ON A RESULT OF WALDSURGER

HERVÉ JACQUET

Abstract. This is a translation of Hervé Jacquet's 1986 paper " Sur un résultat de Waldspurger", published in Annales scientifiques de l'É.N.S. 4e série. The translation was accomplished with AI. Any errors or inaccuracies are my responsibility. For typos, corrections, or suggestions, please contact yluo237@wisc.edu.

0. Introduction

0.1. We will provide a new proof of a remarkable result by Waldspurger ([Wal85, Theorem 2]). Waldspurger's proof relies on the properties of the Weil representation. Ours relies on a variant of the trace formula. We hope it will be of interest.

Let us first recall the result. Let F be a field of numbers, E a quadratic extension of F, η the character of the ideal class group of F attached to E. Let us consider the group GL(2) as an algebraic group G defined over F; let Z be its center. Let A be a maximal split torus in G, say, to fix ideas, the group of diagonal matrices. Let π be an automorphic cuspidal representation of the group $G(F_A)$, trivial on the center $Z(F_A)$. We will say that π satisfies the first Waldspurger condition (abbreviated W1) if there exist automorphic forms ϕ_1 and ϕ_2 in the space of π such that the following integrals are non-zero:

$$\int \phi_1(a)da, \quad \int \phi_2(a)\eta(\det a)da, \quad a \in A(F_{\mathbb{A}})/Z(F_{\mathbb{A}}). \tag{0.1.1}$$

Let us introduce on the other hand the set $X(E:F)$, or simply X, of isomorphism classes of pairs (G',T'), where G' is an inner form of G and T' a maximal torus of G', isomorphic over F to the multiplicative group of E. Recall that such a pair is obtained by means of a pair (H,L), formed of a simple algebra H of rank four over F and a sub-field L of H F-isomorphic to E, by taking for G' the multiplicative group of H and for T' that of L. The center of G' will be denoted Z'. We will identify the set X with one of its systems of representatives. Let us also note $X(\pi)$ the set of triplets (G',T',π'), where the pair (G',T') is in X and π' is a cuspidal automorphic representation of $G'(F_A)$ related to π by the condition of [JL70, Th. (15.1)]; let us recall that this condition can be stated as follows: there exists a finite set S of places of F such that, for v not in S, the groups G_v and G'_v are isomorphic and the representations π_v and π'_v equivalent, after identification of the two groups. We will say that π satisfies the second Waldspurger condition (W2) if there exists

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a triplet (G', T', π') in $X(\pi)$ and an automorphic form ϕ in the space of π' such that the following integral is non-zero:

$$\int \phi(t)dt, \quad t \in T'(F_A)/Z'(F_A). \tag{0.1.2}$$

Then:

Theorem (Waldspurger)**.** *Conditions W1 and W2 are equivalent.*

0.2. Let us outline our demonstration. First, we can identify the set of double classes $A\backslash G/A$ with the disjoint union of double classes $T'\backslash G'/T'$ (§1). To be precise, for this identification we must limit ourselves to "regular" double classes. This leads us to consider a smooth function with compact support f on $G(F_A)/Z(F_A)$ and, for each (G', T'), a smooth function with compact support f' on $G'(F_A)/Z'(F_A)$. In fact, f' will be zero for almost all (G', T'). To the function f is associated the cuspidal kernel K_c and similarly to each function f' is associated a cuspidal kernel K'_c. The conditions imposed on these functions are such that (§7 to §10):

$$\begin{cases} \iint K_c(a,b)da\eta(\det b)db = \sum_{(G',T')} \iint K'_c(s,t)dsdt, \\ a, b \in A(F_A)/A(F)Z(F_A), \quad s,t \in T'(F_A)/T'(F)Z'(F_A) \end{cases} \tag{0.2.1}$$

The relationship between f and the f' is as follows. These functions are naturally products of local functions. If v is a place of F that decomposes in E then for all (G', T') the groups G'_v and G_v are the "same" and we take for f_v and f'_v the "same" function. Suppose on the contrary that v decomposes. Then the set $X(E_v : F_v)$ is still defined but it is reduced to two elements (G_{vi}, T_{vi}), $i = 1, 2$, with G_{v1} split. We can still identify the regular double classes of A_v with the disjoint union of regular double classes of T_1 and T_2. We show that for a given function f there exist functions f_i on G_i such that

$$\iint f(agb)da\eta_v(\det b)db = \iint f_i(sg't)dsdt,$$
$$a, b \in A_v/Z_v, s, t \in T_{iv}/Z_{iv},$$

if g corresponds to g' (§2 to 4) (The exact statement is slightly different since the left-hand side is not quite a function on the set of double classes). If moreover the situation is unramified and f_v is a Hecke function then we can take and do take $f_1 = f_v$ and $f_2 = 0$ (§5). The condition is now that $f'_v = f_i$ if $G'_v = G_i$. Waldspurger's result follows easily from identity ((0.2.1) (cf. §1.1). Paragraph 6 contains auxiliary results.

The method of demonstrating formula (0.2.1) is based on a generalization of the trace formula which can be stated as follows. Let G be a semi-simple group defined over F and A, B subgroups of G defined over F, λ and μ characters of $A(F_A)/A(F)$ and $B(F_A)/B(F)$ respectively. Let us consider a smooth function f on $G(F_A)/G(F)$ with compact support and calculate the following integral:

$$\iint K_c(a,b)\lambda(a)\mu(b)dadb$$

where K_c is the cuspidal kernel attached to f. This kernel has a complicated expression which contains in any case the sum:

$$\sum_\xi f(x^{-1}\xi y), \quad \xi \in G(F).$$

Let us choose a system of representatives for the double classes of groups $A(F)$ and $B(F)$. On the other hand, if η is an element of $G(F)$, let H_η be the subgroup of $A \times B$ formed of pairs (α, β) such that $\alpha^{-1}\eta\beta = \eta$. Then any element of $G(F)$ can be uniquely written in the form:

$$\xi = \alpha^{-1}\eta\beta \quad \eta \in A(F)\backslash G(F)/B(F), \quad (\alpha, \beta) \in H_\eta(F)\backslash A(F) \times B(F).$$

By imitating the usual formal calculation, we immediately arrive at the following expression for the above integral:

$$\sum\nolimits_\eta \mathrm{vol}(H_\eta(F)\backslash H(F_A)) \iint f(a^{-1}\eta b)\lambda(a)\eta(b)da db, \\ a \in A(F)\backslash A(F_A), \quad b \in B(F)\backslash B(F_A),$$

the sum being over all η such that $\lambda(a)\mu(b) = 1$ if $a^{-1}b = 1$. Of course, we have ignored the convergence problems and the existence of other terms in the expression for K_c.

0.3. I wish to thank the Institute for Advanced Study and its permanent members for their hospitality, as the major part of this work was written during my stay at the Institute, during the special year 1983-1984 on L-functions. In particular, I thank Langlands for the interest he has taken in this work. Finally, I owe much gratitude to Piatetski-Shapiro who was also at the Institute that same year. His deep knowledge of Waldspurger's work has been very helpful to me; moreover, a conversation with Piatetski-Shapiro was the starting point of this work.

1. Double classes

1.1. In this paragraph F will be an arbitrary field, say of characteristic zero, and E a quadratic extension of F. We will denote by $N(E : F)$ or simply N the subgroup of norms of E in the multiplicative group of F. The set $X(E : F)$ or simply X introduced in the paragraph is still defined. Let us consider one of its elements (G, T). There exists therefore a simple algebra H of rank 4 over F and a sub-field L of H isomorphic to E such that G is the multiplicative group of H and T that of L. We propose to give a parametrization of the double classes $T\backslash G/T$. To this effect let ε be an element of the normalizer $N(T)$ of T that is not in T. Then every h in H can be written uniquely in the form:

(1.1.1) $$h = h_1 + \varepsilon h_2, \quad \text{where} \quad h_i \in L.$$

On the other hand, if zß${z}^{-1}$ denotes the unique non-trivial F-automorphism of L then:

(1.1.2) $$\varepsilon z \varepsilon^{-1} = z^{-1}.$$

The square $c = \varepsilon^2$ is in Z, or in other words, in F. Moreover, the class of c modulo N is determined by the isomorphism class of the pair (G, T) and, reciprocally, determines it.

Let us define two involutions j^+ and j^- of H by the formulas:

$$j^{\pm}(h) = h_1^- \pm \varepsilon h_2 \quad \text{where} \quad h \quad \text{is as in (1.1.1).} \tag{1.1.3}$$

It is easy to verify that these are the only involutions of H which induce on L the unique non-trivial F-automorphism of L. For h in G we will set:

$$X(h) = \frac{(1/2\,\mathrm{tr}(hj^+(h))}{(1/2\,\mathrm{tr}(hj^-(h))} \tag{1.1.4}$$

Since the denominator of this fraction is nothing other than the reduced norm of h, $X(h)$ is a well-defined element of F depending only on the double class of h modulo T. We will also introduce the function $P(h : T)$ or simply $P(h)$ defined by

$$X(h) = \frac{1 + P(h)}{1 - P(h)}, \tag{1.1.5}$$

or alternatively

$$P(h) = ch_2h_2^-(h_1h_1^-)^{-1}, \quad c = \varepsilon^2. \tag{1.1.6}$$

Thus P is a function with values in the projective line that is constant on the double classes of T in G. Note however that according to the preceding formula, if $P(h)$ is neither zero nor infinite, then it is an element of the class cN determined by the pair (G, T). Moreover, $P(h)$ cannot equal one, as this would make $X(h)$ infinite. We will say that h (or its double class) is T-singular if $P(h)$ is zero or infinite, T-regular in the contrary case.

Proposition. *Two elements h and h' of G have the same double class modulo T if and only if $P(h) = P(h')$. Moreover, if x is in cN and different from 1, then there exists an h in G such that $P(h) = x$.*

The proof is left to the reader.

1.2. The following proposition justifies the use of the term T-regular:

Proposition. *Suppose h is T-regular. Then the relations*

$$sht = hz, \quad s \in T, \quad t \in T, \quad z \in Z$$

imply

$$s \in Z, \quad t \in Z, \quad st = z.$$

The proof is left to the reader.

1.3. What precedes applies “mutatis mutandis” to a pair of the form (G, A) where G is the group GL(2) and A a maximal split torus, say the group of diagonal matrices in G. Then H is the algebra of 2 by 2 matrices, L the subalgebra of diagonal matrices and we can take

$$\varepsilon = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad c = 1.$$

The functions X and $P(\cdot : A)$ (or simply P) are defined as above. In particular:

$$P(h) = bc(ad)^{-1}, \quad \text{if} \quad h = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

They are constant on the double classes of A in G. Again P cannot take the value 1. We will still say that an element h of G is A-singular if $P(h)$ is zero or infinity, A-regular in the contrary case. There are now 6 A-singular double classes: the classes on which P takes the value zero:

$$(1.3.1) \qquad T, \quad Tn_+T, \quad Tn_-T, \quad \text{where} \quad n_+ = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, \quad n_- = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix},$$

and the classes on which P takes the value infinity:

$$(1.3.2) \qquad \varepsilon T, \quad T\varepsilon n_+T, \quad T\varepsilon n_-T.$$

Thus P does not allow us to distinguish these classes from each other.

However P separates the A-regular classes:

Proposition. *Let h and h' be A-regular elements of G. Then h and h' are in the same class if and only if $P(h) = P(h')$. If x is in F different from 1 and 0, then there exists an A-regular element h such that $P(h) = x$.*

We will leave the proof to the reader.

1.4. We also have the analog of proposition 1.2:

Proposition. *Suppose h is A-regular. Then the relations*

$$ahb = hz, \quad a \in A, \quad b \in A, \quad z \in Z$$

imply

$$a \in Z, \quad b \in Z, \quad ab = z.$$

We will leave the proof to the reader.

2. Orbital integrals: case of a compact torus

2.1. Let's keep the notations from paragraph 1 but now suppose that F is a local field. Then T/Z is compact. Let's choose once and for all a non-trivial additive character ψ of F. Let's equip the additive group F with the self-dual measure dx for the character ψ, the multiplicative group $F^\times$ with the measure $L(1, 1_F)|x|^{-1}dx$ (Tamagawa relative measure to ψ). Similarly, let's equip the multiplicative group $E^\times$ with the Tamagawa measure relative to the character $\psi \circ \mathrm{tr}$. By transport of structure, we obtain measures on T and Z. Let's equip T/Z with the quotient measure. Let f be a smooth function with compact support on G/Z. Let us set:

$$H(g : f : T) = \iint f(sgt)dsdt, \quad s, t \in T/Z. \tag{2.1.1}$$

It is clear that $H(g : f : T)$ depends only on the double class of g modulo T. Let x be an element of $F^\times$. If there exists g in G such that $P(g : T) = x$, we will set $H(x : f : T) = H(g : f : T)$. Otherwise, we will set $H(x : f : T) = 0$. We thus obtain a function $H(f : T)$ on $F^\times$ and we propose to characterize the functions H on $F^\times$ that are of the form $H = H(f : T)$ for an appropriate function f.

2.2. Let us therefore consider a function $H = H(f : T)$. By definition H vanishes, thus is smooth, on the complement of cN. Consider a point x of the form $P(h : T)$. Since the norm is a submersive application from $E^\times$ to $F^\times$, the application $g \to P(g : T)$ is a fortiori submersive at point h. It follows that H is smooth at point x. Finally, let us suppose that 1 is in cN (that is to say that the group G is split); we can then suppose that $c = 1$. We are going to show that H is zero in the neighborhood of 1.

Since f has compact support modulo Z, there exists a compact subset C of G such that $H(g : f : T) \neq 0$ implies $g \in TCT$. It will therefore suffice to show the existence of a number K such that the relation $g \in TCT$ implies $|P(g : T) - 1| > K$. Suppose that no such number exists. Then there would exist a sequence g_i of elements of TCT such that $P(g_i : T)$ tends to 1. After enlarging C and multiplying the elements of the sequence by elements of T we can suppose that

$$g_i = 1 + \varepsilon t_i = c_i z_i$$

with t_i in T, c_i in C and z_i in Z. Then

$$P(g_i : T) = t_i t_i^- = 1 + a_i$$

and a_i tends to zero. On the other hand we have:

$$\det g_i = -a_i, \quad \text{and} \quad \det g_i = (z_i)^2 \det c_i.$$

Therefore z_i tends to zero. The same is thus true of g_i. Since the projection of g_i on L is 1, this gives us a contradiction.

