles of F isomorphic to the group of real numbers > 0. Since the absolute value of a is in a compact set, the same goes for t. Thus a is in a compact set and the integral converges. According to lemma (2.6) we have:

$$r^{-1}\begin{vmatrix} c & 0 \\ 0 & 1 \end{vmatrix} r \in G_1;$$

after a change of variables we see that the integral admits:

$$\int \Omega(c)\eta\omega^{-1}(c)\, dc = \int \eta(c)\, dc$$

as a factor. Since this factor is zero, we conclude that the integral is zero. Hence:

PROPOSITION.

$$\lim_{c \longrightarrow \infty} \int_{C^{-1}} K_4 = 0.$$

6.7. We now study the integrals of K_1 and K_2. We recall that these terms are zero unless ε is a norm, thus $\varepsilon = 1$ with the conventions made. They can be written:

$$\int_{C^{-1}}^{C} \int f\left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} n_1 h\right] \Omega(a)\, \omega\eta(\det h)\, dh, \tag{6.7.1}$$

$$\int_{C^{-1}}^{C} \int f\left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} n_1 rh\right] \Omega(a)\omega\eta(\det h)\, dh. \tag{6.7.2}$$

We study the first integral as an example. We will see that it is convergent and has a limit when C tends to infinity, equal to the value at $s = 0$ of the analytic continuation of the integral:

$$\iint f \left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} n_1 h \right] |a|^{-s} \Omega(a) \omega\eta(\det h)\, dh, \tag{6.7.3}$$

integral which itself converges for $\mathrm{Re}s > 1$. For this purpose we introduce a character μ of the group of idele classes of E whose restriction to F is η and we set:

$$f_1(g) = f(g)\ \mu\Omega(\det g).$$

Then the above integral can be written:

$$\iint f_1 \left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} n_1 h \right] |a|^{-s} \mu^{-1}(a)\, da\, dh. \tag{6.7.4}$$

We introduce the function Φ defined by

$$\Phi(a^{-1}) = \int f_1 \left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} n_1 h \right]\ dh. \tag{6.7.5}$$

Then the integral (6.7.4) can be written, after a change of variables:

$$\int \Phi(a) |a|^s \mu(a)\, da. \tag{6.7.6}$$

We write the function Φ as a product over all places of F. Let v be an inert place of F and w the unique place of E above v. Since P is submersive there exists a function f_{0w} on $G(E_w)$, smooth, with compact support, such that:

$$f_{0w}(P(g)) = \int f_{1w}(gh)\, dh. \tag{6.7.7}$$

On the other hand:

$$P \left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} n_1 \right] = \begin{vmatrix} 0 & a^{1-\sigma} \\ -1 & a^{-\sigma} \end{vmatrix}. \tag{6.7.8}$$

It follows that the local factor for the function Φ corresponding to the place v can be written:

$$\Phi_v(a) = f_{0w} \left[\begin{vmatrix} 0 & a^{\sigma-1} \\ -1 & a^{\sigma} \end{vmatrix} \right]. \tag{6.7.9}$$

If we write $a = tb$ with t in F_v and b non-zero, we obtain a Schwartz-Bruhat function of t depending on b. Moreover, for all w not in T, the function f_{0w} is the characteristic function of K_w (and Φ_w the characteristic function of R_w), as follows from the following lemma:

LEMMA 6.7.10. *Suppose v finite, inert and unramified in E. Let x be in K_w such that $xx^t = 1$. Then $x = P(h)$ with h in K_w.*

Proof. It suffices to show that $y = t + xt'$ is in K_w for at least one scalar t which is a unit of E; indeed we have $xy' = y$ from which $x = p(y)$ if y is invertible. Since the matrix y has integer coefficients it suffices to show that it is invertible mod P. We are thus reduced to an analogous problem for a finite field e and a quadratic extension f: given an invertible 2×2 matrix x, there exists a scalar t such that the matrix y is invertible. If this were not the case, the matrix x would admit all numbers $-t^{1-\sigma}$ as eigenvalues. Since there are $q+1 > 2$ such numbers, we would obtain a contradiction. Hence the conclusion. □

We now consider a place v of F which decomposes into v_1 and v_2. We set:

$$f_{0w}(g) = \int f_{v_1}(gh_1) f_{v_2}(h_2) dh_1 \quad (= \Omega_{v_1}\mu_{v_1}(\det g) f_v(g)). \tag{6.7.11}$$

This is a smooth function, with compact support modulo the center, characteristic function of $Z_v K_v$ for almost all v. Then the factor corresponding to place v for the function Φ can be written:

$$\Phi_v(a_1, a_2) = f_{0w}\left[\begin{vmatrix} 0 & a_1^{-1}a_2 \\ -1 & a_2 \end{vmatrix}\right] \tag{6.7.12}$$

Thus Φ is the product of expressions (6.7.9) and (6.7.12). In particular if $\Phi(a)$ is not zero then $a^{1-\sigma}$ is in a compact set of ideles; we can then write $a = mb$, where m is in a fixed compact set and b in the ideles of F. Moreover a is in a compact set of adeles of E. We can therefore regard the function Φ as a Schwartz-Bruhat function of b, depending on the parameter m. It follows that the integral (6.7.6) converges when the real part of s is large enough. Similarly the integral from which we started can be written:

$$\int \Phi(a)\mu(a)\, da, \quad C^{-1} < |a| < C. \tag{6.7.13}$$

It therefore converges. We recall moreover the following result:

