ON THE MODULI DESCRIPTION OF RAMIFIED UNITARY LOCAL MODELS OF SIGNATURE \((n - 1, 1)\)

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Abstract. We give a moduli description for the ramified unitary local model of signature \((n - 1, 1)\) with any parahoric level structure when the hermitian form splits and the residue field has characteristic greater than 2, confirming a conjecture of Smithling [Sm15]. As applications, we can present moduli descriptions for: (1) ramified unitary Pappas-Zhu local models with any parahoric level; (2) the irreducible components of their special fiber with maximal parahoric level; (3) integral model of ramified unitary Shimura varieties for any parahoric level.

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1. Introduction

1.1. Background. Given a Shimura variety \(\text{Sh}_K(G, h)\) defined over the reflex field \(E\). One of the fundamental problems in arithmetic geometry is constructing and studying nice integral models of \(\text{Sh}_K(G, h)\) defined over the ring of integers \(\mathcal{O}_E\). A minimal requirement for such models is to be flat over \(\text{Spec} \mathcal{O}_E\). Local models are one of the most important tools for testing such models. They are certain projective schemes constructed from linear algebra data.

Suppose \(\mathcal{O}\) is the completion of \(\mathcal{O}_E\) at a non-archimedean place. Ideally, given an integral model \(\mathcal{S}_K(G, h)\) of the Shimura variety over \(\text{Spec} \mathcal{O}\), there exist a corresponding local model...
$M_K(G, h)$ over Spec $O$, and vice versa. Moreover, their geometric structures are related by the local model diagram:

\[
\begin{array}{ccc}
\tilde{S}_K(G, h) & \xleftarrow{\pi} & S_K(G, h) \\
& \text{or} & \\
M_K(G, h) & \text{or} & \varphi
\end{array}
\]

where $\pi$ is a principal homogeneous space under the action of a smooth algebraic group scheme over $O$, $\varphi$ is a smooth morphism. These morphisms have the same relative dimension, hence $S_K(G, h)$ and $M_K(G, h)$ are étale locally identified; cf. [RZ96]. Consequently, we can reduce questions of local nature on the integral model $S_K(G, h)$ to the corresponding questions on the local model $M_K(G, h)$. Since local models are constructed from linear algebra data, they should be easier to study than the integral models $S_K(G, h)$ themselves.

Motivated by this, it’s natural to ask the existence of “nice” local models. In the celebrated paper [PZ13], Pappas and Zhu gave a uniform group-theoretic construction of the local models for tamely ramified groups. This construction was further refined by He, Pappas, and Rapoport [HPR20], among others. Such models have been confirmed to be “canonical” in the sense of Scholze-Weinstein; cf. [AGLR22].

On the other hand, since the group-theoretic construction cannot provide a moduli description, it remains an interesting problem to find a moduli description for Pappas-Zhu’s model. In the PEL setting, Rapoport and Zink [RZ96] previously constructed natural moduli functors of integral models of Shimura varieties and their local models based on the lattice models. However, Pappas [Pa00] observed that such functors are not always flat. It has been known now that the Pappas-Zhu model is a closed subscheme of the Rapoport-Zink model ([HPR20, Lemma 4.1]). The question becomes modifying Rapoport-Zink’s moduli functor to get a moduli description for the Pappas-Zhu model.

1.2. Ramified unitary local model. In this paper, we will be interested in the local model associated with the unitary group defined by a hermitian form over a quadratic extension $F/F_0$ of a $p$-adic field. When $F/F_0$ is split or unramified, only algebraic groups of type $A$ are involved, and Görtz [Gö01] has shown the flatness of the Rapoport-Zink models. By contrast, when $F/F_0$ is ramified, the attached Rapoport-Zink models can fail to be flat; cf. [Pa00].