2.3. Let us examine the behavior of the function H in the neighborhood of zero and in the neighborhood of infinity. We are going to show that there exists a neighborhood U of 0 in F and a smooth function A on U such that:

$$H(x) = A(x)(1+\eta(cx)), \quad \text{for} \quad x \in U, \tag{2.3.1}$$

$$2A(0) = \operatorname{vol}(T/Z)\int f(t)dt. \tag{2.3.2}$$

Similarly, we will show that there exists a neighborhood U of 0 in F and a smooth function B on U such that:

$$H(x) = B(x^{-1})(1+\eta(cx)), \quad \text{for} \quad x^{-1} \in U, \tag{2.3.3}$$

$$2B(0) = \operatorname{vol}(T/Z)\int f(\varepsilon t)dt. \tag{2.3.4}$$

Indeed, since $P(\varepsilon g : T) = P(g : T)^{-1}$ we have:

$$\iint f(s\varepsilon gt)dsdt = H(x^{-1} : f : T)$$

or alternatively:

$$H(x^{-1} : f : T) = H(x : f' : T) \quad \text{with} \quad f'(g) = f(\varepsilon g).$$

It will therefore suffice to prove the assertions relating to the zero point. It will be convenient to first deal with the non-archimedean case. Take an x in cN. Then $x = cll^-$ where $x = P(h)$ with $h = 1+\varepsilon l$. We can therefore write:

$$H(x : f : T) = \iint f[t_1(1+\varepsilon l)t_2]dt_1dt_2$$

or after a change of variables:

$$H(x : f : T) = \iint f\left[\left(1+\varepsilon l\frac{\overline{t_1}}{t_1}\right)t_2\right]dt_1dt_2. \tag{2.3.5}$$

Since f is smooth there exists an ideal V of E such that for l in V we have:

$$f(g) = f[(1+\varepsilon)g] \quad \text{for all} \quad g.$$

There exists then an ideal U in F such that $ll^- \in U$ is equivalent to $l \in V$. For x in cU we therefore have $H(x) = 0$ if x is not in cN; if on the contrary x is in cN then $x = cl^-$ with l in V and we have according to formula (2.3.5):

$$H(x) = \operatorname{vol}(T/Z)\int f(t)dt.$$

Our assertion is then immediate.

Let's move on to the archimedean case. Then F is the field of real numbers and L the field of complex numbers. Let $K(x) = H(cx)$. Let V be a disk $\{z \mid zz^- < a\}$ in L such that

$1+\varepsilon V$ is contained in G. Then the second member of (2.3.5) defines a smooth function on V, say $C(l)$, depending only on the norm of l. We have

$$K(x) = 0 \quad \text{if} \quad x < 0,$$
$$K(x) = C(l) \quad \text{if} \quad x > 0 \quad \text{and} \quad x = ll^{-} \quad \text{with} \quad l \text{ in } V.$$

In particular the restriction of C to the real axis is smooth and even and we have

$$K(x) = 0 \quad \text{if} \quad x < 0,$$
$$K(x) = C(y) \quad \text{if} \quad 0 < x < a \quad \text{and} \quad x = y^2 \quad \text{with} \quad l \text{ with } y \text{ real}.$$

The content of our assertion is the existence of a smooth function D on F such that $D(x) = K(x)$ for $a > x > 0$. It is therefore a consequence of a Whitney theorem.

2.4. The preceding properties characterize the functions $H(f:T)$:

Proposition. *Let H be a function on $F^{\times}$. For there to exist a smooth function with compact support f on G/Z such that $H = H(f:T)$, it is necessary and sufficient that the following conditions be satisfied:*

(1) H is zero in the complement of cN;

(2) H is zero in a neighborhood of point 1;

(3) there exists a smooth function A on a neighborhood of 0 in F such that, for x near 0, we have:

$$H(x) = A(x)(1+\eta(cx));$$

(4) there exists a smooth function B on a neighborhood of 0 in F such that for $|x|$ sufficiently large we have:

$$H(x) = B(x^{-1})(1+\eta(cx)).$$

Finally if f, A and B satisfy these conditions then:

$$2A(0) = \mathrm{vol}(T/Z)\int f(t)dt, \quad 2B(0) = \mathrm{vol}(T/Z)\int f(\varepsilon t)dt.$$

We have just shown that conditions (1) to (4) are necessary. We will leave to the reader the task of showing that they are also sufficient. The last assertion of the proposition was proved in number (2.3).

3. Orbital integrals: case of a split torus

3.1. In this paragraph F is a local field, E a quadratic extension, η the quadratic character of $F^{\times}$ attached to E, G the group GL(2) and A the subgroup of diagonal matrices. Let us further equip $F^{\times}$ with the Tamagawa measure, $(F^{\times})^2$ with the tensor product of Tamagawa measures of the factors. By transport of structure we obtain a measure on A. Let us equip

A/Z with the quotient measure. If f is a smooth function with compact support on G/Z and g is A-regular in G we will set:

$$H(g : f : A) = H(g : f : 1) = \iint f(agb)dadb, \quad a, b \in A/Z, \tag{3.1.1}$$

$$H(g : f : \eta) = \iint f(agb)da\eta(\det b)dadb, , \quad a, b \in A/Z. \tag{3.1.2}$$

The first integral only depends on $P(g : A)$ and we shall denote by $H(x : f : A)$ or $H(x : f : 1)$ its value at a point g such that $P(g : A) = x$. We shall also set $H(1 : f : A) = H(1 : f : 1) = 0$. For x in F different from 0 and 1 we shall define a matrix $g(x)$ by:

$$g(x) = \begin{vmatrix} 1 & x \\ 1 & 1 \end{vmatrix}. \tag{3.1.3}$$

Then $P(g(x)) = x$ so that we have defined a section of the space of double classes of A in G. We shall set $H(x : f : \eta) = H(g(x) : f : \eta)$ if x is different from 1 and 0; $H(x : f : \eta) = 0$ if $x = 1$. Let us set:

$$w = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}. \tag{3.1.4}$$

Let N be the group of strictly upper triangular matrices and N' the group of strictly lower triangular matrices. Then we have a covering of G by two open sets:

$$G = ANN' \cup ANwN. \tag{3.1.5}$$

We can therefore write f as a sum $f_1 + f_2$ where f_1 has its support in the first open set and f_2 in the second. Let us set

$$\phi(g) = \int f(ag)da, \quad a \in A/Z \tag{3.1.6}$$

and define ϕ_1 and ϕ_2 similarly. Then ϕ is left invariant under A and has compact support modulo A. The same is true for ϕ_1 and ϕ_2 and $\phi = \phi_1 + \phi_2$. Moreover, the functions ϕ_1 and ϕ_2 defined by

$$\Phi_1(u, v) = \phi_1 \left[\begin{vmatrix} 1 & u \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 0 \\ v & 1 \end{vmatrix} \right], \tag{3.1.7}$$

$$\Phi_2(u, v) = \phi_2 \left[\begin{vmatrix} 1 & u \\ 0 & 1 \end{vmatrix} w \begin{vmatrix} 1 & v \\ 0 & 1 \end{vmatrix} \right] \tag{3.1.8}$$

have compact support on $F \times F$. Since

$$\begin{aligned} g(x) \begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} &= \begin{vmatrix} a(1-x) & 0 \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & a^{-1}(1-x)^{-1}x \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 0 \\ a & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1-x & 0 \\ 0 & a \end{vmatrix} \times \begin{vmatrix} 1 & a(1-x)^{-1} \\ 0 & 1 \end{vmatrix} w \begin{vmatrix} 1 & a^{-1} \\ 0 & 1 \end{vmatrix}, \end{aligned} \tag{3.1.9}$$

for x different from 0 and 1:

$$H(x : f : A) = \int \Phi_1[a^{-1}(1-x)^{-1}x, a]d^x a + \int \Phi_2[a(1-x)^{-1}, a^{-1}]d^x a. \tag{3.1.10}$$

To prove the convergence of our integral, we can assume f positive. Then the previous calculation is justified. In the right-hand side of (3.1.10) the integrands have compact support in $F^\times$ and thus give convergent integrals. Therefore H is convergent and equal to (3.1.10). Similarly, we have, for x different from 0 and 1:

$$H(x : f : \eta) = \int \Phi_1[a^{-1}(1-x)^{-1}x, a]\eta(a)d^x a + \int \Phi_2[a(1-x)^{-1}, a^{-1}]\eta(a)d^x a. \tag{3.1.11}$$

3.2. We shall study the properties of the functions $H(f : \eta)$. Formula (3.1.11) already shows that $H(x : F : \eta)$ is a smooth function at any point x different from 0 and 1. On the other hand, if Φ_1 and Φ_2 have their support in the set of (x, y) such that $|x| < C$, $|y| < C$ then in the second integral we have, on the support of Φ_2, $|a(1-x)^{-1}| < C$ and $|a^{-1}| < C$ which gives: $C^{-2} < |1-x|$ if the second integral is not zero. Similarly, if the first integral is not zero we find $|(1-x)^{-1}x| < C^2$, which also implies $D < |1-x|$ for a suitable constant D. It follows that $H(x : f : \eta)$ is zero in a neighborhood of 1. We can therefore consider formula (3.1.11) as valid for all x not equal to zero.

Let us now study $H(f : \eta)$ in the neighborhood of 0. In (3.1.11) the second integral is evidently a smooth function of x at point 0. To study the first integral we will use the following lemma, whose proof is left to the reader:

Lemma. *Let Φ be a Schwartz-Bruhat function of two variables. Then there exist two Schwartz-Bruhat functions of one variable A_1 and A_2 such that for all x different from* 0 *we have:*

$$\int \Phi(a^{-1}x, a)\eta(a)d^x a = A_1(x) + A_2(x)\eta(x).$$

If F is real and Φ is given with compact support, we can take A_1 and A_2 with compact support.

Let us return to the first integral of (3.1.11). Using the notation from the proof of the lemma, the integral is equal to

$$A_1(x(1-x)^{-1}) + A_2(x(1-x)^{-1})\eta(x(1-x)^{-1}). \tag{3.2.1}$$

If x is sufficiently close to 0 then $1-x$ is a norm and $\eta(x(1-x)^{-1}) = \eta(x)$. Moreover, $A_1(x(1-x)^{-1})$ is a smooth function of x in a neighborhood of 0. Since the second integral of (3.1.11) is evidently smooth at point 0, we conclude that, in a neighborhood of 0, $H(x : f : \eta)$ has the following form:

$$H(x : f : \eta) = A_1(x) + A_2(x)\eta(x) \tag{3.2.2}$$

where A_i, $i = 1, 2$, is smooth.

To study $H(x : f : \eta)$ for large $|x|$ let us note that

$$\varepsilon g(x) = g(x^{-1}) \begin{vmatrix} 1 & 0 \\ 0 & x \end{vmatrix} \quad \text{if} \quad \varepsilon = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

It follows that

(3.2.3) $$H(x^{-1} : f : \eta) = H(x : f' : \eta)\eta(x) \quad \text{with} \quad f'(g) = f(\varepsilon g).$$

There exist therefore two functions B_i, $i = 1, 2$, defined in a neighborhood of 0 and smooth, such that:

(3.2.4) $$H(x : f : \eta) = B_1(x^{-1}) + B_2(x^{-1})\eta(x)$$

for $|x|$ sufficiently large.

3.3. In summary:

Proposition. *Let H be a function on $F^\times$ such that there exists a smooth function with compact support f on G/Z with $H(x : f : \eta) = H(x)$. Then:*

(1) H is smooth on $F^\times$;

(2) H vanishes on a neighborhood of 1;

(3) there exists a neighborhood U of 0 and two smooth functions A_i, $i = 1, 2$, in U such that, for x near 0, we have:

$$H(x) = A_1(x) + A_2(x)\eta(x);$$

(4) there exists a neighborhood U of 0 and two smooth functions B_i, $i = 1, 2$, in U such that, for $|x|$ sufficiently large, we have:

$$H(x) = B_1(x^{-1}) + B_2(x^{-1})\eta(x).$$

3.4. We shall discuss the significance of the zero values of functions A_i and B_i from proposition (3.3). For this purpose, let us first recall that, if Φ is a Schwartz-Bruhat function on F, then the integral:

$$\int \phi(x)|x|^s d^x x,$$

or rather its analytic continuation, has a pole at point $s = 0$; the residue at this point has the form $C\Phi(0)$, where the constant C depends on the choice of the Haar measure on the group $F^\times$. On the other hand, the integral:

$$\int \phi(x)|x|^s \eta(x) d^x x$$

has a holomorphic extension at point zero and its value at this point will still be denoted as an integral:

$$\int \phi(x)\eta(x) d^x x.$$

We shall introduce the following quantities:

$$H(n_+ : f : \eta) = \iint f\left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix}\right] d^x a \eta(b) d^x b, \tag{3.4.1}$$

$$H(n_- : f : \eta) = \iint f\left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 0 \\ b & 1 \end{vmatrix}\right] d^x a \eta(b) d^x b, \tag{3.4.2}$$

$$H(\varepsilon n_+ : f : \eta) = \iint f\left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} \varepsilon \begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix}\right] d^x a \eta(b) d^x b, \tag{3.4.3}$$

$$H(\varepsilon n_- : f : \eta) = \iint f\left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} \varepsilon \begin{vmatrix} 1 & 0 \\ b & 1 \end{vmatrix}\right] d^x a \eta(b) d^x b. \tag{3.4.4}$$

In general these integrals are divergent, but they can be interpreted as recalled above. For example, the first integral is the value at point 0 of the meromorphic function which, for $\operatorname{Re} s > 0$, is given by the convergent integral:

$$\iint f\left[\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix}\right] d^x a \eta(b) |b|^s d^x b. \tag{3.4.5}$$

Let us note however that if f has its support in the open set $ANwN$ then integrals (3.4.1) and (3.4.2) are convergent. In fact integral (3.4.1) is zero since AN does not intersect $ANwN$. On the other hand, the intersection of AN' with a compact set contained in $ANwN$ is a compact set disjoint from A. It follows that in (3.4.2) the integrand has compact support in $F^\times \times F^\times$ and the integral therefore converges trivially. Similarly integrals (3.4.3) and (3.4.4) converge if f has its support in ANN'.

Proposition. *With the above notations and those of proposition* (3.3) *we have:*

$$H(n_+ : f : \eta) = A_2(0) \tag{3.4.6}$$

$$H(n_- : f : \eta) = A_1(0) \tag{3.4.7}$$

$$H(\varepsilon n_+ : f : \eta) = B_1(0) \tag{3.4.8}$$

$$H(\varepsilon n_- : f : \eta) = B_2(0). \tag{3.4.9}$$

Proof. Let us prove assertions (3.4.6) and (3.4.7). According to (3.4.1), it suffices to give the proof when f has its support in ANN' or in $ANwN$. First suppose that f has its support in $ANwN$. Then (cf. (3.1.9) and (3.1.11):

$$H(x : f : \eta) = \int \Phi[a(1-x)^{-1}, a^{-1}]\eta(a) d^x a \tag{3.4.10}$$

where

$$\Phi(u,v) = \phi\left[\begin{vmatrix}1 & u\\0 & 1\end{vmatrix} w \begin{vmatrix}1 & v\\0 & 1\end{vmatrix}\right], \tag{3.4.11}$$

$$\phi(g) = \int f(ag)da, \quad a \in A/Z. \tag{3.4.12}$$

Since H is smooth at zero we have $A_2 = 0$ and $A_1 = H(f:\eta)$. Then:

$$A_1(0) = \int \phi\left[\begin{vmatrix}1 & a\\0 & 1\end{vmatrix} w \begin{vmatrix}1 & a^{-1}\\0 & 1\end{vmatrix}\right]\eta(a)d^x a.$$

Taking into account that f is invariant under the center we can write this in the form:

$$A_1(0) = \iint f\left[b\begin{vmatrix}a^2 & 0\\a & 1\end{vmatrix}\right] db\eta(a)d^x a.$$

A change of variables shows that this is nothing other than $H(n_- : f : \eta)$. Since A_2 and $H(n_+ : f : \eta)$ are zero, relation (3.4.6) is also verified.