LEMMA 6.7.14. *let ϕ be a Schwartz-Bruhat function. Then the following limit exists:*

$$\lim_{C \longrightarrow \infty} \int_{C^{-1}}^{C} \phi(b)\eta(b)\, db.$$

It is equal to the analytic continuation at point 0 *of the integral*

$$\int \phi(b)|b|^s \eta(b)\, db.$$

Finally this limit is equal to the product of the following factors:

(i) $L(0, \eta)$*;*

(ii) for each inert place v:

$$L(0, \eta_v)^{-1} \int \phi_v(b)\eta_v(b)\, db;$$

(iii) for each place v decomposed in E:

$$\lim_{s \longrightarrow 0} L(s, 1_v)^{-1} \int \phi_v(b)|b|^s\, db = |a_v|^{1/2}\phi_v(0);$$

where a denotes a differential idele of F. Almost all factors are equal to 1.

We apply this lemma, or rather a variant of this lemma with parameters, to the integral (6.7.13). We first obtain that the integral (6.7.1), that is the integral of K_1, has a limit when C tends to infinity, equal to the value of the integral (6.7.4) at point 0. Moreover we can calculate this limit as the product of the following factors, almost all equal to 1:

(i) $L(0, \eta)$;

(ii) for each inert place v below a place w of E:

$$L(0, \eta_v)^{-1} \int f_{0w} \left[\begin{vmatrix} 0 & a^{\sigma-1} \\ -1 & a^{\sigma} \end{vmatrix}\right] \mu_w(a)\, da;$$

(iii) for each place v decomposed in E:

$$|a_v|^{1/2} \int f_{0w} \left[\begin{vmatrix} 0 & a \\ -1 & 0 \end{vmatrix}\right] \mu_{v_1}(a)^{-1}\, da. \tag{6.7.15}$$

Similarly using the fact that

$$P\left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} n_1 r\right] = \begin{vmatrix} -2a & a^{1-\sigma} \\ -1 & 0 \end{vmatrix}$$

we find that the integral of K_2 has a limit when C tends to infinity, equal to the product of the following factors:

(i) $L(0, \eta)$;

(ii) for each inert place v below a place w of E:

$$L(0, \eta_v)^{-1} \int f_{0w} \left[\begin{vmatrix} a & a^{\sigma-1} \\ -1 & 0 \end{vmatrix}\right] \mu_w(a)\, da;$$

(iii) for each place v decomposed in E:

$$|a_v|^{1/2} \int f_{0w} \left[\begin{vmatrix} 0 & a \\ -1 & 0 \end{vmatrix}\right] \mu_{v_1}(a)^{-1}\, da. \tag{6.7.16}$$

We can also formulate these results in terms of the functions U_v for v inert and the functions f_v for v decomposed. We know that

$$U_v(x) = M_{1v}(x^{-1}) + \eta_v(-x) M_{2v}(x^{-1}), \tag{6.7.17}$$

where the functions M_{iw} are smooth near 0 and we have (Prop 3.5):

$$M_{1v}(0) = \int f_{0w} \left[\begin{vmatrix} 0 & a^{\sigma-1} \\ -1 & a^{\sigma} \end{vmatrix}\right] \mu_w(a)\, da,$$

$$M_{2v}(0) = \int f_{0w} \left[\begin{vmatrix} a & a^{\sigma-1} \\ -1 & 0 \end{vmatrix}\right] \mu_w(a)\, da.$$

Similarly the integral (6.7.15) (*iii*) above is nothing but

$$|a_v|^{1/2}\int f_v\left[\begin{vmatrix}0 & a\\ -1 & 0\end{vmatrix}\right]\Omega_{v_1}(a)\,da,$$

and an analogous remark applies to (6.7.16) (*iii*).

6.8. We can summarize the preceding discussion as follows: for each inert place v we have the orbital integral function U_v defined by (6.3.5) and the numbers $M_{iw}(0)$ defined by (6.3.8); for each decomposed place v we have the function f_v on the group $\mathrm{GL}(2,F_v)$ defined by (6.3.8), as well as the function U_v defined in terms of f_v by (6.3.9).

PROPOSITION. *The integral*

$$\int_{C^{-1}}^{C}\int K\left(resp \int_{C^{-1}}^{C}\int T_2^*K \;\; if\ \varepsilon=1\right)$$

has a limit when C tends to infinity (resp. when c tends to infinity and then C tends to infinity). The limit is equal to the sum of the following terms:

(1) $\sum_{\mu\neq 0}\Pi_v U_v(\mu)\Pi\Omega_{v_2}(-\varepsilon)$;

(2) a term present only if Ω is the lift of a character λ of $F^\times$; it is the product of the following factors:

(i) $vol(N_{1A}/N_{1F})$;

(ii) for each inert place v of F:

$$\lim_{x\longrightarrow 0, x\in -\varepsilon N_{0v}} U_v\lambda_v^{-1}(x)vol(N_{1v})^{-1}\lambda(-\varepsilon);$$

(iii) for each decomposed place v:

$$\int f_v(a)\lambda_v(a)da, \quad a\in A(F_v)/Z(F_v);$$

(3) two terms present only if $\varepsilon=1$; each term is the product of the following factors where index i takes values 1 or 2 depending on the term:

(i) $L(0,\eta)$;

(ii) for each inert place v:

$$M_{iv}(0)L(0,\eta_v)^{-1};$$

(iii) for each decomposed place v:

$$|a_v|^{1/2}\int f_v\begin{vmatrix}0 & a\\ -1 & 0\end{vmatrix}\Omega_{v_1}(a)da.$$