To be more precise, let $F/F_0$ be a ramified quadratic extension of $p$-adic fields with $p \neq 2$. Let us note that the $p = 2$ case is fundamentally more difficult; we do not omit it merely for simplicity. We will keep this assumption throughout the paper. Let $n \geq 3$ be an integer, and let $m := \lfloor n/2 \rfloor$. We will also fix a pair $(r, s)$ with $r + s = n$, called the signature. In this paper, we are interested in the case when $(r, s) = (n - 1, 1)$. Suppose the hermitian form is split. In this case, the unitary group is quasi-split and we can label the conjugacy class of
the parahoric subgroups by nonempty subsets of $I \subset \{0, \ldots, m\}$ such that

\[(1.2.1) \quad \text{if } n \text{ is even, then } m - 1 \in I \Rightarrow m \in I \text{ (for odd } n, \text{ no condition on } I \text{ is imposed)}.\]

Such index sets are related to the local Dynkin diagram; cf. [PR08, 4.a] or §5.1.1. Given such a set, we define the Rapoport-Zink local model $M_I^{\text{naive}}(r,s)$ associated with the quasi-split unitary group, called the “naive” local model, since it is not generally flat; see §2.2 for the definition.

In [PR08, §11], Pappas and Rapoport defined the local model $M_I^{\text{loc}}(r,s)$ as the scheme-theoretic closure of the generic fiber of $M_I^{\text{naive}}(r,s)$. This model identifies with the corresponding Pappas-Zhu local model, cf. Proposition 6.1.3. As a first step toward a moduli description of the $M_I^{\text{loc}}(r,s)$, Pappas [Pa00] added the wedge condition to the moduli problem of the naive model; cf. §2.3. Denoting by $M_I^\wedge(r,s)$ the closed subscheme of $M_I^{\text{naive}}(r,s)$ cut out by the wedge condition, we get a chain of closed subschemes of the naive model, which are equalities over the generic fiber:

$$M_I^{\text{loc}}(r,s) \subset M_I^\wedge(r,s) \subset M_I^{\text{naive}}(r,s).$$

In the self-dual maximal parahoric case, i.e., when $I = \{0\}$, Pappas conjectured that $M_{I(0)}^\wedge(r,s) = M_{I(0)}^{\text{loc}}(r,s)$, and verified this for the signature $(r,s) = (n-1,1)$.

However, for the other indices $I$, the wedge condition is not enough. The next advance came in [PR09] where Pappas and Rapoport introduced the spin condition; cf. §2.3. Denote by $M_I^{\text{spin}}(r,s)$ the closed subscheme of $M_I^\wedge(r,s)$ cut out by the spin condition. Pappas and Rapoport conjectured that $M_I^{\text{spin}}(r,s) = M_I^{\text{loc}}(r,s)$. Later, Rapoport, Smithling and Zhang [RSZ17] verified that when $n$ is even, the inclusion $M_I^{\text{loc}}_{\{m\}}(n-1,1) \subset M_I^{\text{spin}}_{\{m\}}(n-1,1)$ is an equality. In [Sm11, Sm14], Smithling showed that the spin models have the same underlying topological space as the local models under the inclusion $M_I^{\text{loc}}(r,s) \subset M_I^{\text{spin}}(r,s)$. However, in [Sm15], Smithling, in response to a question of Rapoport, showed that $M_I^{\text{spin}}_{\{m\}}(n-1,1)$ is not equal to $M_I^{\text{loc}}_{\{m\}}(n-1,1)$ for odd $n \geq 5$. In the same paper, he came up with the strengthened spin condition, which defines a closed subscheme $M_I(r,s)$; cf. §2.4. The strengthened spin model fits into the chain of closed immersions:

$$M_I^{\text{loc}} \subset M_I \subset M_I^{\text{spin}} \subset M_I^\wedge \subset M_I^{\text{naive}}.$$  

Smithling conjectured that $M_I(r,s) = M_I^{\text{loc}}(r,s)$, and showed that $M_{\{m\}}(n-1,1) = M_{\{m\}}_{\{m-1,m\}}(n-1,1)$ for odd $n \geq 5$. Later in Yu’s thesis [Yu19], she verified that when $n = 2m$,

we have $M_{\{m-1,m\}}^{\text{loc}}(n-1,1) = M_{\{m-1,m\}}(n-1,1)$.