Suppose now that f has its support in ANN'. Then (cf. (3.1.7) and (3.1.11)):

$$H(x:f:\eta) = \int \Phi(a^{-1}(1-x)^{-1}x, a)\eta(a)d^\times a \tag{3.4.13}$$

where:

$$\Phi(u,v) = \phi\left[\begin{pmatrix}1 & u\\0 & 1\end{pmatrix} \times \begin{pmatrix}1 & 0\\v & 1\end{pmatrix}\right], \tag{3.4.14}$$

$$\phi(g) = \int f(ag)da, \qquad a \in A/\mathbb{Z}. \tag{3.4.15}$$

Let us recall the definition of A_1 and A_2:

$$H(x) = A_1(x) + A_2(x)\eta(x); \tag{3.4.16}$$

On the other hand, according to lemma 3.2:

$$\int \Phi(a^{-1}x, a)\eta(a)d^\times a = C_1(x) + C_2(x)\eta(x). \tag{3.4.17}$$

Comparing with (3.4.4), we see that $C_i((1-x)^{-1}x) = A_i(x)$. Therefore C_i and A_i have the same value at zero. Taking the Mellin transform of the preceding equation, we obtain:

$$\iint \Phi(x,a)|x|^s\eta(a)|a|^s d^\times a = \int C_1(x)|x|^s d^\times x + \int C_2(x)|x|^s\eta(x)d^\times x.$$

Calculating the residue of both sides at s, we find:

$$\int \Phi(0,a)\eta(a)d^\times a = C_1(0).$$

According to (3.4.14) and (3.4.15), the left-hand side is none other than $H(n_- : f : \eta)$. On the other hand, the right-hand side is none other than $A_1(0)$. Relation (3.4.7) is thus established. Relation (3.4.6) can be established in the same way.

Relations (3.4.8) and (3.4.9) follow from relations (3.4.6) and (3.4.7) applied to the function f' defined by $f'(g) = f(\varepsilon g)$. □

4. Matched functions

4.1. In this paragraph E is a local field, E a quadratic extension of F, η the quadratic character attached to E. We will still consider the pair (G, A) formed by the group GL(2) and the subgroup of diagonal matrices and the set $X = X(E : F)$. It is reduced to two elements (G_i, T_i), $i = 1, 2$, with say G_1 deployed. Let f be a smooth function with compact support on G/Z and f_i, $i = 1, 2$, a smooth function with compact support on G_i/Z_i. We will say that f and the pair (f_1, f_2) are matched if the following condition is satisfied. For all x in F different from 1 and 0, let i and g in G_i be such that $x = P(g : T_i)$ ($i = 1$ if x is a norm of E, $i = 2$ otherwise); then:

$$H(x : f : \eta) = H(g : f_i : T_i).$$

Proposition. *Given a function f, there exists a pair (f_1, f_2) matched to f. Moreover, if f and the pair (f_1, f_2) are matched we have:*

$$\mathrm{vol}(T_i/Z_i) \int f_i(t_i)dt_i = H(n_+ : f : \eta) \pm H(n_- : f : \eta),$$

$$\mathrm{vol}(T_i/Z_i) \int f_i(\epsilon t_i)dt_i = H(\epsilon n_- : f : \eta) \pm H(\epsilon n_+ : f : \eta)$$

with the $+$ sign if $i = 1$, the $-$ sign if $i = 2$.

Proof. This follows immediately from propositions (2.4), (3.3) and (3.4). □

4.2. If F is real we will denote by K the orthogonal subgroup in G. We will denote by U the set of pairs (f_1, f_2) that are matched to a smooth function with compact support f on G/Z, K-finite if F is real.

Let U_1 (resp. U_2) be the first (resp. second) projection of U. Then the sets U_i have a density property.

Proposition. *Let ϕ be a continuous function on G_i/Z_i biinvariant under T_i. Suppose that the integral of ϕ against any function in U_i is zero. Then ϕ is zero.*

The proof will occupy the rest of this paragraph.

4.3. Let f be a continuous function with compact support on G/Z; then:

(4.3.1) $$\int f(g)dg = c \int H(x : f : A)|1 - x|^{-2}dx, \qquad x \neq 0.$$

where c is a constant independent of f.

We will also need an estimate on the functions $H(x : f : A)$ where f is continuous with compact support on G/Z:

Lemma. *Let f be a function with compact support on G/Z and $H(x) = H(x : f : A)$. Then H vanishes in a neighborhood of point 1 and is $O(\log|x|)$ for $|x|$ small or large.*

Proof of the lemma. It is analogous to that given in (3.2) except that lemma (3.2) is replaced by the following assertion: if Φ is a Schwartz-Bruhat function of two variables, then there exist two Schwartz-Bruhat functions of one variable A_i, $i = 1, 2$, such that:

$$\int \Phi(a^{-1}x, a)d^*a = A_1(x) + A_2(x)\log|x|.$$

Finally, we will need integration formulas for the groups G_i analogous to (4.3.1):

$$\int f_1(g)dg = c_1 \int H(x : f_1 : T_1)|1-x|^{-2}dx, \qquad x > 0 \tag{4.3.2}$$

$$\int f_2(g)dg = c_2 \int H(x : f_2 : T_2)|1-x|^{-2}dx, \qquad x < 0 \tag{4.3.3}$$

where c_i is a constant and f_i is a continuous function with compact support on G_i/Z_i. □

4.4.

Proof of proposition 4.2. To fix ideas, let's suppose $i = 1$. Let $H(x) = H(x : \phi : T_1)$. In particular, $H(x) = 0$ if x is not a norm. We will calculate up to multiplicative constants. Now suppose f and (f_1, f_2) are paired with f K-finite if F is real. Then according to (4.3.2):

$$\int \phi(g_1)f_1(g_1)dg_1 = \int H(x)H(x : f_1 : T_1)|1-x|^{-2}dx$$

On the other hand, let ϕ_0 be the function on G defined by:

$$\phi_0(g) = H(x)\eta(\det b) \quad \text{if } g = ag(x)b.$$

Given the properties of H and the integration formula (4.3.1), ϕ_0 is locally integrable and:

$$\int \phi_0(g)f(g)dg = \int H(x)H(x : f : \eta)|1-x|^{-2}dx.$$

Since $H(x : f : \eta) = H(x : f_1 : T_1)$ if x is a norm, we have:

$$\int \phi_0(g)f(g)dg = \int \phi(g_1)f_1(g_1)dg_1.$$

The right-hand side is zero by hypothesis. Therefore ϕ_0 is orthogonal to all smooth functions (resp. all K-finite smooth functions if F is real). It follows that ϕ_0 is zero. The same therefore holds for H; since ϕ is T_1-biinvariant, ϕ is completely determined by H and we obtain $\phi = 0$. □

5. Orbital integrals: the unramified situation

5.1. In this paragraph F is a non-archimedean local field, E a non-ramified quadratic extension of F. We will assume that the residual characteristic of F is not 2 and that the order of character ψ is 0. We will consider the pair (G, A) formed by the group GL(2) and the diagonal subgroup. We will denote R as the ring of integers of F, P the maximal ideal of R, ω a uniformizer and K the group $\mathrm{GL}(2, R)$. The set $X = X(E : F)$ is reduced to two elements (G_1, T_1) and (G_2, T_2). We will assume $G_1 = G$, T_1 contained in the subgroup ZK. We will simply write T for T_1. The measures of $A \cap K/Z \cap K$ and $T \cap K/Z \cap K$ are thus equal to 1. The aim of this paragraph is to prove the following proposition:

Proposition. *Let f be a bi-K-invariant function with compact support on the group G/Z. Then f and the pair $(f, 0)$ are matched. Moreover:*

$$H(n_+ : f : \eta) = H(n_- : f : \eta) = \frac{1}{2}\operatorname{vol}(T/Z)\int f(t)dt.$$

It will be convenient to consider functions with compact support on G rather than functions with compact support on G/Z. Of course, the measures of the sets $A \cap K$, $T \cap K$, $Z \cap K$ are thus equal to 1. If f is a bi-K-invariant function with compact support on G, then we will set:

$$H(g : f : T) = \iint f(sgt)dsdt, \quad s \in T, \quad t \in T/Z. \tag{5.1.1}$$

Since T is contained in ZK this reduces to:

$$H(g : f : T) = \int f(zg)dz, \quad z \in Z. \tag{5.1.2}$$

Similarly, we will set:

$$H(g : f : \eta) = \iint f(agb)da\eta(\det b)db, \quad a \in A, \quad b \in A/Z. \tag{5.1.3}$$

We will also write $H(x : f : \eta)$ for $H(g(x) : f : \eta)$. The relations to prove are thus:

$$H(x : f : \eta) = \int f(zg)dz \quad \text{if } v(x) \text{ is even and } P(g : T) = x, \tag{5.1.4}$$

$$H(x : f : \eta) = 0 \quad \text{if } v(x) \text{ is odd}. \tag{5.1.5}$$

By linearity we can assume that f is either the characteristic function f_0 of K, or the characteristic function f_m of the set

$$K\begin{vmatrix}\omega^m & 0\\ 0 & 1\end{vmatrix}K, \quad \text{where } m > 0. \tag{5.1.6}$$

Note that $f_m(g) \neq 0$ if and only if the following conditions are verified:

(5.1.7) the coefficients of g are integers;

(5.1.8) $v(\det g) = m$;

(5.1.9) at least one of the coefficients of g is a unit.

Of course if $m = 0$ condition (5.1.9) is superfluous.

5.2. We will first calculate $H(x : f_m : \eta)$ which, for simplicity, we will denote $H(x : m)$. We will begin with the case $m > 0$.

Proposition. *Suppose $m > 0$. Then $H(x : m)$ is given by the following formulas:*

(1) if $v(x)$ is odd $H(x : m) = 0$;

(2) if $v(x)$ is even $H(x : m) = 0$, unless $v(x) = 0$ and $v(1 - x) = m$ in which case $H(x : m) = 1$.

Proof We will use the following lemma:

Lemma. *Let $S = (-1)^{i+j}$ where the sum is over all pairs of integers (i, j) belonging to the border of the rectangle defined by the inequalities*

$$0 \leq i \leq P, \quad 0 \leq j \leq Q.$$

Then S is given by the following formulas:

(5.2.1) *if $PQ > 0$, then $S = 0$;*

(5.2.2) *if $P = 0$ and $Q > 0$, then $S = 1$ if Q is even and $S = 0$ if Q is odd;*

(5.2.3) *if $Q = 0$ and $P > 0$, then $S = 1$ if P is even and $S = 0$ if P is odd;*

(5.2.4) *if $P = 0$ and $Q = 0$, then $S = 1$.*

Let us now prove the proposition. It will be convenient to write $\mathrm{Mat}[a, b, c, d]$ for the matrix whose coefficients are the numbers a, b, c, d. With this notation we have:

$$H(x : m) = \sum f_m(\mathrm{Mat}[\omega^{i+k}, x\omega^{j+k}, \omega^i, \omega^j])(-1)^{i+j}, \tag{5.2.5}$$

where the sum is over all triplets of integers (i, j, k). Since the determinant of matrix (5.2.5) has a valuation equal to $i + j + k + v(1 - x)$ condition (5.1.8) shows that in the above sum we can restrict ourselves to triplets such that:

$$i + j + k + v(1 - x) = m.$$

This allows us to eliminate k and, taking into account (5.1.7) and (5.1.9), to write:

$$H(x : m) = \sum (-1)^{i+j} \tag{5.2.6}$$

where the sum is over all pairs of integers (i, j) such that:

(5.2.7) $$0 \le i \le m - v(1-x) + v(x)$$

(5.2.8) $$0 \le j \le m - v(1-x)$$

(5.2.9) $$\text{if } [m - v(1-x) + v(x) - i]^{\mathbf{K}} m - v(1-x) - j] = 0.$$

The sum is empty and $H(x : m)$ is zero unless:

(5.2.10) $$m - v(1-x) \ge 0 \quad \text{and} \quad m - v(1-x) + v(x) \ge 0.$$

Suppose conditions (5.2.10) are satisfied. Then we can apply the lemma. We therefore have $H(x : m) = 0$ unless

(5.2.11) $$[m - v(1-x)][m - v(1-x) + v(x)] = 0.$$

The verification of the proposition is then elementary.

5.3. Let us now calculate $H(x : 0)$.

Proposition. *$H(x : 0)$ is given by the following formulas:*

(1) if $v(x)$ is odd then $H(x : 0) = 0$;

(2) if $v(x)$ is even then $H(x : 0) = 1$, unless $v(x) = 0$ and $v(1-x) > 0$ in which case $H(x : 0) = 0$.

Proof. We still have:

(5.3.1) $$H(x : 0) = \sum f_0(\mathrm{Mat}[\omega^{i+k}, x\omega^{j+k}, \omega^i, \omega^j])(-1)^{i+j},$$

where the sum is over all triplets of integers (i, j, k). As above, taking into account conditions (5.1.7) and (5.1.8), we can eliminate k and write:

(5.3.2) $$H(x : 0) = \sum (-1)^{i+j}$$

where the sum is over all pairs of integers (i, j) such that:

(5.3.3) $$0 \le i \le v(x) - v(1-x)$$

(5.3.4) $$0 \le j \le -v(1-x).$$

The verification of the proposition is then elementary. □

5.4. Let us now calculate the integral $\int f_m(zg)dz$. It only depends on $x = P(g : T)$ and we will denote its value $H(x : m : T)$. Recall that by definition x is a norm, in other words the valuation of x is even. We will begin with the case $m > 0$.

Proposition. *Suppose $m > 0$. Then $H(x : m : T) = 0$, unless $v(x) = 0$ and $v(1-x) = m$ in which case $H(x : m : T) = 1$.*

Proof. We can assume that E is the extension generated by the square root of τ, where τ is a unit. Then we can take for T the multiplicative group of the algebra:

$$L = \left\{ \begin{vmatrix} a & b \\ b\tau & a \end{vmatrix} \right\} \tag{5.4.1}$$

and for ε the matrix:

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.4.2}$$

Let's calculate $H(x : m : T)$ where $x = P(g : T)$. We can assume that:

$$g = I_2 + \varepsilon \begin{vmatrix} u & v \\ v\tau & u \end{vmatrix} \tag{5.4.3}$$

Then $\det g = 1 - x$ and $x = u^2 - v^2\tau$. We have:

$$H(b : m : T) = \sum_k f_m(\omega^k g) \tag{5.4.4}$$

where

$$\omega^k g = \begin{vmatrix} \omega^k(1+u) & \omega^k v \\ -\omega^k v\tau & \omega^k(1-u) \end{vmatrix} \tag{5.4.5}$$

Given condition (5.1.8), this sum thus has at most one term whose index k is determined by the equation

$$k = 1/2[m - v(1-x)] \tag{5.4.6}$$

In particular $H(b : m : T) = 0$ or 1. Given conditions (5.1.7) to (5.1.9), $H(x : m : T) = 1$ if and only if the following conditions are verified:

$$m \equiv v(1-x) \mod 2 \tag{5.4.7}$$

(5.4.8) the coefficients of matrix (5.4.5) where k is given by (5.4.6) are integers;

(5.4.9) at least one of the coefficients of this matrix is a unit.