7. Comparison

7.1. Now we fix an element ε_0 of $F^\times$. If ε_0 is a norm we assume $\varepsilon_0 = 1$. We write G_0 for the group G_ε with $\varepsilon = \varepsilon_0$. We consider a finite set S of places of F containing all places at infinity, all places that ramify in E, and finally all finite places where ε_0 is not a unit. We denote by T the set of places of E which are above a place of E contained in S. We furthermore assume that S is large enough so that ω is unramified outside S and Ω is unramified outside T. As in §6, we consider a smooth function f on $G(E_A)$, transforming by the inverse of ω' under the center and with compact support modulo the center; the function f is a product of local functions f_w. For w not in T we assume that f_w is K_w-invariant. From the function f, we define for every inert place v a function U_v on $F_v^\times$ and for every decomposed place v a function f_v on G_v as well as a function U_v; to do this we choose if v is decomposed a numbering of the two places of E above v (6.3).

We choose on the other hand a system of representatives of the norm classes of the group N_0 in the multiplicative group of F. We assume that $-\varepsilon_0^{-1}$ is in this system. For each ε in this system we choose a function f_ε on the adelic group of G_ε, transforming by the inverse of the character ω under the center and with compact support modulo the center. The function f_ε is a product of local functions $f_{\varepsilon v}$. If v decomposes into v_1 and v_2 then $G_{\varepsilon v}$ is the group of pairs of matrices of $G_v = GL(2, F_v)$ of the form:

$$(h_1, h_2) \text{ with } h_2 = \begin{vmatrix} 0 & \varepsilon \\ 1 & 0 \end{vmatrix} h_1 \begin{vmatrix} 0 & \varepsilon \\ 1 & 0 \end{vmatrix}^{-1}. \tag{7.1.1}$$

By transporting f_v to $G_{\varepsilon v}$ by the isomorphism of $G_{\varepsilon v}$ onto G_v which sends the pair (h_1, h_2) to h_1 we can regard f_v as a function on $G_{\varepsilon v}$. We will take:

$$f_{\varepsilon v} = f_v \Omega_{v_2}(-\varepsilon_0)\Omega_{v_1}(\varepsilon). \tag{7.1.2}$$

If v is an inert place and w the unique place of E above v then we consider the orbital integral function $H_{\varepsilon w}$ defined by $f_{\varepsilon w}$:

$$H_{\varepsilon w}(x) = \Omega_w(u) \iint f_{\varepsilon w} \begin{vmatrix} 1 & u\varepsilon \\ u\sigma & 1 \end{vmatrix} \Omega_w(t_1)\Omega_w^\sigma(t_2) dt_1 dt_2, \tag{7.1.3}$$

if x is of the form $\varepsilon N(u)$ for at least one u,

$$H_{\varepsilon w}(x) = 0 \text{ otherwise.}$$

We choose $f_{\varepsilon w}$ such that

$$H_{\varepsilon w}(x) = U_v(x) \text{ if } x \in \varepsilon N_{0v}. \tag{7.1.4}$$

This is possible according to the results of §3. In particular if v is inert but not in S then the function f_w is K_w-invariant. It follows that $U_v(x) = 0$ if the valuation of x is odd (lemma (4.7.2)). According to the choice of $f_{\varepsilon w}$, if ε is not a norm at place v, that is if $v(\varepsilon)$ is odd, then $H_{\varepsilon w} = 0$. We will in fact take $f_{\varepsilon w} = 0$. Let f_v be on the other hand the K_v-invariant function which is the image of f_w by the homomorphism (4.1.2); then f_v

transforms by the inverse of the character ω_v under the center. If ε is a norm then we choose an isomorphism of $GL(2, F_v)$ onto $G_{\varepsilon w}$ and we take for $f_{\varepsilon w}$ the image of f_v by this isomorphism, as it is possible to do according to Prop. (4.1). We denote by $K_{\varepsilon w}$ the image of K_w by this isomorphism; it is thus a maximal compact subgroup of $G_{\varepsilon w}$ and the function $f_{\varepsilon w}$ is invariant under this group. If moreover ε is a unit at place v then we choose the isomorphism such that $K_{\varepsilon w}$ is the intersection of K_w with $G_{\varepsilon w}$ ("privileged isomorphism").

We note that $f_\varepsilon = 0$ if ε is not a norm in at least one place of F which is not in S. We can therefore ignore such ε. The remaining ε, that is those which are norms at all places not in S, form a finite set.

7.2. Let τ be a cuspidal representation of $G_\varepsilon(F_A)$ of central character ω; it is thus a representation of infinite dimension, by definition if ε is not a norm. Let ϕ_i be an orthonormal basis of the representation space τ. We set:

$$K_\tau(x, y) = \sum \varrho(f_\varepsilon)\phi_i(x)\phi_i(y)^*.$$

LEMMA 7.2.1. *The series*

$$\sum K_\tau(x, y)$$

converges uniformly on any compact. Moreover if ε is a norm then the series converges uniformly.