The main result of this paper is the following:

**Theorem 1.2.1.** Suppose $n \geq 3$. For any $I \subset \{0, \ldots, m\}$ satisfying (1.2.1), the inclusion $M_I^{\text{loc}}(n-1,1) \subset M_I(n-1,1)$ is an equality. In other words, for signature $(n-1,1)$, the strengthened spin model is a moduli description of the quasi-split ramified unitary local model for any parahoric level structure.
We further obtain the moduli description for the ramified unitary local model associated with any hermitian form by some descent arguments. See §6.1 for the precise formalism. The result is the following:

**Theorem 1.2.2.** Suppose $(G, \{\mu\}, K)$ is a local model triple [HPR20, §2.1] where $G = \text{GU}(V, \phi)$ is the unitary similitude group of some hermitian space $(V, \phi)$ of a quadratic extension $F/F_0$, which is not necessarily split, with $\dim V \geq 3$. Assume that the geometric cocharacter class $\{\mu\}$ gives the signature $(n - 1, 1)$, and $K$ is a parahoric subgroup of $G(F)$. The Pappas-Zhu local model $M^\text{loc}_K(G, \{\mu\})$ admits a moduli description as in §6.1.4, using the strengthened spin condition.

### 1.3. Ramified unitary local models of strongly non-special parahoric subgroups.

Using the theory of (partial) affine flag varieties and their relation to local models, we can reduce the proof of Theorem 1.2.1 into cases when the parahoric subgroups are maximal; see §5 for details of the reduction. Furthermore, we will focus on the following cases.

**Definition 1.3.1.** Suppose $n \geq 3$. A non-empty subset $I \subset \{0, 1, \cdots, m\}$ is called strongly non-special if: (1) it equals a single set $\{\kappa\}$; (2) when $n = 2m$, $\kappa$ is not equal to $0, m - 1, m$; (3) when $n = 2m + 1$, $\kappa$ is not equal to $0, m$.

In §2.1.2, we will relate non-empty index sets $I$ with parahoric subgroups. A strongly non-special index set $I$ will correspond to a maximal parahoric subgroup which is neither special nor self-dual.

We will prove that the strengthened spin models represent local models in the strongly non-special cases. The main result is the following:

**Theorem 1.3.2.** Suppose $n \geq 5$, the signature is $(n - 1, 1)$, $I = \{\kappa\}$ is strongly non-special, and the local model $M^\text{loc}_{\{\kappa\}}$ is defined over the discrete valuation ring $\mathcal{O}_F$ with residue field $k$.

(i) The strengthened spin model $M_{\{\kappa\}}$ is equal to the local model $M_{\{\kappa\}}$. It is flat, normal, and Cohen-Macaulay of dimension $n$.

(ii) The special fiber of the local model $M^\text{loc}_{\{\kappa\}, s}$ is reduced. It consists of two irreducible components $Z_1$ and $Z_2$. Both are normal and Cohen-Macaulay of dimension $n - 1$.

(iii) The intersection of the irreducible components $Z_1 \cap Z_2$ is an irreducible quadratic cone of dimension $n - 2$, which is normal and Cohen-Macaulay.

(iv) Let the “worst point” be the only closed Schubert cell that lies in the special fiber of the local model; cf. Lemma 3.1.1. The closed subvarieties $Z_1$, $Z_2$, and $Z_1 \cap Z_2$ are all singular, with the only singular point being the worst point. In particular, the local model $M^\text{loc}_{\{\kappa\}}$ does not admit a semi-stable reduction.

(v) (§6.2) The irreducible components $Z_1$ and $Z_2$, as well as their intersection $Z_1 \cap Z_2$, admit moduli descriptions.
Remarks 1.3.3. (i) We exclude the cases $I = \{0\}$ and $I = \{m\}$ for all $n$, and the case $I = \{m-1,m\}$ when $n$ is even, since they already appear in [Pa00, Sm15, Yu19], and they have different geometric structures. But our proof generalizes to all these cases except when $I = \{m\}$ for $n = 2m$; cf. Remarks 4.3.6.

(ii) All those properties, including the moduli descriptions for the irreducible components of the special fiber, also apply to any ramified unitary Pappas-Zhu model $M^K_{\text{loc}}(G, \{\mu\})$ of signature $(n-1,1)$, when $K$ is a strongly non-special parahoric subgroup.

(iii) Some results in Theorem 1.3.2 have been proved using the group-theoretic data. For instance, by [PZ13] (resp. [HR22]), all the ramified unitary local models are normal (resp. Cohen-Macaulay). By [HPR20], the ramified unitary local models with a strongly non-special parahoric level structure do not admit semi-stable reductions. Comparing to their proofs, ours are more ad hoc and direct.