Suppose first $v(x) < 0$. Then $v(1-x) = v(x)$. Since $x = u^2 - v^\tau$ and τ is not a square, $v(x)$ is even. According to (5.4.7) $H(x : m : T) = 0$ unless m is also even. Let's suppose therefore that this is the case. We can write:

$$u = u_0\omega^{1/2v(x)}, \quad v = v_0\omega^{1/2v(x)} \tag{5.4.10}$$

where u_0 and v_0 are integers, at least one being a unit. Then the coefficients of matrix (5.4.5) are the numbers:

$$\begin{aligned} &\omega^{1/2(m-v(x))}(1 + u_0\omega^{1/2v(x)}), && \omega^{1/2m}v_0, \\ &-\omega^{1/2m}v_0\tau, && \omega^{1/2(m-v(x))}(1 - u_0\omega^{1/2v(x)}) \end{aligned}$$

All these numbers are in the ideal P thus $H(x:m:T)=0$. Suppose $v(x)>0$. Then $v(1-x)=0$. According to (5.4.7) $H(x:m)=0$ unless m is even. Suppose this is the case. Then $k=1/2m$, u and v are integers. The coefficients of matrix (5.4.5) are now the numbers:

$$\begin{matrix} \omega^{1/2m}(1+u) & \omega^{1/2m}v \\ -\omega^{1/2m}v\tau & \omega^{1/2m}(1-u) \end{matrix}$$

All these numbers are in the ideal P thus $H(x:m:T)=0$.

Finally, suppose $v(x)=0$. Then $v(1-x)\geq 0$. If $m-v(1-x)$ is odd then $H(x:m:T)=0$. Suppose therefore $m-v(1-x)$ is even. Then the coefficients of matrix (5.4.5) are the numbers:

$$\begin{matrix} \omega^{1/2[m-v(1-x)]}(1+u) & \omega^{1/2[m-v(1-x)]}v \\ -\omega^{1/2[m-v(1-x)]}v\tau & \omega^{1/2[m-v(1-x)]}(1-u) \end{matrix}$$

Since x is a unit, u and v are integers and at least one is a unit. If $1+u$ and $1-u$ were both in P we would have $2\in P$, a contradiction. Thus at least one of the numbers $1+u$ and $1-u$ is a unit. If therefore $H(x:m:T)$ is not zero then conditions (5.4.8) and (5.4.9) imply that $m=v(1-x)$. The coefficients of matrix (5.4.5) thus reduce to:

$$1+u, \quad v, \quad -v\tau, \quad 1-u$$

These are integers and at least one is a unit. Thus $H(x:m:T)=1$.

We have therefore completely calculated H and our result agrees with the proposition.

5.5. Finally, let's calculate $H(x:0:T)$. Recall once again that $v(x)$ is even.

Proposition. $H(x:0:T)=1$, *unless* $v(x)=0$ *and* $v(1-x)>0$ *in which case* $H(x:0:T)=0$.

Proof. As above we have:

$$H(x:m:T)=\sum_k f_0(\omega^k g) \tag{5.5.1}$$

The sum in fact has at most one term whose index k is given by:

$$k=-1/2v(1-x) \tag{5.5.2}$$

In particular $H(x:0:T)=0$ or 1. Moreover $H(x:0:T)=1$ if and only if $v(1-x)$ is even and the matrix

$$\omega^k g=\begin{vmatrix} \omega^k(1+u) & \omega^k v \\ -\omega^k v\tau & \omega^k(1-u) \end{vmatrix} \tag{5.5.3}$$

with k given by (5.5.2) is in $\mathrm{GL}(2,R)$.

Suppose $v(x)<0$ and $v(1-x)$ even. Then $v(1-x)=v(x)$, $v(x)$ is even and

$$u=u_0\omega^{1/2v(x)}, \quad v=v_0\omega^{1/2v(x)}$$

where u_0 and v_0 are integers, at least one being a unit. Then the coefficients of matrix (5.5.3) are the numbers:

$$\omega^{-1/2v(x)} + u_0, \quad v_0, \quad -v_0\tau, \quad \omega^{-1/2v(x)} - u_0.$$

These are integers. Since the determinant of matrix (5.5.3) is a unit according to the choice of k, matrix (5.5.3) is in $\mathrm{GL}(2, R)$ and $H(x : 0 : T) = 1$.

Suppose $v(x) \geq 0$ and $v(1-x) = 0$ (of course $v(x) > 0$ implies $v(1-x) = 0$). Then $k = 0$ and the coefficients of matrix (5.5.3) reduce to the numbers:

$$1 + u, \quad v, \quad -v\tau, \quad 1 - u$$

Since u and v are integers, these numbers are also integers and matrix (5.5.3) is in $\mathrm{GL}(2, R)$. Thus $H(x : 0 : T) = 1$.

Finally suppose $v(x) = 0$ and $v(1-x) > 0$. Then H is zero unless $v(1-x)$ is even. Suppose this is the case. Then the coefficients of matrix (5.5.3) are the numbers:

$$\begin{matrix} \omega^{-1/2v(1-x)}(1+u) & \omega^{-1/2v(1-x)}v \\ \omega^{-1/2v(1-x)}v\tau & \omega^{-1/2v(1-x)}(1-u) \end{matrix}$$

Since $1 + u$ or $1 - u$ is a unit, at least one of these numbers is not an integer thus (5.5.3) is not in $\mathrm{GL}(2, R)$ and $H(x : 0 : T) = 0$.

We have therefore completely calculated H and our result agrees with the proposition.

5.6. By bringing together propositions (5.2), (5.3), (5.4) and (5.5) we see that we have proved the identities (5.1.4) and (5.1.5). This thus completes the proof of the first assertion of proposition (5.1). The second then follows from proposition (4.1).

5.7. To establish the convergence of global orbital integrals we will need an additional result, whose proof we will leave to the reader:

Lemma. *Suppose h in KZ and let $x = P(h : A)$. Suppose $v(x) = 0$ and $v(1-x) = 0$. Then the relations*

$$ahb \in KZ, \quad a \in A, \quad b \in A$$

imply

$$a \in Z(K \cap A), \quad b \in Z(K \cap A)$$

The lemma obviously implies the following proposition:

Proposition. *Let f be the characteristic function of KZ. Suppose the quadratic extension E is unramified and T contained in KZ. Let h be an element of KZ and $x = P(h : A)$. If x and $1 - x$ are units then:*

$$H(h : f : A) = 1, \quad H(h : f : \eta) = 1$$

6. Reminders about local representations of GL(2)

6.1. Let F be a local field and E a quadratic extension of F. We will again consider the set X which is reduced to two elements (G_1, T_1) and (G_2, T_2), with say G_1 deployed. It will be convenient to use the following result:

Proposition. *Let π be an irreducible unitary representation of G_1/Z_1. Then the dimension of the space of continuous linear T_1-invariant forms on the space of smooth vectors of π is at most one. Moreover, such a form is given by the scalar product with a T_1-invariant smooth vector.*

If F is not archimedean then the assertion about the dimension is proved in [Wal91, Propositions 9]. If F is real, it is classical. The rest of the proposition is evident.

6.2. Similarly:

Proposition. *Let π be an irreducible unitary representation of G_1/Z_1 of infinite dimension. Then the dimension of the space of continuous linear A-invariant forms (respectively invariant relative to the character $\eta \circ \det$ of A) on the space of smooth vectors of π is one.*

These are propositions 9 and 10 of [Wal80].

6.3. Consider, for $i = 1, 2$, an irreducible unitary representation π_i of G_i/Z_i. We will assume that the pair (π_1, π_2) satisfies the conditions of theorem (15.1) of [JL70]; in particular π_1 is in the discrete series.

Proposition. *The representations π_i cannot both have a non-zero vector invariant under the group T_i.*

If F is non-archimedean then our assertion is found in theorem 2 of [Wal91]. If F is real, it is classical.

7. Global orbital integrals: case of a split torus

7.1. In the remainder of this work, F will be a number field and E a quadratic extension of F, η the quadratic character of the ideal class group of F attached to E. In this paragraph and the next, we will consider the pair (G, A) and a smooth function with compact support f on $G(F_A)/Z(F_A)$. To the function f is attached the cuspidal kernel K_c. Let φ_j be an orthonormal basis of the space of cuspidal forms for the group G/Z.

Then, by definition:

$$K_c(x, y) = \sum \rho(f)\phi_j(x)\overline{\phi_j(y)} \tag{7.1.1}$$

where:

$$\rho(f)\phi(x) = \int f(g)\phi(xg)dg \tag{7.1.2}$$

In this paragraph and the next, we propose to give a useful expression for the integral:

(7.1.3) $$\iint K_c(a,b)\eta(\det b)dadb, \quad a,b \in A(F_A)/A(F)Z(F_A)$$

We have chosen a non-trivial character ψ of F_A/F. We thus have at each place v the Tamagawa measure attached to ψ_v and by transport of structure on A_v and Z_v. We therefore have the product measure on $A(F_A/F)$ and the quotient measure on $A(F_A)/Z(F_A)$. We will denote by S a finite set of places containing the infinite places, the ramified places in E and the places of residual characteristic 2. For any place v of F, we will denote by K_v. The usual maximal compact subgroup. In particular $K_v = \mathrm{GL}(2, R_v)$ if v is finite. We will take the function f as a product of local functions f_v which are K_v-finite at all places. We will assume that f_v is bi-K_v-invariant for all v not in S. Indeed, f_v is actually the characteristic function of K_vZ_v for almost all v not in S. We have a decomposition of K_c as a sum:

(7.1.4) $$K_c(x,y) = \sum f(x^{-1}\gamma y) - K_{sp}(x,y) - K_{ei}(x,y),$$

where the sum is over all γ in $G(F)/Z(F)$, K_{sp} denotes the special kernel and K_{ei} the Eisenstein kernel (the definition is recalled later). We can write the first term of this sum as the sum of two other terms K_r and K_s where:

(7.1.5) $$K_r(x,y) = \sum f(x^{-1}\gamma, y), \quad \gamma \text{ A-regular;}$$

(7.1.6) $$K_s(x,y) = \sum f(x^{-1}\gamma, y), \quad \gamma \text{ A-singular.}$$

Then K_c can be written as

(7.1.7) $$K_c = K_r + K_s - K_{sp} - K_{ei}.$$

7.2. We first consider the integral of K_r. Any element γ of $G(F)/Z(F)$ can be uniquely written in the form:

(7.2.1) $\gamma = \alpha g(\xi)\beta$, with α and $\beta \in A(F)/Z(F)$ and ξ different from 0 and 1.

(cf. (3.1.3) for notation and §(1)). It follows immediately:

(7.2.2) $$\iint K_r(a,b)\eta(\det b)da\,db = \sum H(\xi : f : \eta), \quad \xi \neq 0 \text{ and } 1.$$

where we have set:

(7.2.3) $$H(\xi : f : \eta) = \iint f(ag(\xi)b)\eta(\det b)da\,db, \quad a,b \in A(F_A)/Z(F_A).$$

Let us justify our formal calculations. First, the support of f meets only a finite number of regular classes. Indeed, the function X introduced in paragraph 1 (cf. (1.1.4)) defines a continuous function from the group $G(F_A)/Z(F_A)$ to F_A. It therefore takes only a finite number of values on the intersection of the support of f with the set of rational points; the same is thus true for the function $P(\cdot : A)$, which gives us our assertion. On the other hand, each of the integrals (7.2.3) converges absolutely: it is sufficient to prove this for the integral

(7.2.4) $$H(\xi : f : A) = \iint f(ag(\xi)b)da\,db, \quad a, b \in A(A)/Z(A).$$

Each of the local integrals $H(\xi : f_v : A_v)$ converges; almost all are equal to 1 (cf. (5.7)). Therefore (7.2.4) converges. The same is thus true for (7.2.3) and (7.2.3) is the product of the corresponding local integrals:

(7.2.5) $$H(\xi : f : \eta) = \prod H(\xi : f_v : \eta_v).$$

In this product almost all the integrals are equal to 1 (cf. (5.7)).

7.3. Let's consider the integral of K_s. It is not absolutely convergent, but it is "weakly" convergent in the following sense. Let c be a number greater than 1. Let us denote by

(7.3.1) $$\int_{c^{-1}}^{c}\int_{c^{-1}}^{c} K_s(a,b)\eta(\det b)dadb, \quad a, b \in A(F_A)/A(F)Z(F_A)$$

the integral of $K_s(a,b)\eta(\det b)$ over the set of pairs (a,b) satisfying $c^{-1} < |a_1/a_2| < c$, $c^{-1} < |b_1/b_2| < c$; where a_1 and a_2 for example denote the diagonal coefficients of a. Since the integral is taken over a compact set, it exists. We will see that the integral (7.3.1) tends to a limit as c tends to infinity. This limit will be, by definition, the weak integral of $K_s(a,b)\eta(\det b)$. We saw in section (1.3) that there were 6 singular double classes for A, namely the double classes of the following elements: $e, n_+, n_-, \varepsilon, \varepsilon n_+, \varepsilon n_-$. Let's number them from 1 to 6. Then we have a decomposition of K_s into 6 terms K_i, $1 \le i \le 6$, where K_i is the sum of $f(x^{-1}\gamma y)$ for all γ in the i-th double class. Let's study the integral of K_1 for example. We have

$$K_1(x,y) = \sum f(x^{-1}\alpha y), \quad \alpha \in A(F)/Z(F).$$

Hence:

$$\int_{c^{-1}}^{c}\int_{c^{-1}}^{c} K_1(a,b)\eta(\det b)dadb = \int_{c^{-1}}^{c}\int_{c^{-1}}^{c} f(ab)\eta(\det b)dadb;$$

in the left integral, a and b vary in the compact subset of $A(F_A)/A(F)Z(F_A)$ defined above; in the right integral, b still varies in the compact subset of $A(F_A)/A(F)Z(F_A)$ defined by $c^{-1} < |b_1/b_2| < c$, but a varies in the subset of $A(F_A)/Z(F_A)$ defined by $c^{-1} < |a_1/a_2| < c$. Let's change a to ab^{-1} in the right integral. We obtain a double integral, where the inner integral depends only on $|b_1/b_2|$. This inner integral can be written as:

$$\int_{c^{-1}}^{c} \eta(\det b)db, \quad b \in A(F_A)/A(F)Z(F_A).$$

It is zero because the restriction of η to the group of idéles of absolute value 1 is not trivial. The integral of K_1 is therefore weakly convergent and its value is 0. The same applies to the integral of K_4.