Proof. The series converges in any case in the space L^2. On the other hand we know that f_ε is a finite sum of convolution products $f_1 * f' * f_2$. By linearity we can assume that f_ε is such a product. Denoting by K'_τ the kernel associated to f' we have:

$$K_\tau(x, y) = \int K'_\tau(h_1.x, h_2.y) f_1(h_1) f_2(h_2^{-1}) dh_1 dh_2.$$

In other words, the series associated to f_ε is obtained by applying to the series associated to f' convolution operators with smooth functions of compact support. In general these operators send the space L^2 into the space of smooth functions; if moreover ε is a norm then these operators send the space of L^2 and cuspidal functions into the space of bounded continuous functions. The result follows. □

7.2.2. We will set:

(dihedral kernel)

$$K_{\varepsilon,\text{die}} = \sum K_\tau,\ \tau \text{ dihedral};$$

(proper kernel)

$$K_{\varepsilon,\text{pro}} = \sum K_\tau,\ \tau \text{ cuspidal non-dihedral};$$

(special kernel)

$$K_{\varepsilon,\text{spe}} = \sum K_\tau,\ \dim \tau = 1;$$

(geometric kernel)

$$K_\varepsilon(x, y) = \sum f_\varepsilon(x^{-1}\xi y),\ \text{all } \xi;$$

(regular kernel)

$$K_{\varepsilon,\text{reg}}(x,y)=\sum f_\varepsilon(x^{-1}\xi y),\ \xi \text{ regular};$$

(singular kernel)

$$K_{\varepsilon,\sin}(x,y)=\sum f_\varepsilon(x^{-1}\xi y),\ \xi \text{ singular}.$$

Finally, we will denote by $K_{1,\text{eis}}$ the Eisenstein kernel associated to the group G_ε and the function f_ε, where ε is the unique element that is a norm in E. We thus have:

$$K_{\varepsilon,\text{pro}}=K_\varepsilon-K_{\varepsilon,\text{spe}}-K_{\varepsilon,\text{die}}-K_{1,\text{eis}}, \tag{7.2.3}$$

the last term being present only if ε is a norm.

We have on the other hand the kernel K_{cusp} associated to function f. The main result of this work can then be stated as follows:

THEOREM 7.2. *With the above notations we have:*

$$\sum_\varepsilon \iint K_{\varepsilon pro}(t_1,t_2)\Omega^{-1}(t_1)\Omega(t_2)dt_1dt_2$$

$$=\iint K_{cusp}\begin{bmatrix}a & 0\\ 0 & 1\end{bmatrix}, h\Omega^{-1}(a)\eta(\det h)dadh,$$

$$t_i\in T(F_\lambda)/T(F)Z(F_\lambda),\quad a\in E_\lambda^\times/E^\times,\quad h\in G_0(F)Z(F_\lambda)\backslash G_0(F_\lambda)$$

7.3. We will calculate the left-hand side. For this, we integrate each term in the right-hand side of (7.2.3) and sum over ε. We begin with the regular kernel. We have:

$$K_{\varepsilon\text{reg}}(x,y)=\sum_{N(\xi)}\sum_{\tau_1,\tau_2} f_\varepsilon\left[x^{-1}\tau_1\begin{vmatrix}1 & \xi\varepsilon\\ \xi^\sigma & 1\end{vmatrix}\tau_2 y\right];$$

in the sum, each τ_i runs through $T(F)/Z(F)$ and $N(\xi)$ runs through the norm group deprived of point 1 if ε is a norm. It follows that the integral of the regular kernel is written:

$$\iint K_{\text{reg}}=\sum \Omega(\xi)\int f_\varepsilon\left[t_1\begin{vmatrix}1 & \xi\varepsilon\\ \xi^\sigma & 1\end{vmatrix}t_2\right]\Omega(t_1)\Omega'(t_2)dt_1dt_2,$$

the factor $\Omega(\xi)$ being in fact equal to 1. This integral decomposes into a product of local integrals. The factor corresponding to an inert place v of F in E is by definition the orbital integral of $f_{\varepsilon v}$ evaluated at the point $N(\xi)\varepsilon$. By the very choice of $f_{\varepsilon v}$, this factor equals $U_v(N(\xi)\varepsilon)$. If on the contrary v is a place that decomposes into v_1 and v_2, then the corresponding local factor is written:

$$\Omega_{v_1}(\xi_1)\Omega_{v_2}(\xi_2)\Omega_{v_2}(-\varepsilon_0)\Omega_{v_1}(\varepsilon)$$

$$\int f_v\left[\begin{vmatrix}a_1 & 0\\ 0 & a_2\end{vmatrix}\begin{vmatrix}1 & \xi_1\varepsilon\\ \xi_2 & 1\end{vmatrix}\begin{vmatrix}b_1 & 0\\ 0 & b_2\end{vmatrix}\right]$$

$$\Omega_{v_1}(a_1)\Omega_{v_2}(a_2)\Omega_{v_1}(b_2)\Omega_{v_2}(b_1)dadb.$$

Since the restriction of Ω to F is ω, the product $\Omega_{v_1}\Omega_{v_2}$ equals ω_v. After a change of variables, this integral is written:

$$\Omega_{v_2}(-\varepsilon_0)\Omega_{v_1}(\varepsilon\xi_1,\xi_2)\int f_v\left[\begin{vmatrix}a & 0\\0 & 1\end{vmatrix}\begin{vmatrix}1 & \xi_1\xi_2\varepsilon\\1 & 1\end{vmatrix}\begin{vmatrix}b & 0\\0 & 1\end{vmatrix}\right]\Omega_{v_1}(a)\Omega_{v_2}(b)dadb.$$