(iv) As a by-product, we verified that the local model $M_{\{\kappa\}}^{\text{loc}}$ is Gorenstein when $n = 4\kappa + 2$. It remains an interesting question to determine whether the rest cases are Gorenstein or not.

To compare the strengthened spin model with the local model, we only need to verify that the special fiber of the strengthened spin model is reduced since they are topologically flat; cf. §2.5. Using the equivariant action of the loop group, we can further focus on an affine neighborhood $U_{\{\kappa\}}$ of the worst point of the strengthened spin local model $M_{\{\kappa\}}$; cf. §3.1. The proof of Theorem 1.3.2 now follows from the following explicit computation of $U_{\{\kappa\}}$:

Theorem 1.3.4. Suppose $n \geq 5$, the signature is $(n-1,1)$ and $I = \{\kappa\}$ is strongly non-special. Denote by $\mathcal{O} = \mathcal{O}_F$ the discrete valuation ring over which the local model is defined and by $k$ its residue field. Let $\pi \in F$ and $\pi_0 \in F_0$ be uniformizers of $F$ and $F_0$ respectively, such that $\pi^2 = \pi_0$. Let $U_{\{\kappa\}}$ be the open affine chart in the previous paragraph.

(i) The open affine chart $U_{\{\kappa\}}$ is isomorphic to the spectrum of the ring

$$R_O \simeq \frac{\mathcal{O}[A,B]}{\wedge^2(A + \pi H, B), A - A^t, \text{tr}(AH) + (n-2\kappa-2)\pi}.$$

Where $A$, resp., $B$\footnote{The $A$ and $B$ we define here is the same as in (4.2.1), but different from $A$ and $B$ in (3.1.4).} is a matrix of indeterminates of size $(n-2\kappa) \times (n-2\kappa)$, resp., $(n-2\kappa) \times 2\kappa$, and $H$ is the anti-diagonal unit matrix of size $(n-2\kappa) \times (n-2\kappa)$.

(ii) The special fiber of the open affine chart $U_{\{\kappa\},s}$ is isomorphic to the spectrum of the ring

$$R_s \simeq \frac{k[A,B]}{\wedge^2(A, B), A - A^t, \text{tr}(AH)}.$$

The ring $R_s$ is reduced and Cohen-Macaulay of dimension $n-1$. 
(iii) The affine scheme $U_{\{\kappa\},s} \simeq \text{Spec } R_s$ has two irreducible components, isomorphic to the spectra of the rings

$$R_{s,1} \simeq \frac{k[A,B]}{A, \sqrt{2}B}, \text{ resp., } R_{s,2} \simeq \frac{k[A,B]}{\sqrt{(A,B), A - A^t, \text{tr}(AH), B^tHB}}.$$ 

Both are normal and Cohen-Macaulay of dimension $n - 1$, with the singular locus defined by $A = 0, B = 0$.

(iv) The intersection of $\text{Spec } R_{s,1}$ and $\text{Spec } R_{s,2}$ is isomorphic to the spectrum of the ring

$$R_{s,12} \simeq \frac{k[A,B]}{A, \sqrt{2}B, B^tHB}.$$ 

The scheme $\text{Spec } R_{s,12}$ is normal, Cohen-Macaulay, and irreducible of dimension $n - 2$, with the singular locus defined by $A = 0, B = 0$.

The main body of the paper will focus on the proof of Theorem 1.3.4 (ii) and (iv). Part (i) of the theorem is irrelevant to the main result, and its proof will be postponed to Section 8. Part (iii) of the theorem will be a consequence of (ii) and (iv).

To get the defining equations in (ii), we mimic Arzdorf’s computation [Ar09], which is further generalized in [Sm15, Yu19]. We will explicitly write down the defining equations of the ring $R_s$ using the “worst term”; cf. Definition 3.3.10. Then, we show that the ring $R_s$ is Cohen-Macaulay and generically smooth. By Serre’s criterion, it is reduced; cf. Theorem 4.2.1. This part is based on Conca’s work about the symmetric determinant varieties. Next, we will show $R_{s,12}$ is normal. This is based on the Gröbner basis computations of the ring. The remaining parts will follow from some conceptual arguments.

Remark 1.3.5. The commutative algebra results in Theorem 1.3.2, in particular the normal and Cohen-Macaulay properties of $R_{s,2}$ and $R_{s,12}$, may be of independent interests.