Let's examine the integrals of the other terms, K_2 for example. We have:

$$K_2(x,y) = \sum f(x^{-1}\alpha n_+ \beta y), \quad \alpha,\beta \in A(F)/Z(F). \tag{7.3.2}$$

It follows that:

$$\int_{c^{-1}}^{c}\int_{c^{-1}}^{c} K_2(a,b)\eta(\det b)dadb = \int_{c^{-1}}^{c}\int_{c^{-1}}^{c}\sum f(\alpha n_+\beta b)\eta(\det b)dadb;$$

in the right integral, b still varies in the compact subset of $A(F_A)/A(F)Z(F_A)$ defined by $c^{-1} < |b_1/b_2| < c$, but a varies in the subset of $A(F_A)/Z(F_A)$ defined by $c^{-1} < |a_1/a_2| < c$. Let's now introduce the function ϕ on $(F^x)_A \times F_A$ defined by:

$$\phi(x,y) = f\begin{bmatrix} x & 0 \\ 0 & 1\end{bmatrix} \times \begin{bmatrix} 1 & y \\ 0 & 1\end{bmatrix}. \tag{7.3.3}$$

It has compact support. Our integral can be written as:

$$\int\sum_{\zeta}\int \phi(ab^{-1}, b\zeta)\eta(b)dadb,$$

$$\zeta \in F^x, \quad a \in (F^x)_A, \quad c^{-1} < |a| < c, \quad b \in (F^x)_A/F^x, \quad c^{-1} < |b| < c.$$

Using Poisson's formula with respect to the second variable and taking the Fourier transform with respect to the second variable, we obtain for this integral the expression:

$$\int\sum_{\zeta}\int \phi(ab^{-1}, b\zeta)\eta(b)dadb + \int\sum_{\zeta}\int \phi^\wedge(ab, b\zeta)|b|\eta(b)dadb, \quad c^{-1} < |a| < c, \quad 1 < |b| < c.$$

It is evident that the same two integrals extended to the domain:

$$a \in F_A^x, \quad b \in F_A^x/F_A^x, \quad 1 < |b|,$$

converge absolutely. Moreover, in the integrals extended to the preceding domain, we can change a to $ab^{\pm 1}$. We conclude that the integral of K_2 is weakly convergent and that its value is the sum:

$$\iint\sum_{\zeta}\phi(a, b\zeta)\eta(b)dadb + \iint_{\zeta}\phi^\wedge(a, b\zeta)|b|\eta(b)dadb,$$

$$a \in F_A^x, \quad b \in F_A^x/F_A^x, \quad 1 < |b|.$$

This is nothing other than the value at $s = 0$ of the analytic continuation of the following integral:

$$\iint \phi(a,b)|b|^s\eta(b)dadb, \quad a \in F_A^x, \quad b \in F_A^x. \tag{7.3.4}$$

This value will also be written as an integral:

$$\iint \phi(a,b)\eta(b)dadb, \quad a \in F_A^x, \quad b \in F_A^x. \tag{7.3.5}$$

With this convention, we can write that the weak integral of K_2 is equal to:

(7.3.6) $$H(n_+ : f : \eta) = \iint f \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \eta(b) da db.$$

An analogous result holds for the integrals of the other K_i. Finally, we see that the weak integral of K_s exists and is equal to the sum:

(7.3.7) $$H(n_+ : f : \eta) + H(n_- : f : \eta) + H(\varepsilon n_+ : f : \eta) + H(\varepsilon n_- : f : \eta);$$

the first term is defined by (6) and the other terms are defined analogously:

(7.3.8) $$H(n_- : f : \eta) = \iint f \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \eta(b) da db;$$

(7.3.9) $$H(\varepsilon n_+ : f : \eta) = \iint f \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \varepsilon \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \eta(b) da db;$$

(7.3.10) $$H(\varepsilon n_- : f : \eta) = \iint f \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \varepsilon \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \eta(b) da db.$$

7.4. Let's now consider the integral of K_{sp}. Recall the definition of K_{sp}:

$$K_{sp}(x,y) = \mathrm{vol}^{-1} \sum \int f(g)\chi(\det g) d\varepsilon \chi(\det x)\chi^{-1}(\det y),$$

where the sum is over all quadratic characters χ of the idele class group of F and Vol is the volume of the quotient $G(F_A)/G(F)Z(F_A)$. If χ is such a character, then either χ or $\chi\eta$ has a non-trivial restriction to the groups of idele classes of norm 1. Reasoning as for K_1, we immediately see that K_{sp} is weakly integrable and its integral is zero.

8. The kernel of Eisenstein

8.1. We continue with the notations from paragraph 7. We will see that the integral

(8.1.1) $$\iint K_{ei}(a,b)\eta(\det b) da db, \quad a,b \in A(F_A)/A(F)Z(F_A).$$

is weakly convergent. Regarding the value of the integral, as in the now-classical applications of the trace formula, we will only need a rather weak result. Let us indeed choose a place u outside of S that decomposes in E. Let us fix the components of f at other places and consider the integral (8.1.1) as a function of f_u. Let us denote by $f_u^\wedge$ the Satake transform of f_u. We will prove the following:

Proposition. *There exists an integrable function on the real line ϕ and a constant c such that:*

(8.1.2) $$\iint K_{ei}(a,b)\eta(\det b) da db = \int \phi(t) f_u^\wedge(q_u^{-it}) dt + c f_u^\wedge(q^{-1}).$$

8.2. We will need standard results about the Mellin transform of an Eisenstein series. We will fix once and for all a subgroup C of F_A isomorphic to the group of real numbers > 0 such that F_A is the product of C and F^1, the group of ideles of module 1. The group C is equipped with the reciprocal image measure of the measure $t^{-1}dt$ by the application $c \to |c|$ and F^1 with the quotient measure. Unless explicitly stated otherwise, all characters of the group of ideal classes will be assumed trivial on C. Let χ be such a character and $V(\chi)$ the space of functions ϕ on K (the product of K_v) such that:

(8.2.1) $$\phi \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} k = \chi(ab^{-1})\phi(k)$$

if $\begin{vmatrix} a & x \\ 0 & b \end{vmatrix}$ is in K.

Let us now consider a function ϕ on $K \times C$, such that for each complex number u the function $\phi(\cdot, u)$ is in $V(\chi)$. The function will be assumed to be holomorphic, or even meromorphic with respect to u; for example, it can be independent of u. We will extend ϕ to a function $\phi(g, u, \chi)$ on $G(F_A)$ such that:

(8.2.2) $$\phi \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} g, u, \chi = \chi(ab^{-1})|ab^{-1}|^{u+1/2}\phi(g, u, \chi).$$

The Eisenstein series is then the analytic continuation of the series:

(8.2.3) $$E(g, \phi, u, \chi) = \sum \phi(\gamma g, u, \chi), \quad \gamma \in G(F)/A(F)N(F).$$

The series converges absolutely if $\operatorname{Re} u > 1/2$. The constant term of E along N, the group of strictly upper triangular matrices, is by definition the integral

(8.2.4) $$E_N(g, \phi, u, \chi) = \int E(ng, \phi, u, \chi)dn, \quad n \in N(F_A)/N(F).$$

It has the form:

(8.2.5) $$E_N(g, \phi, u, \chi) = \phi(g, u, \chi) + M(u, \chi)\phi(g, -u, \chi^{-1})$$

where $M(u, \chi)$ is the intertwining operator that goes from $V(\chi)$ to $V(\chi^{-1})$. We will also need another Fourier coefficient of E, namely:

(8.2.6) $$W(g, \phi, u, \chi) = \int E \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g, \phi, u, \chi\psi(-x)dx, \quad x \in F_A/F,$$

where ψ is the fixed character of the group F_A/F. The Fourier series of E can thus be written:

(8.2.7) $$E(g, \phi, u, \chi) = \phi(g, u, \chi) + M(u, \chi)\phi(g, -u, \chi^{-1}) + \sum W(\alpha g, \phi, u, \chi)$$

where the sum is for α in $A(F)/Z(F)$. We can also consider a Fourier series for the group N' of strictly lower triangular matrices. Since

(8.2.8) $$N' = wNw^{-1}, \quad \text{with } w = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix},$$

it is actually written:

(8.2.9) $$E(g, \phi, u, \chi) = \phi(wg, u, \chi) + M(u, \chi)\phi(wg, -u, \chi^{-1}) + \sum W(\alpha wg, \phi, u, \chi).$$

The Mellin transform $L(s, \lambda : \phi : u, \chi)$ of E is defined by the following integral (or its analytic continuation):

(8.2.10) $$L(s, \lambda : \phi : u, \chi) = \int \{E[\mathrm{diag}(a,1)] - E_N[\mathrm{diag}(a,1)]\}|a|^{s-1/2}\lambda(a)da, \quad a \in F_A^*/F^*.$$

We have removed from the notation the dependence of E on variables other than the one. By replacing E with its Fourier series, we immediately obtain for the Mellin transform the expression:

(8.2.11) $$\int W[\mathrm{diag}(a,1)]|a|^{s-1/2}\lambda(a)da.$$

We can also write the Mellin transform of E as follows:

(8.2.12) $$L(s, \ldots) = \int_1^{+\infty} (E - E_N) + \int_0^1 (E - E_{N'}) + \int_1^{+\infty} E_N + \int_0^1 E_{N'}.$$

In each of these integrals, the function is evaluated at the point $\mathrm{diag}(a, 1)$ and integrated against $|a|^{s-1/2}\lambda(a)$ over a subset of the group of ideal classes. For the first integral, for example, the subset is defined by the inequality $1 < |a|$. Using the Fourier series of E, we obtain without difficulty another expression for the Mellin transform of E:

(8.2.13) $$\begin{aligned} &\int_1^{+\infty} W[\mathrm{diag}(a,1)]|a|^{s-1/2}\lambda(a)da \\ &+ \int_1^{+\infty} W[\mathrm{diag}(a,1)w]|a|^{s-1/2}\lambda(a)da \\ &+ \int_1^{+\infty} [|a|^{s+u}\lambda\chi(a)\phi(e) + |a|^{s-u}\chi^{-1}(a)M(u,\chi)\phi(e)]da \\ &+ \int_0^1 [|a|^{s-u-1}\chi^{-1}(a)\phi(w) + |a|^{s+u-1}\chi(a)M(u,\chi)\phi(w)]da. \end{aligned}$$

The first two integrals converge for all s and the last two for $\mathrm{Re}\, s > 1/2$. The last two integrals can be easily calculated. In particular, for $s = 1/2$ and u purely imaginary, we obtain the following expression for the Mellin transform of E at the point $s = 1/2$:

$$(8.2.14) \qquad L(1/2, \lambda : \phi : u, \chi) = \int_1^{+\infty} W[\mathrm{diag}(a,1)]\lambda(a)da$$
$$+\int_1^{+\infty} W[\mathrm{diag}(a,1)w]\lambda(a)da$$
$$-\frac{1}{u+1/2}[\phi(w)\delta(\lambda\chi^{-1}) + \phi(e)\delta(\lambda\chi)]$$
$$+\frac{1}{u-1/2}[M(u,\chi)\phi(w)\delta(\lambda\chi) + M(u,\chi)\phi(e)\delta(\lambda\chi^{-1})].$$

where we have defined for any character χ of the group of ideal classes

$$\delta(\chi) = \int \chi(a)da, \quad a \in F_A^1/F^*.$$

We will need to calculate the difference between the Mellin transform and the following integral:

$$(8.2.15) \qquad \int_{c^{-1}}^{c} E[\mathrm{diag}(a,1)]\lambda(a)da.$$

Let us recall that this notation means that the integral is taken over the compact subset of ideal classes a such that $c^{-1} < |a| < c$. Instead of (8.2.12), we have for integral (8.2.15) the expression:

$$(8.2.16) \qquad \int_1^c (E - E_N) + \int_{c^{-1}}^1 (E - E_{N'}) + \int_1^c E_N + \int_{c^{-1}}^1 E_{N'}.$$

By replacing E again with its Fourier series, we obtain for (8.2.15) the expression:

$$(8.2.17) \qquad \int_1^c W[\mathrm{diag}(a,1)]\lambda(a)da + \int_1^c W[\mathrm{diag}(a,1)w]\lambda(a)da$$
$$+\int_1^c [|a|^{1/2+u}\chi\lambda(a)\phi(e) + |a|^{1/2-u}\chi^{-1}\lambda(a)M(u,\chi)\phi(e)]da$$
$$+\int_{c^{-1}}^1 [|a|^{-u-1/2}\chi^{-1}\lambda(a)\phi(w) + |a|^{u-1/2}\chi\lambda(a)M(u,\chi)\phi(w)]da.$$

By calculating the last two integrals and comparing with (8.2.14), we finally obtain the expression we were aiming for:

$$(8.2.18) \qquad \int_{c^{-1}}^c E[\mathrm{diag}(a,1)]\lambda(a)da = L(1/2, \lambda : \phi : u, \chi)$$
$$+\frac{c^{u+1/2}}{u+1/2}\delta(\chi\lambda)\phi(e) + \frac{c^{-u+1/2}}{-u+1/2}\delta(\chi^{-1}\lambda)M(u,\chi)\phi(e)$$
$$+\frac{c^{u+1/2}}{u+1/2}\delta(\chi^{-1}\lambda)\phi(w) + \frac{c^{-u+1/2}}{-u+1/2}\delta(\chi\lambda)M(u,\chi)\phi(w) + R(c).$$

where $R(c)$ is given by:

$$(8.2.19) \qquad -R(c) = \int_c^{+\infty} W[\mathrm{diag}(a,1)]\lambda(a)da + \int_c^{+\infty} W[\mathrm{diag}(a,1)w]\lambda(a)da.$$

It is clear that $R(c)$ tends to zero as c tends to infinity.

8.3. We will need precise estimates for $R(c)$. Recall that R depends not only on c, but also on u, λ and ϕ. Our estimates will be a consequence of the following lemma:

Lemma. *Suppose the function ϕ is independent of u. Then there exists a Schwartz-Bruhat function ϕ such that, for all imaginary u, we have:*

$$|W[diag(a,1),\phi,u,\chi]| \leq \Phi(a)|a|^{-1/2}|L(2u+1,\chi^{2S})|^{-1}.$$

The notation $L(s,\chi^S)$ denotes the product of local factors $L(s,\chi_v)$ for all v not in S. Furthermore, we assume ϕ is invariant under K_v for all v not in S.

Proof. There exists a Schwartz-Bruhat function in two variables ϕ such that:

$$\phi(g,u,\chi) = \int \Phi[(0,t)g]\chi^2(t)|t|^{2u+1}dt \times \chi(\det g)|\det g|^{u+1/2} \times L(2u+1,\chi^{2S})^{-1}.$$

A formal calculation (done in detail in [JL70], §3) gives:

$$W[\mathrm{diag}(a,1),\dots] = L(2u+1,\chi^{2S})^{-1} \times \int \Phi^\wedge(ta,t^{-1})\chi^2(t)|t|^{2u+1}dt\chi(a)|a|^{u+1/2},$$

where $\Phi^\wedge$ is the Fourier transform with respect to the second variable. It will therefore suffice to prove the following assertion: given a Schwartz-Bruhat function $\Phi \geq 0$ in two variables, there exists a Schwartz-Bruhat function in one variable $\phi \geq 0$ such that for all idele a we have:

$$\int \Phi(at,t^{-1})dt \leq \phi(a)|a|^{-1}.$$

Let us consider the analogous local problem. More precisely, let us first consider the case where the local field F is non-archimedean and the function Φ is the characteristic function of integers. Then the integral is nothing but the volume of the set defined by the inequalities $|a| \leq |t| \leq 1$. The integral is thus 0 unless a is an integer. Assuming this is the case, the integral equals $1+v(a)$. Since $q \geq a$ this is less than $q^{v(a)}$. Therefore our integral is at most $\phi(a)|a|^{-1}$, where ϕ is the characteristic function of integers. If F and Φ are arbitrary, the integral, viewed as a function of a, has the form:

$$\int \Phi(at,t^{-1})dt = \phi_1(a) + \phi_2(a)\log|a|$$

where the ϕ_i are Schwartz-Bruhat functions (cf. (4.3)). It is clear that the right-hand side is majorized by $\phi(a)|a|^{-1}$, where ϕ is a suitable Schwartz-Bruhat function. By multiplying these local majorizations, we easily obtain the required global majorization.