This is nothing other than

$$\Omega_{v_2}(-\varepsilon_0)U_v(\varepsilon\xi_1,\xi_2).$$

Writing μ for $\varepsilon N(\xi)$, we see that the integral of the kernel $K_{\varepsilon\mathrm{reg}}$ is written

$$\sum_\mu\prod_v U_v(\mu)\prod_v\Omega_{v_2}(-\varepsilon_0),$$

where the sum is over all elements of εN_0 different from 0; the first product is over all places v of F and the second over those that decompose in E. Summing over all ε, we find:

$$\sum_\varepsilon\iint K_{\varepsilon\mathrm{reg}}=\sum_\mu\prod_v U_v(\mu)\prod_v\Omega_{v_2}(-\varepsilon_0),\tag{7.3.1}$$

the sum being over all elements μ of F except point 1. This is nothing other than term (1) in proposition (6.8). We have thus proved the following proposition:

PROPOSITION 7.3. *We have*

$$\iint K_{reg}=\sum_\varepsilon\iint K_{\varepsilon reg}.$$

7.4. We now move on to the integral of the singular kernel. According to the results of §2:

$$K_{\varepsilon\sin}(x,y)=\sum_\tau f_\varepsilon(x^{-1}\tau y)+\sum_\tau f_\varepsilon\left(x^{-1}\tau\begin{vmatrix}0 & \varepsilon\\1 & 0\end{vmatrix}y\right),$$

where the sum is over all τ in $T(F)/Z(F)$. Therefore we find:

$$\iint K_{\varepsilon\sin}=\iint f_\varepsilon(t_1t_2)\Omega(t_1)\Omega'(t_2)dt_1dt_2+\iint f_\varepsilon\left[t_1\begin{vmatrix}0 & \varepsilon\\1 & 0\end{vmatrix}t_2\right]\Omega(t_1)\Omega'(t_2)dt_1dt_2,$$

where t_1 is integrated over $T(F_\lambda)/Z(F_\lambda)$ and t_2 over $T(F_\lambda)/T(F)Z(F_\lambda)$. Since Ω' is in fact the transform of Ω by the Galois automorphism σ, the second integral is written:

$$\mathrm{vol}(T(F_\lambda)/Z(F_\lambda)T(F))\int f_\varepsilon\left[t\begin{vmatrix}0 & \varepsilon\\1 & 0\end{vmatrix}\right]\Omega(t)dt.\tag{7.4.1}$$

Similarly in the first integral, the integral of the character $\Omega^{1-\sigma}$ is a factor; the integral is therefore zero unless Ω is invariant by σ, that is unless it is the lift of a character λ. The integral is then written

$$\mathrm{vol}(T(F_\lambda)/Z(F_\lambda)T(F))\int f_\varepsilon[t]\lambda(\det t)dt.\tag{7.4.2}$$

We will calculate these integrals in terms of the local data f_v and U_v. The integral (7.4.2) is a product of local integrals. The factor corresponding to an inert place v is written:

$$\int f_{\varepsilon v}[t]\lambda_v(\det t)dt.$$

According to proposition (3.1) this is nothing other than the limit of

$$\mathrm{vol}(T_v/Z_v)^{-1}H_{\varepsilon v}(x)\lambda_v(x)^{-1}$$

for x tending to 0 in εN_{0v}; by the very choice of $f_{\varepsilon v}$, this is the limit of

$$\mathrm{vol}(T_v/Z_v)^{-1}U_v(x)\lambda_v(x)^{-1}$$

for x tending to 0 in εN_{0v}. However if x is small enough $U_v(x) = 0$, unless x is in $-\varepsilon_0 N_{0v}$ (Prop. (3.4)). It follows that (7.4.2) is 0 unless ε is in $-\varepsilon_0^{-1}N_{0v}$ for all inert places v. This condition means in fact that ε is in $-\varepsilon_0^{-1}N_0$; there is thus only one such ε, namely ε_0^{-1}. On the other hand, the factor corresponding to a place v decomposed into v_1 and v_2 is written:

$$\Omega_{v_2}(-\varepsilon_0)\Omega_{v_1}(\varepsilon)\int f_v\left[\begin{vmatrix} a_1 & 0 \\ 0 & a_2 \end{vmatrix}\right]\Omega_{v_1}(a_1)\Omega_{v_2}(a_2)da_1da_2,$$

the integral being taken modulo the center. Since:

$$\varepsilon\varepsilon_0 = -1 \text{ and } \mu_{v_1} = \mu_{v_2} = \lambda_v$$

this is nothing other than:

$$\int f_v(a)\lambda_v(\det a)da, \quad a \in A_v/Z_v.$$

In total, the sum over all ε of the integrals (7.4.2) is zero unless Ω is the lift to E of a character λ of F. If this is the case, this sum reduces to one term; in turn this term is written as the product of the following factors:

(i) $\mathrm{vol}(E_\lambda^\times/F_\lambda^\times E^\times)$;

(ii) for each inert place v:

$$\mathrm{vol}(E_v^\times/F_v^\times)^{-1}\lim U_v\lambda_v^{-1}(x)\lambda(-\varepsilon_0), x \longrightarrow 0, x \in -\varepsilon_0 N_{0v};$$

(iii) for each place v decomposed in E:

$$\int f_v(a)\lambda_v(\det a)da, \quad a \in A_v/Z_v. \tag{7.4.3}$$

We now compare this result with expression (2) in proposition (6.8). Taking into account the exact sequence:

$$1 \longrightarrow F^\times \longrightarrow E^\times \longrightarrow N_1 \longrightarrow 1$$

where the second arrow is

$$x \longrightarrow x^{1-\sigma},$$

we see that the volumes are identical in the two expressions. The two expressions are therefore themselves identical.