It is classical that the function $L(2u+1,\chi^{2S})^{-1}$ has at most polynomial growth on the line $\mathrm{Re}(u)=0$. On the other hand, if ϕ is a Schwartz-Bruhat function, there exists for all $N>0$ a constant $C(N)$ such that

$$\int_c^{+\infty} \phi(a)|a|^{-1}da \leq C(N)c^{-N}.$$

Comparing with the definition (8.2.19) of R, we immediately obtain:

Lemma. *For all N there exist constants $C(N)$ and M such that for all imaginary u we have:*

$$|R(c,u)| \leq C(N)|c|^{-N}|u|^M.$$

Similarly, using expression (8.2.14) for the Mellin transform and the fact that the operator $M(u,\chi)$ is unitary on the imaginary axis, we obtain the following estimate:

Lemma. *On the imaginary axis $M(u,\chi)\phi(k)$ and $L(1/2,\lambda:\phi:u,\chi)$ have at most polynomial growth.*

8.4. Let us now study the integral of the kernel K_{ei}. Let us recall its definition. For any character χ, let us choose an orthonormal basis ϕ_i of the Hilbert space $V(\chi)$; let us denote by $\rho(u,\chi)$ the representation of $\mathrm{G}(F_A)$ by right translations in the space of functions ϕ such that

$$\phi\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} g = \chi(ab^{-1})|ab^{-1}|^{u+1/2}\phi(g). \tag{8.4.1}$$

We can identify the space of $\rho(u,\chi)$ with $\mathrm{V}(\chi)$ and set:

$$F(u,\chi:i,j) = (\rho(u,\chi)\phi_i,\phi_j). \tag{8.4.2}$$

We will write $E_{ei}(x,i,\ldots)$ for $E_{ei}(x,\phi_i,\ldots)$. With these notations:

$$K_{ei}(x,y) = \sum_{\chi} K_\chi(x,y) \tag{8.4.3}$$

where, for each character χ of the group of ideal classes,

$$K_\chi(x,y) = (2\pi i)^{-1}\sum_{i,j}\int_{-i\infty}^{+i\infty} F(u,\chi:i,j)E(x,j,u,\chi)E(y,i,u,\chi)^{-}du. \tag{8.4.4}$$

For a given f, the sums (8.4.3) and (8.4.4) are finite. Let us set

$$I(c,\chi) = \int_{c^{-1}}^{c}\int_{c^{-1}}^{c} K_\chi(a,b)\eta(\det b)dadb. \tag{8.4.5}$$

We can obviously exchange the order of integration for u and the pair (a, b). Taking into account (8.2.18), we obtain for $I(c, \chi)$ the following expression:

(8.4.6)

$$(i\pi)^{-1} \sum_{i,j} \int_{-i\infty}^{+i\infty} F(u, \chi : i, j)$$

$$\times \left[L(1/2, 1 : j, u, \chi) + R(c, u) + \frac{c^{u+1/2}}{u+1/2}\delta(\chi)\phi_j(e) + \frac{c^{-u+1/2}}{-u+1/2}\delta(\chi^{-1})M(u, \chi)\phi_j(e) \right.$$

$$\left. + \frac{c^{u+1/2}}{u+1/2}\delta(\chi^{-1})\phi_j(w) + \frac{c^{-u+1/2}}{-u+1/2}\delta(\chi)M(u, \chi)\phi_j(w) \right]$$

$$\times \left[L(1/2, \eta, i, u, \chi)^{-} + R'(c, u) + \frac{c^{-u+1/2}}{-u+1/2}\delta(\eta\chi)\phi_i^{-}(e) + \frac{c^{u+1/2}}{u+1/2}\delta(\chi^{-1}\eta)M(u, \chi)\phi_i^{-}(e) \right.$$

$$\left. + \frac{c^{-u+1/2}}{-u+1/2}\delta(\chi^{-1}\eta)\phi_i^{-}(w) + \frac{c^{u+1/2}}{u+1/2}\delta(\chi\eta)M(u, \chi)\phi_i^{-}(w) \right] \times du.$$

For each (i, j), the terms $R(c, u)$ and $R'(c, u)$ satisfy the conclusions of lemma (8.3). For a given f, $F(u, \chi : i, j)$ is zero except for a finite number of pairs (i, j). In particular, $F(u, \chi : i, j)$ is zero unless ϕ_i and ϕ_j are both invariant under all K_v with v not in S. Moreover, on the imaginary axis, $F(u, \chi : i, j)$ decreases rapidly (faster than the inverse of any polynomial in u). On the contrary, according to (8.3), the terms $L(\ldots)$ and the terms containing powers of c have slow growth. It follows that when we develop expression (8.4.6), we find a certain number of terms that when c tends to infinity, these terms tend to zero. We can ignore these terms. Among the remaining terms, there is an integral independent of c:

(8.4.7)
$$\sum_{i,j} \int_{-i\infty}^{+i\infty} F(u, \chi, i, j) \times L(1/2, 1 : j : u, \chi) \times [L(1/2, \eta, i, u, \chi)]^{-} du.$$

The other terms are only present if $\chi = 1$ or $\chi = \eta$. Each of these terms is of one of the following types:

(8.4.8)
$$\int F(u, 1 : i, j) L(1/2, \eta : i : u, 1)^{-} \frac{c^{1/2+u}}{1/2+u}(\phi_j(e) + \phi_j(w)) du,$$

(8.4.9)
$$\int F(u, \eta : i, j) L(1/2, 1 : j : u, \eta) \frac{c^{1/2-u}}{1/2-u}(\phi_i^{-}(e) + \phi_i^{-}(w)) du,$$

(8.4.10)
$$\int F(u, 1 : i, j) L(1/2, \eta : i : u, 1)^{-} \frac{c^{1/2-u}}{1/2-u}(M(u, 1)\phi_j(e) + M(u, 1)\phi_j(w)) du,$$

(8.4.11)
$$\int F(u, \eta : i, j) L(1/2, 1 : j : u, \eta) \frac{c^{1/2+u}}{1/2+u}(M(u, \eta)\phi_i^{-}(e) + M(u, \eta)\phi_i^{-}(w)) du.$$

Integral (8.4.7) clearly has the properties required by proposition (8.1). To prove this proposition, it will suffice to show that each of expressions (8.4.8) to (8.4.11) has a limit when c tends to infinity and, moreover, this limit is zero if the Satake transform of the function f_u is zero at point q^{-1}. This last condition means that the integral of f_u on G_u/Z_u is zero and implies that $F(u,1:i,j)$ and $F(u,\eta:i,j)$ vanish at points $u=1/2$ and $u=-1/2$.

8.5. Let us examine term (8.4.8) from number (8.4). We will shift the integration contour from the line $\mathrm{Re}\, u=0$ to the line $\mathrm{Re}\, u=-1/2$; however on this latter line, we will replace the segment joining point $-1/2-i\varepsilon$ to point $-1/2+i\varepsilon$ by the semicircle with center $-1/2$ and radius ε passing through points $-1/2-\varepsilon i$, $\varepsilon-1/2$ and $-1/2+i\varepsilon$. Let us verify that this contour displacement is legitimate. The factor

$$F(u)=F(u,1:i,j)(\phi_j(e)+\phi_j(w))$$

as well as its derivatives, is holomorphic and rapidly decreasing in the vertical strip $-1/2\leq \mathrm{Re}\, u\leq 0$. The exponential function remains bounded. The factor $(1/2+u)^{-1}$ also remains bounded at infinity in this vertical strip. Let us examine the Mellin transform. Recall that we can find an integral representation of $\phi_i(g,u,1)$:

$$\phi_i(g,u,1)=\int \Phi[(0,t)g]|t|^{2u+1}dt\times|\det g|^{u+1/2}L(2u+1,1^S)^{-1}.$$

A simple formal calculation then gives for the Mellin transform (denoted $L(u)$ in abbreviated form):

$$L(u)=L(2u+1,1^S)^{-1}\int \Phi^\wedge(a,b)|a|^{1/2+u}\eta(a)|b|^{1/2-u}\eta(b)dadb, \tag{8.5.1}$$

where $\Phi^\wedge$ is the Fourier transform of Φ with respect to the second variable. Taking the complex conjugate of both sides, we obtain:

$$L(-u^-)^-=L(-2u+1,1^S)^{-1}T(u), \tag{8.5.2}$$

where

$$T(u)=\int \Phi_1(a,b)|a|^{1/2-u}\eta(a)|b|^{1/2+u}\eta(b)dadb. \tag{8.5.3}$$

In this expression Φ_1 is a Schwartz-Bruhat function; the double "Tate" integral $T(u)$, as well as all its derivatives, is bounded in the vertical strip $-1/2\leq \mathrm{Re}(u)\leq 0$. Finally, in the strip in question we have $1\leq \mathrm{Re}(-2u+1)\leq 2$ and the function $L(-2u+1,1_S)^{-1}$ is holomorphic and bounded by a polynomial in $\mathrm{Im}(u)$. As our integral is written:

$$\int F(u)L(1-2u,1^S)^{-1}T(u)c^{1/2+u}(1/2+u)^{-1}du.$$

Our displacement of the integration contour is indeed legitimate. Replacing u by $u - 1/2$, we obtain for term (8.4.8) the following expression:

$$\int F(u-1/2)L(2-2u,1^S)^{-1}T(u-1/2)c^u u^{-1}du. \tag{8.5.4}$$

In (8.5.4) the integration contour is now the line $\mathrm{Re}(u) = 0$, except that the segment joining point $-i\varepsilon$ to point $i\varepsilon$ is replaced by the semicircle of center 0 passing through points $-i\varepsilon$, ε, $i\varepsilon$. Let us now let ε tend to 0. Then the integral over the semicircle tends to

$$i\pi F(-1/2)L(2,1^S)^{-1}T(-1/2),$$

while the integral over the rectilinear part of the contour tends to a "Cauchy principal value". Using a real integration variable t we therefore obtain that (8.5.4) is also equal to:

$$\int_{-\infty}^{+\infty} F(it-1/2)L(2-2it,1^{S-1})T(it-1/2)c^{it}t^{-1}dt + i\pi F(-1/2)L(2,1^S)^{-1}T(-1/2). \tag{8.5.5}$$

For real t, the function $L(-2it+2,1^S)$ is given by an absolutely and uniformly convergent infinite product (or Dirichlet series). Its derivatives are thus bounded and its inverse is also bounded. The derivatives of the factor $L(-2it+2,1^S)^{-1}$ are therefore bounded. In (8.5.5) the product of the first three terms is thus a Schwartz function of t. When c tends to infinity the Cauchy integral tends to $i\pi$ times the value of the Schwartz function at point 0. In total we see that (8.5.5), i.e. term (8.4.8), tends to a finite limit when c tends to infinity, namely:

$$2i\pi F(-1/2)L(2,1^S)^{-1}T(-1/2);$$

this limit vanishes at the same time as $F(-1/2, 1 : i, j)$. This is indeed what we needed to prove. An analogous conclusion applies to term (8.4.9).

8.6. Let us now examine term (8.4.10). For simplification, we will set:

$$F(u) = F(u, 1 : i, j).$$

We will use a slightly different expression from what we have used until now for the Mellin transform.

Let us write ϕ for ϕ_j and assume, as is permissible, that ϕ is a product of local functions ϕ_v. We can also assume that, for each place v, ϕ_v is either K_v invariant, or conversely has zero integral over K_v. Let T be the set of places where this latter condition is satisfied. Then T is finite and contains S. We can find an integral representation for $\phi(g, u, 1)$ of the form:

$$\phi(g,u,1) = \int \Phi[(0,t)g]|t|^{2u+1}dt \times |\det g|^{u+1/2}L(2u+1,1^T)^{-1}. \tag{8.6.1}$$

We conclude, as above, that the Mellin transform appearing in (8.4.10) can be written:

$$L(-2u+1,1^T)^{-1}T(u). \tag{8.6.2}$$

where $T(u)$ is defined by a double Tate integral, holomorphic for all u. On the other hand, we can write the intertwining operator $M(u, 1)$ as a product:

$$M(u,1) = L(2u,1)L(2u+1,1)^{-1}N(u,1) \tag{8.6.3}$$

where N is the normalized intertwining operator. Now the quotient of $L(2u, 1)$ by $L(-2u+1, 1)$ is an exponential function ab^u. It follows that the product of factors (8.6.2) and (8.6.3) reduces to:

$$ab^u L(-2u+1, 1_T)L(2u+1, 1_T)^{-1}L(2u+1, 1^T)^{-1}T(u)N(u,1). \tag{8.6.4}$$

Thus term (8.4.10) is given by the following integral:

$$\int F(u)L(2u+1, 1^T)^{-1}T(u)c^{1/2-u}(1/2-u)^{-1}A(u)du, \tag{8.6.5}$$

with

$$A(u) = ab^u L(-2u+1, 1_T)L(2u+1, 1_T)^{-1}[N(u,1)\phi(e) + N(u,1)\phi(w)].$$

We will shift the integration contour. The present contour is the line $\mathrm{Re}(u) = 0$. The new contour will be the line $\mathrm{Re}(u) = 1/2$, except that the segment joining the points $1/2-i\varepsilon$ and $1/2+i\varepsilon$ will be replaced by the semicircle passing through points $1/2-i\varepsilon$, $1/2-\varepsilon$, $1/2+i\varepsilon$. The end of the proof will then be the same as in the previous case, except that we must show that the factor $A(u)$ is holomorphic and slowly growing in the strip $0 \le \mathrm{Re}(u) \le 1/2$. The ratio of L-factors appearing in A is the product of ratios

$$L(-2u+1, 1_v)L(2u+1, 1_v)^{-1}$$

for all v in T. If v is a finite place, then this ratio is a rational function in q_v^{-u} and thus is slowly growing. If v is infinite, Stirling's formula shows that this ratio has slow growth. Recall that ϕ_v equals one on all K_v for all v not in T. For such a v, we have $N(u, 1_v)\phi_v(k_v) = 1$ for all u. Thus $N(u, 1)\phi(e)$ is in fact the product over all v in T of:

$$N(u, 1_v)\phi_v(e)$$

If v is finite, this still has slow growth. If v is infinite, this is a polynomial in u. Therefore A has slow growth. Let us finally prove the holomorphy of A at the poles of the factor $L(-2u+1, 1_T)$ in the strip. Let us prove for example the holomorphy at $1/2$ of:

$$L(-2u+1, 1_T)L(2u+1, 1_T)^{-1}N(u,1)\phi(e)$$

The preceding product can in fact be written as:

$$\prod_{v\in T} L(-2u+1, 1_v)L(2u+1, 1_v)^{-1}N(u, 1_v)\phi_v(e)$$

Take a v in T. Since the integral of ϕ_v over K_v is zero, $N(u, 1_v)\phi_v(e)$ vanishes at point $u = 1/2$ and this zero compensates for the pole of the factor $L(-2u+1, 1_v)$ at the same point. The product is therefore holomorphic at point $1/2$ and this concludes our discussion for term (8.4.10). An analogous discussion applies to term (8.4.11). The assertions of number (8.1) are therefore completely proved.

9. Global orbital integrals: the case of a compact torus

9.1. In this paragraph F is still a number field and E is a quadratic extension of F. We will fix an element (G, T) of the set $X(E : F)$ and an element ε of $N(T) - T$. Then the square c of ε is an element of F^x and the class cN of the norm group N of E determines the isomorphism class of (G, T). Let f be a smooth function with compact support on the group $G(F_A)/Z(F_A)$. To the function f is attached the cuspidal kernel K_c. Let ϕ_i be an orthonormal basis of the space of automorphic forms that are cuspidal and orthogonal to the functions $g \to \chi(\det g)$, where χ is a trivial square character of the ideal class group of F. Then, by definition:

$$K_c(x, y) = \sum \rho(f)\phi_j(x)\phi_j^-(y). \tag{9.1.1}$$

We propose to give a useful expression for the integral

$$\iint K_c(s, t)dsdt, \quad s, t \in T(F_A)/T(F)Z(F_A). \tag{9.1.2}$$

Of course $\psi \circ \mathrm{Tr}$ is a character of E_A/E and we thus have for each place v of E the Tamagawa measure on the group E_v^x and, by transport of structure on the group T_v. We also have the product measure on the group $T(E_A)$ and the quotient measure on $T(F_A)/Z(F_A)$. We will denote by S a finite set of places of F containing the infinite places, the places ramified in E, the places where G is not split, the places where ψ_v is not of order 0 and the places of residual characteristic 2. We will choose for all v a maximal compact subgroup K_v of G_v such that T_v is contained in $K_v Z_v$ if v does not decompose in E and $G(F_A)$ is the restricted product of the G_v with respect to the K_v. We assume that f is the product of local functions f_v, smooth and with compact support, on G_v/Z_v. We will assume f_v is bi-K_v-invariant for each v not in S. We do not change integral (9.1.1) if for v not decomposed in E we replace f_v by the function f_v' defined by:

$$f_v'(g) = \mathrm{vol}(T_v)^{-1} \int f(s_v g t_v) ds_v dt_v.$$

We can therefore assume that for each v that does not decompose in E the function f_v is bi-T_v-invariant, in particular bi-K_v-finite. Finally, we will assume f_v bi-K_v-finite at places v that decompose in E. We then have:

$$K_c(x, y) = \sum f(x^{-1}\gamma y) - K_{\mathrm{sp}}(x, y) - K_{\mathrm{ei}}(x, y), \tag{9.1.3}$$

where the sum is over all γ in $G(F)/Z(F)$, K_{sp} denotes the special kernel and K_{ei} the Eisenstein kernel. The Eisenstein kernel is of course null if G is not split. The kernel K_{sp} is defined by the following sum:

$$K_{\mathrm{sp}}(x, y) = \sum \mathrm{vol}^{-1} \int f(\det g) dg \chi(\det x)\chi^{-1}(\det y); \tag{9.1.4}$$

the sum is over all characters χ of trivial square of the ideal class group of F and Vol is the volume of the quotient $G(F_A)/G(F)Z(F_A)$. We can define two other kernels:

(9.1.5) $$K_r(x,y) = \sum f(x^{-1}\gamma y), \quad \gamma\ T\text{-regular},$$

(9.1.6) $$K_s(x,y) = \sum f(x^{-1}\gamma y), \quad \gamma\ T\text{-singular}.$$

Then K_c is the following sum:

(9.1.7) $$K_c = K_r + K_s - K_{\mathrm{sp}} - K_{\mathrm{ei}}.$$

Since the quotient $T(F_A)/T(F)Z(F_A)$ is compact, integral (9.1.2) is simply the sum of the integrals of each term in (9.1.7).

9.2. Let us study the integral of K_r. Each element γ of $G(F)/Z(F)$ that is T-regular can be written uniquely in the form:

(9.2.1) $$\gamma = \sigma^{-1}\mu\tau,$$

where σ and τ range over $T(F)/Z(F)$ and μ a set of representatives for the regular double classes $T(F)$ in $G(F)$ (Prop. (1.2)). We therefore immediately obtain:

(9.2.2) $$\iint K_r(s,t)dsdt = \sum \iint f(s^{-1}\mu t)dsdt,$$

the integrals on the right-hand side now both being over $T(F_A)/Z(F_A)$. The double integral on the right-hand side depends only on $\zeta = P(\mu : T)$ and we will denote its value by $H(\zeta : f : T)$. We can therefore write:

(9.2.3) $$\iint K_r(s,t)dsdt = \sum H(\zeta : f : T), \quad \zeta \in cN - 1,$$

since the function P parameterizes the regular double classes and its values, on regular elements, are all points of the class cN associated with the pair (G,T) minus point 1 (Prop. (1.1)). Of course, the orbital integral $H(\zeta : f : T)$ is the product of local orbital integrals:

(9.2.4) $$H(\zeta : f : T) = \prod H(\zeta : f_v : T_v).$$

Almost all factors are equal to 1. Indeed, let v be a place of F that is not in S; suppose that f_v is the characteristic function of Z_vK_v. If v does not decompose in E, then T_v is contained in Z_vK_v and the integral equals 1. If v decomposes in E then the local integral still equals 1 according to proposition (5.7).

9.3. Let us move on to the integral of term K_r. There are only two singular double classes, $T(F)$ and $\varepsilon T(F)$. Therefore we have:

(9.3.1) $$\iint K_s(s,t)dsdt = \mathrm{vol}\int f(t)dt + \mathrm{vol}\int f(\varepsilon t)dt,$$

where vol denotes the volume of the quotient $T(F_A)/T(F)Z(F_A)$ and each of the integrals is over the quotient $T(F_A)/Z(F_A)$.

9.4. Let us move on to the integral of term $K_{\rm sp}$. According to (9.1.4), we have:

$$\iint K_{\rm sp}(s,t)dsdt = \sum \mathrm{Vol}^{-1} \int f(g)\chi(\det g)dg \int \chi(\det s)ds \int \chi^{-1}(\det t)dt; \tag{9.4.1}$$

each of the integrals over the quotient $T(F_A)/T(F)Z(F_A)$ is 0 unless $s \to \chi(\det s)$ is trivial on $T(F_A)$; this is the case if and only if $\chi = 1$ or $\chi = \eta$. The integral of $K_{\rm sp}$ thus reduces to two terms:

$$\iint K_{\rm sp}(s,t)dsdt = \mathrm{Vol}^{-1}\,\mathrm{vol}^2 \left[\int f(g)\eta(\det g)dg + \int f(g)dg\right]. \tag{9.4.2}$$

In particular, let us choose as in (8.1) a place z of F not in S, fix the components of f at places other than z and consider the integral as a function of $f_z^\wedge$. Then integral (9.1.5) is of the form $cf_z^\wedge(q_z^{-1})$, where c is a constant.

9.5. Let us move on to term $K_{\rm ei}$. It is null if G is not split over F. Suppose G is split and let us return to the notations of (8.4). We have:

$$K_{\rm ei}(x,y) = (i\pi)^{-1} \int_{-i\infty}^{+i\infty} A(x,y,u)du, \tag{9.5.1}$$

where

$$A(x,y,u) = \sum_{\chi,j} [\rho(f)E](x,j,u,\chi)E(y,j,u,\chi)^{-}. \tag{9.5.2}$$

Note that the maximal compact subgroup implicit in the definition of Eisenstein series is now the product of groups K_v, with $K_vZ_v = T_v$ if v is infinite. In particular series (9.5.2) is finite. Since we are integrating over a compact set we obtain:

$$\iint K_{\rm ei}(s,t)dsdt = (i\pi)^{-1} \int_{-i\infty}^{+i\infty} A(u:f)du, \tag{9.5.3}$$

where

$$A(u:f) = \sum_{\chi,j} \int [\rho(f)E](s,j,u,\chi)ds \int E(t,j,u,\chi)^{-}dt. \tag{9.5.4}$$

Now $[\rho(f)E](x,j,u,\chi)$ is null unless ϕ_j is K_v-invariant for all places v not in S. In particular, let us choose as above a place z of F that is not in S and decomposes in E. Then $f = f^z \cdot f_z$ where f^z is the product of f_v for $v \neq z$ and

$$[\rho(f)E](x,j,u,\chi) = f_z^\wedge(q_z^{-2iu})[\rho(f^z)E](x,j,u,\chi). \tag{9.5.5}$$

Therefore we have:

$$\iint K_{\rm ei}(s,t)dsdt = (2i\pi)^{-1} \int_{-i\infty}^{+i\infty} f_z^\wedge(q_z^{-2iu})A(u:f^z)du, \tag{9.5.6}$$

where $A(u:f^z)$ is integrable, a result that will be sufficient for our purpose.

10. The fundamental identity

10.1. In this paragraph F is always a number field, E a quadratic extension of F, η the quadratic character attached to E. We will further consider the pair (G, A) formed by the group GL(2) and the subgroup of diagonal matrices A; we will fix an element ε of the normalizer of A that is not in A. We will denote by K_v the usual maximal compact subgroup of G_v, and we will assume ε is contained in K_v for all v. We will give ourselves a finite set S of places of F, containing the infinite places, the places that ramify in E, the places where ψ is not of order 0 and the places of residual characteristic 2. It will be convenient to assume that S has an even number of elements. Let $X(S)$ be the set of pairs (G', T') in $X(E : F)$ such that G' splits outside of S. For each (G', T') in $X(S)$ and each place v, we will choose a maximal compact subgroup K'_v of G'_v such that $G'(F_A)$ is the restricted product of the G'_v with respect to the K'_v. We will assume that if v does not decompose in E then T'_v is contained in $K'_v Z_v$. For all v not in S the measures of $T'_v \cap K'_v / K'_v \cap Z'_v$ and $A_v \cap K_v / K_v \cap Z_v$ are 1. We will fix an element ε' of the normalizer of T' that is not in T' and we will assume that ε' is in K'_v for all v not in S. We will give ourselves a smooth function with compact support f on $G(F_A)/Z(F_A)$ and, for each (G', T') in $X(S)$, a smooth function with compact support f' on $G'(F_A)/Z'(F_A)$. Of course these functions will be assumed to be products of local functions. We will furthermore make the following hypotheses:

10.1.1. Let v be a place in S that does not decompose in E. Then f'_v is T'_v-biinvariant. Moreover if x is an element of F_v different from 1 and 0, (G', T') an element of $X(S)$ and g' an element of G'_v such that $x = P(g' : T_v)$ then (cf. §4):

$$H(x : f_v : \eta_v) = H(g' : f'_v : T'_v).$$

10.1.2. Let v be a place of S that decomposes in E. Then f_v is K_v-finite and f'_v is K'_v-finite. Let g be an A_v-regular element of G_v. If $(G', T') \in X(S)$ and $g' \in G'_v$ are such that

$$P(g : A_v) = P(g' : T'_v)$$

then:

(i). $H(g : f_v : A_v) = H(g' : f'_v : T'_v)$;

(ii). $\int f_v(a_v) da_v = \int f'_v(t'_v) dt'_v$;

(iii). $\int f_v(\varepsilon a_v) da_v = \int f'_v(\varepsilon' t'_v) dt'_v$.

10.1.3. If v is not in S then f_v is K_v-biinvariant, f'_v K'_v-biinvariant and any isomorphism of the pair (G_v, K_v) onto the pair (G'_v, K'_v) transforms f_v into f'_v.

10.1.4. *Remark.* In the situation of condition (10.1.2) there exists an isomorphism of the pair (G_v, A_v) onto the pair (G'_v, T'_v). Condition (10.1.2) is satisfied if we take for f'_v the image of f_v under this isomorphism. Indeed this is clear for ((10.1.2).i) and ((10.1.2).ii). For ((10.1.2).iii), the integral on the right-hand side does not change if we replace ε' by the image of ε under the isomorphism in question and then our assertion is evident.

To the function f is associated the cuspidal kernel K_c for the group G. Similarly, for each (G', T'), to the function f' is associated the cuspidal kernel K'_c for the group G'. We will prove in this paragraph the following result:

Theorem. *With the preceding hypotheses and notations:*

$$\iint K_c(a,b)\eta(\det b)dadb = \sum_{(G',T')} \iint K'_c(s,t)dsdt, \quad (G',T') \in X(S). \tag{10.1.1}$$

10.2. To prove our identity we will write, as in paragraphs 7 and 9:

$$K_c = K_r + K_s - K_{sp} - K_{ei}, \tag{10.2.1}$$

$$K'_c = K'_r + K'_s - K'_{sp} - K'_{ei}. \tag{10.2.2}$$

We will first prove the following identities:

$$\iint K_r(a,b)\eta(\det b)dadb = \sum_{(G',T')} \iint K'_r(s,t)dsdt, \tag{10.2.3}$$

$$\iint K_s(a,b)\eta(\det b)dadb = \sum_{(G',T')} \iint K'_s(s,t)dsdt. \tag{10.2.4}$$

Supposing these identities proven, let us show how the theorem follows. Consider the difference:

$$\iint K_c(a,b)\eta(\det b)dadb - \sum_{(G',T')} \iint K'_c(s,t)dsdt. \tag{10.2.5}$$

Given (10.2.3) and (10.2.4), it can be written as:

$$- \iint K_{sp}(a,b)\eta(\det b)dadb + \sum_{(G',T')} \iint K'_{sp}(s,t)dsdt$$

$$- \iint K_{ei}(a,b)\eta(\det b)dadb + \sum_{(G',T')} \iint K'_{ei}(s,t)dsdt.$$

Let us recall that for group G these are weak integrals.

Let us now choose a place z of E that is not in S and that decomposes in E. Let us fix the components of f and f' at other places. At place z, the Satake transforms of f_z and f'_z

are the same. We can therefore consider our integrals as functions of $f_z^\wedge$. Then according to (8.1), (9.3) and (9.4), the sum above has the form:

$$\int_{-\infty}^{+\infty} \phi(t) f_z^\wedge(q_z^{-2it})dt + c f_z^\wedge(q_z^{-1}), \tag{10.2.6}$$

where ϕ is integrable. We complete the proof as in [Lan80] using the fact that the integrals of K_c and K_c' also have the form:

$$\sum_t a_t f_z^\wedge(t), \tag{10.2.7}$$

where the complex numbers t are either on the unit circle or on the real axis between q_z^{-1} and q_z, and the series of a_t is absolutely convergent. The uniqueness of the decomposition of a measure into an atomic measure and a continuous measure implies that the difference (10.2.5) is in fact zero.

10.3. Let's prove equality (10.2.3). The left-hand side can be written as:

$$\sum_\zeta H(\zeta : f : \eta), \quad \zeta \neq 0, 1,$$

while the right-hand side can be written as a double sum:

$$\sum_{(G',S')} \sum_\zeta H(\zeta : f' : T')$$

where the inner sum is over all ζ in the class cN, excluding point 1, determined by the pair (G', T'). We can recombine the two sums and write the right-hand side as a sum

$$\sum_\zeta H(\zeta : f' : T), \quad \zeta \in N(S) - 1,$$

where $N(S)$ denotes the union of the classes cN determined by the elements of $X(S)$.

According to class field theory, the elements of $F^x - N(S)$ are exactly the ζ in F^x that satisfy the following condition: there exists a place v of F, which is not in S, which does not decompose in F and such that ζ is not a norm of the quadratic extension E_v of F_v. According to Proposition (5.1), we have, for such a ζ, $H(\zeta : f_v : \eta_v) = 0$ if v is the place in question. It follows that $H(\zeta : f : \eta) = 0$. Therefore, it suffices to prove the equality of the orbital integrals $H(\zeta : f : \eta)$ and $H(\zeta : f' : T')$ if ζ is in $N(S)$. Let us decompose these integrals into products of local integrals $H(\zeta : f_v : \eta_v)$ and $H(\zeta : f_v' : T_v')$ respectively. For v in S, the equality of these integrals follows from hypotheses (10.1.1) and (10.1.2). For v not in S, the equality follows from hypothesis (10.1.3) and proposition (5.1). The equality of global orbital integrals, and formula (10.2.3), are thus established.