7.5. We now move on to (7.4.1). The factor corresponding to an inert place v is written:

$$\int f_{\varepsilon v}\left[\begin{vmatrix} a & 0 \\ 0 & a^\sigma \end{vmatrix}\begin{vmatrix} 0 & \varepsilon \\ 1 & 1 \end{vmatrix}\right]\Omega_v(a)da.$$

As we have seen (Prop. (3.1)) this is nothing other than the limit of

$$\mathrm{vol}(T_v/Z_v)^{-1}H_{\varepsilon v}(x)$$

for x tending to infinity in εN_{0v}. Still according to the choice of function $f_{\varepsilon v}$ this is also the limit of

$$\mathrm{vol}(T_v/Z_v)^{-1}U_v(x),$$

when x tends to infinity in εN_{0v}. Now we know that for x large enough, $U_v(x)$ is zero unless ε_0 is a norm (Prop. (3.4)). We conclude that (7.4.1) is zero unless ε_0 is a norm at each inert place v, that is unless ε_0 is a norm; with the conventions made this implies $\varepsilon_0 = 1$. Under this hypothesis we know (loc. cit.) that the function U_v has the form:

$$U_v(x) = M_{1v}(x^{-1}) + \eta_v(-x)M_{2v}(x^{-1}),$$

where the functions M_{iv} are smooth near 0. The above limit therefore has the form:

$$M_{1v}(0) + \eta_v(-\varepsilon)M_{2v}(0).$$

On the other hand, the factor corresponding to a place v that decomposes into v_1 and v_2 is written:

$$\Omega_{v_2}(-\varepsilon_0)\Omega_{v_1}(\varepsilon)\int f_v\left[\begin{vmatrix} a_1 & 0 \\ 0 & a_2 \end{vmatrix}\begin{vmatrix} 0 & \varepsilon \\ 1 & 0 \end{vmatrix}\right]\Omega_{v_1}(a_1)\Omega_{v_2}(a_2)da_1da_2;$$

Since $\varepsilon_0 = 1$, we obtain after a change of variables:

$$\int f_v\left[\begin{vmatrix} 0 & a \\ -1 & 0 \end{vmatrix}\right]\Omega_{v_1}(a)da.$$

Finally, we see that the integral (7.4.1) is zero unless $\varepsilon_0 = 1$. If this is the case, this integral equals the product of the following factors:

(i) $\mathrm{vol}(E_\lambda^\times/E^\times F_\lambda^\times)$;

(ii) for each place v, decomposed:

$$\int f_v\left[\begin{vmatrix} 0 & a \\ -1 & 0 \end{vmatrix}\right]\Omega_{v_1}(a)da; \tag{7.5.1}$$

(iii) for each inert place v under the place w of E:

$$\mathrm{vol}(E_w^\times/F_v^\times)^{-1}[M_{1v}(0) + M_{2v}(0)\eta_v(-\varepsilon)]$$

We will now show that the sum of the integrals (7.4.1) for all ε equals the sum of the two terms 3 in proposition (6.8). By examining the terms in this proposition and comparing them with the above expression, we see that it suffices to prove the following identity:

$$L(0,\eta)\left\{\prod_v M_{1v}(0)L(0,\eta_v)^{-1}+\prod_v M_{2v}(0)L(0,\eta_v)^{-1}\right\}\prod_{v\text{ decomposed}}|a_v|^{1/2}$$
$$=\mathrm{vol}(E_\lambda^\times/F_\lambda^\times E^\times)\sum_\varepsilon\prod_v\{M_{1v}(0)+M_{2v}(0)\eta_v(-\varepsilon)\}\mathrm{vol}(E_w^\times/F_v^\times)^{-1}.$$

In this formula, the products containing the functions $M_{iv}(0)$ are over the inert places. Note that for almost all inert v we have $U_v(x)=1$ if $v(x)$ is even and x is large enough, $U_v(x)=0$ if $v(x)$ is odd and x is large enough and $\eta_v(-\varepsilon)=1$. These relations imply that $M_{iv}(0)=1/2$. We also have $L(0,\eta_v)=1/2$. We thus see that the products in the left member have almost all their factors equal to 1. Let U be a set of inert places such that the preceding conditions are satisfied for v not in U. For a given ε if v is not in U and $\eta_v(-\varepsilon)=1$ then the corresponding factor in the infinite product of the right member corresponding to ε is 1. This already shows that the infinite products of the right member have a meaning. On the other hand if $\eta_v(-\varepsilon)=-1$ for an inert v not in U, then the corresponding factor is zero. Thus the product is zero for all $-\varepsilon$ that is not a norm in at least one place not in U. The sum is therefore finite.

To prove this identity we introduce a differential ideal $\mathfrak{b}$ for E. The local volume that appears in the formula is nothing other than:

$$2L(0,\eta_v)|\mathfrak{b}_w|^{1/2}|a_v|^{-1/2}$$

while the global volume is $2L(1,\eta)$. Setting

$$A_v=M_{1v}(0)L(0,\eta_v)^{-1},\quad B_v=M_{2v}(0)L(0,\eta_v)^{-1}$$

we see that the relation to be proved is a consequence of the following two relations:

$$|a|^{1/2}L(0,\eta)=L(1,\eta)\prod_v|\mathfrak{b}_w|^{1/2},$$

$$1/2\left\{\prod_v A_v+\prod_v B_v\right\}=\sum_\varepsilon\prod_v 1/2\{A_v+\eta_v(-\varepsilon)B_v\}.$$

The first follows from the functional equation of the L-functions. The second is an exercise in group theory (Cf. [Jac86, §10]). We have thus finally proved the following result:

PROPOSITION 7.5. *The sum of the integrals of the kernels $K_{\varepsilon sin}$ equals the sum of terms* (2) *and* (3) *in proposition 6.8.*

7.6. The difference between the two members of the equality in theorem 7.2 is thus equal to the difference of the contributions of the Eisenstein, special and dihedral terms. More precisely, let U be a set of places of F containing S and let V be the set of places of E above U. We suppose that ε_0 and all the ε that effectively enter the formula are units at all places not in U. We write f_V for the product of f_w with w in V and f^V for the product of f_w with w not in V. Let π be a cuspidal representation of $G(E_\lambda)$ with central character ω' and admitting an invariant vector under the group K^V, that is the product of groups K_w for w not in V. Let ϕ_i be an orthonormal basis of the space of vectors fixed by K^V; we set

$$K_\pi(x,y) = \sum_i \varrho(f_V)\phi_i(x)\phi_i(y)^*.$$

We also denote by $f^V(\pi)$ the eigenvalue of the Hecke algebra associated to π:

$$\varrho(f^V)\phi_i = f^{\widehat{V}}(\pi)\phi_i.$$

Then the cuspidal kernel has the following expression:

$$K_{\text{cusp}}(x,y) = \sum_\pi f^{\widehat{V}}(\pi)K_\pi(x,y),$$

the series being uniformly convergent. The integral of the cuspidal kernel can therefore be written:

$$\iint K_{\text{cusp}} = \sum_\pi f^{\widehat{V}}(\pi)A(\pi;f_V) \tag{7.6.1}$$

with

$$A(\pi;f_V) = \iint K_\pi\left[\begin{vmatrix} a & 0 \\ 0 & 1\end{vmatrix}, h\right]\Omega^{-1}(a)d\eta\omega(\det h)dh. \tag{7.6.2}$$

Similarly, the contribution of the Eisenstein kernel is the sum of two terms (Cf. (5.6)). The first is written:

$$\sum_\chi \int A(\mu,\chi,\chi')f^{\widehat{V}}(\mu,\chi,\chi')d\mu \tag{7.6.3}$$

where $A(\cdot,\cdot,\cdot)$ denotes an appropriate function and the second factor is the eigenvalue $f^{\widehat{V}}(\pi)$ for the representation $\pi = \pi(\alpha^u\chi, \alpha^u\chi')$, the sum being over all characters χ which are lifts and χ' is the product of ω' by the inverse of χ. The other term is written:

$$\sum_\chi f^{\widehat{V}}(0,\chi,\chi')A(\chi,\chi'), \tag{7.6.4}$$

the sum being over all characters χ which are not lifts and whose restriction to F is $\omega\eta$ and $A(\cdot,\cdot)$ denotes an appropriate constant. The contribution of the special kernel is written:

$$f^V(\Omega)A \tag{7.6.5}$$

where we write Ω for the representation Ω^0 det and A is an appropriate constant.

Similarly for each ε we consider the functions $f_{\varepsilon U}$ and f_ε^U. We have a decomposition of the kernel $K_{\varepsilon\text{cus}}$:

$$K_{\varepsilon\text{cus}} = \sum_V f_\varepsilon^U(\sigma_\varepsilon) K_{\sigma_\varepsilon}, \quad K_{\sigma_\varepsilon}(x,y) = \sum_i \varrho(f_\varepsilon^U)\phi_i(x)\phi_i^*(y),$$

where the first sum is over all cuspidal representations σ_ε having an invariant vector under the compact group K_ε^U and ϕ_i runs through an orthonormal basis of the space formed by these vectors. By integrating this kernel we therefore obtain:

$$\iint K_{\varepsilon\text{cus}} = \sum_{\sigma_\varepsilon} f_\varepsilon^{\widehat{U}}(\sigma_\varepsilon) a(\sigma_\varepsilon, f_{\varepsilon U}), \tag{7.6.6}$$

where we have set:

$$a(\sigma_\varepsilon, f_{\varepsilon U}) = \iint K_{\sigma_\varepsilon}(t_1,t_2)\Omega(t_1)\Omega'(t_2)dt_1dt_2. \tag{7.6.7}$$

We now sum the integrals (7.6.4) over all ε. For a given representation σ_ε, there exists a cuspidal representation ε of $G(F_\lambda)$ which corresponds. For v not in U and inert under w, we have denoted f_v the image of f_w by the Hecke homomorphism. For v not in S and decomposed into v_1 and v_2, we have defined a function f_v on G_v. Given the choices made, we see that:

$$f^{\widehat{U}}(\sigma) = f_\varepsilon^{\widehat{U}}(\sigma_\varepsilon). \tag{7.6.8}$$

Making the announced sum, we get:

$$\iint K_{\varepsilon\text{pro}} = \sum_\sigma f^{\widehat{U}}(\sigma) \sum_\varepsilon a(\sigma_\varepsilon, f_{\varepsilon U}). \tag{7.6.9}$$

The first sum is over all cuspidal representations σ of $G(F_\lambda)$ that are not dihedral for the extension E and the second over all ε that are norms at all places not in S and such that there exists a cuspidal representation of G_ε corresponding to σ.

We have also defined dihedral kernels. They have expressions analogous to the above and the sum of the integrals of the dihedral kernels is written:

$$\sum_\varepsilon \iint K_{\varepsilon\text{die}} = \sum_\sigma f^{\widehat{U}}(\sigma) \sum_\varepsilon a(\sigma_\varepsilon, f_{\varepsilon U}). \tag{7.6.10}$$

where the first sum is now over all dihedral cuspidal representations.