10.4. Let us now prove equality (10.2.4). We can use formula (7.3.7) to calculate the left-hand side and formula (9.3.1) to calculate the right-hand side. Equality (10.2.4) will then be a consequence of the following two equalities:

$$H(n_+ : f : \eta) + H(n_- : f : \eta) = \sum_{(G',T')} vol(T'(F_A)/T'(F)Z'(F_A)) \int f'(t)dt, \tag{10.4.1}$$

$$H(\varepsilon n_+ : f : \eta) + H(\varepsilon n_- : f : \eta) = \sum_{(G',T')} vol(T'(F_A)/T'(F)Z'(F_A)) \int f'(\varepsilon' t)dt. \tag{10.4.2}$$

The second identity follows from the first identity applied to the function f_1 defined by $f_1(g) = f(\varepsilon g)$ and to the functions f_1' defined by $f_1'(g) = f'(\varepsilon' g)$. It is indeed easy to verify that conditions (10.1.1) to (10.1.3) are satisfied by f_1 and the f_1'. Let us therefore prove the first identity.

Let us calculate the right-hand side of (10.4.1). Let us introduce a differential ideal a of E and b of F. The analytic continuation of the Tate integral

$$\int \phi(t)|t|^s \eta(t)dt, \tag{10.4.3}$$

where ϕ is a Schwartz-Bruhat function, takes at point $s = 0$ the value:

$$L(0,\eta) \prod_{v\in T} \int \phi_v(t_v)\eta(t_v)dt_v L(0,\eta_v)^{-1} \prod_{v\in V} \phi_v(0)|a_v|^{1/2},$$

where T is the set of places of F that do not decompose in E and V the set of those that do decompose.

Let us apply this formula to the functions ϕ_+ and ϕ_- defined by:

$$\phi_+(x) = \int f\left[a\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right] da, \quad a \in A(F_A)/Z(F_A),$$

$$\phi_-(x) = \int f\left[a\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}\right] da, \quad a \in A(F_A)/Z(F_A).$$

The local components of ϕ_+ and ϕ_- are defined analogously in terms of the local components of f. Then the right-hand side of (10.4.1) is nothing but the sum of the values of the Tate integrals (10.4.3) of ϕ_+ and ϕ_- at point $s = 0$. Moreover, we obviously have for each v in V:

$$\phi_{+,v}(0) = \phi_{-,v}(0) = \int f_v(a_v)da_v, \quad a_v \in A_v/Z_v.$$

On the other hand, for each v in T, the values at point 0 of the Tate integrals of $\phi_{+,v}$ and $\phi_{-,v}$ are nothing but the singular orbital integrals of points n_+ and n_-. Let us therefore set, to simplify the notation:

$$M_v = \int f_v(a_v)da_v \quad \text{for } v \text{ in } V,$$

$$M_{v\pm} = 2H(n_\pm : f_v : \eta_v) \quad \text{for } v \text{ in } T.$$

Then the left-hand side of equality (10.4.1) can be written:

$$L(0,\eta)\prod_T 1/2L(0,\eta_v)^{-1}\prod_V |a_v|^{1/2}\times\prod_V M_v[\prod_T M_{v+}+\prod_T M_{v-}]. \tag{10.4.4}$$

Note that in each of the infinite products, almost all factors are equal to 1.

Let's move to the second member of (10.4.1). The volume that appears there is none other than $2L(1,\eta)$, as is well known. The integral is obviously the product of analogous local integrals:

$$\int f'(t)dt=\prod_v\int f'_v(t_v)dt_v.$$

If v is in V, the local integral is none other than M_v according to hypothesis (10.1.2)(ii). If v is in T, then the integral is equal to

$$\mathrm{vol}(T_v/Z_v)^{-1}1/2[M_{v-}+\eta_v(c)M_{v+}]$$

according to proposition (4.1) and proposition (5.1). The volume that appears in this formula is none other than $|b_w|^{1/2}|a_v|^{-1/2}$, where w is the unique place of E above v. In total, the right-hand side of (10.4.1) is equal to the following product:

(10.4.5)

$$2L(1,\eta)\sum_{c\in N(S)/N}\prod_{v\in T}1/2L(0,\eta_v)^{-1}\prod_{v\in T}|b_w|^{-1/2}|a_v|^{1/2}\prod_{v\in V}M_v\prod_{v\in T}1/2[M_{v-}+\eta_v(c)M_{v+}].$$

Comparing with (10.4.4), we see that it suffices to prove the following equalities:

$$L(0,\eta)\prod_{v\in V}|a_v|^{1/2}=L(1,\eta)\prod_{v\in T}|b_w|^{-1/2}|a_v|^{1/2}, \tag{10.4.6}$$

$$\prod_{v\in T}M_{v+}+\prod_{v\in T}M_{v-}=2\sum_{c\in N(S)/N}\prod_{v\in T}1/2[M_{v-}+\eta_v(c)M_{v+}]. \tag{10.4.7}$$

Equality (10.4.6) follows immediately from the functional equations of functions $L(s,1_E)$ and $L(s,1_F)$ and their relation with $L(s,\eta)$.

Let's move to equality (10.4.7). For $v\in T-S$ we have $\eta_v(c)=1$ according to the definition of $N(S)$ and $M_{v+}=M_{v-}$ (Prop. (5.1)); moreover, for almost all $v\in T-V$, $M_{v+}=M_{v-}=1$. Setting $U=T\cap S$, we see that identity (10.4.7) reduces to:

$$\prod_{v\in U}M_{v+}+\prod_{v\in U}M_{v-}=2\sum_{c\in N(S)/N}\prod_{v\in U}1/2[M_{v-}+\eta_v(c)M_{v+}].$$

Let H be the group $\{1,-1\}^U$. For each v in U, let us define a character χ_v of H by the formula $\chi_v(h)=h_v$. Let H' be the subgroup of H defined by the equation $\prod_{v\in U}\chi_v(h)=1$. Then the mapping $c\to(\eta_v(c))$ defines a bijection from $N(S)/N$ to H'. With this notation, the formula to be proved becomes

$$\prod_{v\in U}M_{v+}+\prod_{v\in U}M_{v-}=2\sum_{h\in H}\prod_{v\in U}1/2[M_{v-}+\chi_v(h)M_{v+}].$$

Since $[H:e] = 2[H':e]$, the right-hand side can be written:

$$[H':e]^{-1} \sum_{Y} \sum_{h \in H'} \prod_{v \in Y} \chi_v(h) \prod_{v \in Y} M_{v+} \prod_{v \in U-Y} M_{v-},$$

where the outer sum is over all subsets Y of U. The character $\prod_{v \in Y} \chi_v$ has a non-trivial restriction to H', unless Y is empty or equal to U. In the preceding sum, therefore, only the terms corresponding to the empty set and U remain; this gives us our equality.

11. The result of Waldspurger

11.1. We will finally prove Waldspurger's result using the identity from paragraph (10). We will denote by S a finite set of places of F satisfying the conditions of paragraph (10). Since we can take S arbitrarily large, there will be no issue in only considering cuspidal representations of G that are unramified outside S, couples (G', T') belonging to $X(S)$, and for such a couple, cuspidal representations of G' that are unramified outside S. We will denote by K (resp. K^S) the product of compact subgroups K_v for all v (resp. all v not in S) and G^S the restricted product of G_v for v not in S. For a couple (G', T') in $X(S)$, the notations K', K'^S and G'^S have analogous meanings.

To specify the first Waldspurger condition, let us note that if the integrals

$$\int \phi(a)da \quad \text{and} \quad \int \phi(bc)\eta(\det b)db$$

are not zero for a couple of smooth vectors, they are not zero for a couple of K-finite vectors (ϕ, ϕ'); moreover if S is large enough, then ϕ and ϕ' are K^S-invariant. Similarly if there exists a couple (G', T') in X, a cuspidal representation π' and a smooth vector ϕ in the space of π' such that the integral $\int \phi(t)dt$ is not zero, then we can take ϕ K'-finite; moreover, if S is large enough, (G', T') is in $X(S)$ and ϕ is invariant under K'^S.

11.2. Let us therefore consider a set S and functions f and f' satisfying the conditions of paragraph (10). In particular f_v (resp. f'_v) is bi-invariant under K_v (resp. K'_v). Consider the kernel K_c. We can write it as:

$$K_c = \sum_{\pi} K_\pi, \tag{11.2.1}$$

where, for each cuspidal automorphic representation (unramified outside S) π, we have defined:

$$K_\pi(x,y) = \sum_j \rho(f)\phi_j(x)\phi_j^-(y), \tag{11.2.2}$$

ϕ_j denoting an orthonormal basis of the subspace of K^S-invariant vectors in the space of π. We will assume the ϕ_j are K-finite. The series (11.2.1) converges not only in the Hilbert space of square-integrable functions on the quotient $G(F_A)/G(F)Z(F_A)$, but also in the space of rapidly decreasing functions on the quotient $G(F_A)/G(F)Z(F_A)$. Moreover, since f_v is K_v-finite for infinite v, the series (11.2.2), for a given f, has only a finite number of non-zero terms. Let us denote by $H(S)$ the Hecke algebra of group G^S relative to subgroup K^S. Let us write $f = f_S f^S$, where f_S (resp. f^S) is the product of f_v for v in S (resp. not in S). Let Λ_π be the character of $H(S)$ attached to a representation π. Then we have:

$$\iint K_c(a,b)\eta(\det b)dadb = \sum_\pi a(\pi, f_S)\Lambda_\pi(f^S), \tag{11.2.3}$$

where we have defined:

$$a(\pi, f_S) = \sum_j \int \rho(f_S)\phi_j(a)da \int \phi_j^-(b)\eta(\det b)db. \tag{11.2.4}$$

11.3. Let us similarly consider a couple (G', S') in $X(S)$. We still have a decomposition

$$K'_c = \sum_{\pi'} K_{\pi'}, \tag{11.3.1}$$

$$K_{\pi'}(x,y) = \sum_j \rho(f')\phi_j(x)\phi_j^-(y), \tag{11.3.2}$$

where ϕ_j is an orthonormal basis of the space of K'^S-invariant vectors of π. The series (11.3.1) still converges in the space of rapidly decreasing functions and series (11.3.2) is finite. By integrating term by term we find:

$$\iint K_{\pi'}(s,t)dsdt = a(\pi', f'_S)\Lambda_{\pi'}(f'^S), \tag{11.3.3}$$

where we have defined

$$a(\pi', f'_S) = \sum_j \int \rho(f')\phi_j(s)ds \int \phi_j^-(t)dt. \tag{11.3.4}$$

The total integral is therefore:

$$\iint K'_c(s,t)dsdt = \sum_{\pi'} a(\pi', f'_S)\Lambda_{\pi'}(f'^S).$$

11.4. Let us now use our fundamental identity. Note that if π' is a cuspidal representation of G' and π the cuspidal representation of G that corresponds to it, then $\Lambda_\pi(f_S) = \Lambda_{\pi'}(f'_S)$. Since π' determines π we can write $a(\pi, f'_S)$ for $a(\pi', f'_S)$. On the other hand, for a given representation π for group G, it will be convenient to set $a(\pi, f'_S) = 0$ if there does not exist a representation π' for G' corresponding to π. Then our fundamental identity becomes:

$$\sum_\pi a(\pi, f_S)\Lambda(\pi, f^S) = \sum_\pi [\sum_{(G',T')} \sum a(\pi, f'_S)]\Lambda(\pi, f^S). \tag{11.4.1}$$

In this formula f^S is an arbitrary element of $H(S)$. Let v be a place in S. Then the function f_v is arbitrary K_v-finite. The function f'_v is related to f_v by conditions (10.1.1) or (10.1.2). If v decomposes, f'_v is in fact arbitrary K'_v-finite (cf. (10.1.1)). If on the contrary v does not decompose, then f'_v is no longer arbitrary but satisfies a density condition: if a continuous function h on G_v/Z_v is bi-T_v-invariant and orthogonal to all possible f'_v then h is zero (Prop. (4.2)).

Suppose that π satisfies the first Waldspurger condition; then there exist K-finite vectors ϕ and ϕ' in the space of π such that:

$$\int \phi(a)da \neq 0 \quad \text{and} \quad \int \phi'(b)\eta(\det b)db \neq 0,$$

and we can suppose ϕ and ϕ' are K^S-invariant. Let us choose the basis ϕ_j such that $\phi_1 = \phi'/\|\phi'\|^{-1}$. There exists f_S such that $\rho(f_S)\phi_1 = \phi$ and $\rho(f_S)\phi_j = 0$ if $j \neq 1$. Then we have:

$$a(\pi, f_S) = \int \phi(a)da \left[\int \phi_1(b)\eta(\det b)db\right]^{-} \neq 0.$$

According to the principle of "infinite" linear independence of characters of $H(S)$ ([Lan80, p. 211]), there exists at least one (G', T') such that $a(\pi, f'_S)$ is not zero. It evidently follows that there exists at least one ϕ in the space of π' such that $\int \phi(t)dt$ is not zero. Thus π satisfies the second Waldspurger condition.

Suppose now that there exists a couple (G', T'), a representation π' and a K'-finite vector ϕ in the space of π' such that the integral $\int \phi(t)dt$ is not zero; we can suppose that (G', T') is in $X(S)$ and ϕ is invariant under K'^S. We will see that we can choose f'_S such that $a(\pi', f'_S)$ is not zero. The integral over T defines a continuous linear form on the space of smooth vectors of π' fixed by K'^S. Let us write it as the scalar product with a "generalized" vector e_T:

$$\int \phi(t)dt = (\phi, e_T).$$

If h is a smooth function with compact support on G_S/Z_S, then $\pi'(h)(e_T)$ is defined: it is a smooth vector such that $(\phi, \pi'(h)e_T) = (\pi'(h^*)\phi, e_T)$ for all vectors ϕ, smooth or not. With this notation we have:

$$a(\pi, f'_S) = (\pi'(f'_S)e_T, e_T).$$

The subspace of π' formed by K'^S-invariant vectors is isomorphic to the tensor product of the spaces of π'_v with v in S. For each v in S, there exists a non-zero continuous linear form e_v on the space of smooth vectors of π'_v that is invariant under T_v. This form is unique up to a scalar factor (cf. (6.1) and (6.2)), and we can therefore write:

$$a(\pi', f'_S) = (\pi'(f'_S)e_T, e_T) = C \prod_{v \in S} (\pi_v(f'_v)e_v, e_v),$$

where C is a non-zero constant. It remains to see that we can choose f'_v such that $(\pi_v(f'_v)e_v, e_v)$ is not zero. This is evident if v decomposes since f'_v is then arbitrary K'_v-finite. If v decomposes, e_v is in fact an ordinary vector since T_v is compact. Then $(\pi_v(f'_v)e_v, e_v)$ is the scalar product of the function f'_v with the continuous matrix coefficient $(\pi'(g)e_v, e_v)$; it therefore cannot be zero for any choice of f'_v according to the density property of f'_v. On the other hand, if (G'', T'') is another element of $X(S)$, then $a(\pi, f''_S) = 0$; otherwise there would exist at least one place v in S where the groups G'_v and G''_v are not isomorphic and the representations π'_v and π''_v admit non-zero vectors invariant under T'_v and T''_v respectively. But this is impossible (Proposition (6.3)). The coefficient of Λ_π in the second member of (11.4.1) is therefore non-zero, for a suitable choice of f'_S (i.e. f_S). It follows as above that $a(\pi, f_S)$ is not zero. This evidently implies that π satisfies the first Waldspurger condition. This therefore completes the demonstration of the equivalence of the two Waldspurger conditions.

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