The integral of the Eisenstein kernel (for ε a norm) is similarly written:

$$\iint K_{1\text{eis}} = \sum_\chi \int f^{\widehat{U}}(u,\chi,\chi')a(u,\chi,\chi')du. \tag{7.6.11}$$

Finally the special kernel is written:

$$K_{\varepsilon\text{spe}}(x,y) = \sum_\lambda \int \lambda(\det g) f_\varepsilon(g)dg\lambda(\det x)\lambda^*(\det y)V^{-1}, \tag{7.6.12}$$

where the sum is over all characters λ whose square is ω and V denotes the volume of the quotient $G(F_\lambda)/G(F)Z(F_\lambda)$. Integrating this we find 0 unless Ω is the lift to E of a character λ. Then Ω is also the lift of $\eta\lambda$ and we can write the integral as a sum:

$$\iint K_{\varepsilon\text{spe}} = f^{\widehat{U}}(\lambda)a_1 + f^{\widehat{U}}(\lambda\eta)a_2. \tag{7.6.13}$$

where a_1 and a_2 are appropriate constants. We now note that if σ lifts to π then

$$f^{\widehat{V}}(\pi) = f^{\widehat{U}}(\sigma). \tag{7.6.14}$$

Indeed if v is a place of F, inert and not in U, under w then we have:

$$\hat{f}_w(\pi_w) = \hat{f}_v(\sigma_v),$$

because π_w is the lift of σ_v. If on the contrary v decomposes into v_1 and v_2 then taking into account that ε is a unit at place v we have:

$$f_v(g) = \int f_{v_1}(gh)f_{v_2}(h)\omega_v(\det h)dh.$$

Since the contragredient representation of σ_v is its tensor product with the inverse of its central character and since

$$\pi_{v_1} \simeq \pi_{v_2} \simeq \sigma_v$$

we see that

$$\hat{f}_v(\pi_v) = \hat{f}_{v_1}(\pi_v)f_{v_2}(\pi_v) = \hat{f}_{v_1}(\pi_{v_1})\hat{f}_{v_2}(\pi_{v_2}).$$

The relation (7.6.14) follows from these identities. Similarly, we have:

$$f^{\widehat{V}}(u,\chi,\chi') = f^{\widehat{U}}(u,\lambda,\lambda')$$

if χ is the lift of λ, and

$$f^{\widehat{V}}(\Omega) = f^{\widehat{U}}(\lambda)$$

if Ω is the lift of λ. Finally if σ is dihedral, there exists a character χ of E whose restriction to F is $\omega\eta$ and such that $\pi(0,\chi,\chi')$ is the lift of σ. We then have:

$$f^{\widehat{V}}(0,\chi,\chi') = f^{\widehat{U}}(\sigma).$$

We now consider the difference of the two members in proposition (7.2). According to what precedes, it is a linear combination of the differences of expressions (7.6.1) and (7.6.9), (7.6.3) and (7.6.11), (7.6.4) and (7.6.10), (7.6.5) and (7.6.13). According to the preceding equalities, the difference of (7.6.3) and (7.6.11) has the form (7.6.3), with a different function A. Similarly, the difference of (7.6.5) and (7.6.13) has the form (7.6.5), with another constant A. Again similarly the difference of (7.6.4) and (7.6.10) has the form (7.6.4) with a different constant A. Finally since every σ admits a lift the difference of (7.6.1) and (7.6.9) also has the form (7.6.1). First, as in the "Base Change", we use the fact that a continuous measure cannot be equal to a discrete measure to show that the difference of (7.6.3) and (7.6.11) is zero. We then apply the principle of infinite linear independence of characters to show that each of the differences is in fact zero. In particular the terms (7.6.1) and (7.6.9) are equal which finally proves the sought assertion.

7.6. We now prove the fact that the first Waldspurger condition implies the second. We thus consider a proper cuspidal representation σ of $G(F_A)$ of central character ω and we denote π its lift to E. We suppose that $L(1/2, \pi \otimes \Omega^{-1})$ is not zero. There thus exists a K-finite function ϕ in the space of π such that the following integral is not zero:

$$\int \phi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \Omega^{-1}(a)da.$$

On the other hand [HLR86] we know that there exists a K-finite function ϕ' in the space of π such that the following integral is non zero:

$$\int \phi'(h)\eta\omega^{-1}(\det h)dh, \quad h \in G_1(F_A)/Z(F_A)G_1(F).$$

We apply the above result with $\varepsilon_0 = 1$. We choose S and U large enough so that the functions ϕ and ϕ' are K^V invariant. It is then clear that we can choose the function f_V such that $A(\pi, f_V)$ is not zero. We know that the set of representations of which π is the lift is composed of σ and its tensor product with the character η. By applying again the linear independence of characters we obtain:

$$A(\pi, f_v) = \sum a(\sigma_v, f_U) + \sum a(\sigma_v \otimes \eta, f_U).$$

Each sum extends over all ε such that there exists a cuspidal representation σ_ε of G_ε corresponding to σ. There exists therefore at least one ε such that $a(\sigma_\varepsilon, f_\varepsilon U)$ or $a(\sigma_\varepsilon \otimes \eta, f_\varepsilon U)$ is non zero. In both cases this implies the existence of a function ϕ in the space of σ_ε such that the following integral is non zero:

$$\int \phi(t)\Omega^{-1}(t)dt.$$

The second Waldspurger condition is therefore satisfied.

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