Remark 1.3.6. It is quite interesting to point out that the defining equations of the local models $M_{\{\kappa\}}^{\text{loc}}$ given in Theorem 4.2.1 look the same for even and odd $n$. From the group-theoretic point of view, these two cases are expected to be different. For instance, they have different types of local Dynkin diagrams. The same phenomenon also appears in the orthogonal setting: the even and odd dimensional special orthogonal groups fall into two classes of (local) Dynkin diagrams. But the defining equations of the local models (over an affine neighborhood of the worst point) have the same form in both cases; see [Za23].

Another common phenomenon is about the singularity. The only singular point in both cases are the worst point in the special fiber. Moreover, if we blow-up the worst point, we will get a regular integral model with semi-stable reduction. The orthogonal case is proved in [PZa22]. The ramified unitary case has been confirmed when the level is self-dual or special; cf. §1.4. The strongly non-special case will be shown in the forthcoming work.
1.4. Moduli description of the local model of the unitary group. We briefly summarize the moduli description of the unitary local model of signature \((n-1,1)\) with references. The axioms \(\text{LM1-LM8}\) used here can be found in Section 2.

We start with the global setting. Let \(F\) be a CM field with the totally real subfield \(F^+\), and let \((V, \langle , \rangle)\) be a hermitian space over \(F\), such that for each embedding \(\phi : F \hookrightarrow \mathbb{C}\), the base change of the hermitian space to \(\mathbb{C}\) has signature \((r_{\phi}, s_{\phi})\). Consider now the unitary similitude group \(G := \text{Res}_F^+ \mathbb{Q} \text{GU}(V, \langle , \rangle)\). Fix a prime \(p \neq 2\), and let \(V_p\) be the set of places of \(F^+\) lying above \(p\). Let \(F^+ := F^+_v, F_v := F \otimes_{F^+} F^+_v\) and \(V_v = V \otimes_F F_v\) with induced hermitian form. Now one has

\[
G := G_{\mathbb{Q}_p} = \prod_{v \in V_p} \text{Res}_{F^+_v/\mathbb{Q}_p} \text{GU}(V_v).
\]

Assume now \(F^+_v/\mathbb{Q}_p\) are all unramified. By unramified descent, the study of the local models of \(G\) reduces to the study of the local models of \(\text{GU}(V_v)\) for each \(v \in V_p\). Let us write \(F = F_v, F^+ = F^+_v, V = V_v\) and \(G = \text{GU}(V)\). Now we have three situations:

(i) The prime \(v\) splits in \(F/F^+\). Then \(G \simeq \text{GL}_n \times \mathbb{G}_m\).

(ii) The prime \(v\) is inert in \(F/F^+\). Then \(G\) becomes isomorphic to \(\text{GL}_n \times \mathbb{G}_m\) after the unramified base extension from \(F^+\) to \(F\).

(iii) The prime \(v\) is ramified in \(F\).

For the case (i) and (ii), since local models (and Bruhat-Tits buildings) behave well under unramified base change, the naive local model defined using \(\text{LM1-LM5}\) (see §2.2) is flat by [Gö01]. Note that this part is valid for any signature \((r,s)\).

For the case (iii), since the hermitian form will split after some unramified extension, let’s use an index \(I\) to parameterize parahoric subgroups (cf. §6.1.4). In this case, the naive local model fails to be flat due to the failure of the Kottwitz condition \(\text{LM5}\). The moduli description will be given by adding some variants of the Kottwitz condition:

(a) In the self-dual case when \(I = \{0\}\), a moduli description is given by [Pa00] using the wedge condition \(\text{LM6}\); cf. §2.3.1. A regular semi-stable integral model is studied in [Kr03].

(b) In the \(\pi\)-modular case when \(I = \{m\}\) and \(n = 2m\) is even, a moduli description is given in [RSZ17] using the wedge condition and the spin condition \(\text{LM7}\); cf. §2.3.2. Note that in loc.cit., a variant of the spin condition is given, which is easier to verify in practice; see loc. cit. Remark 3.11. The local model, in this case, is smooth.

(c) In the almost \(\pi\)-modular case when \(I = \{m\}\) and \(n = 2m + 1\) is odd, a moduli description is given by [Sm15] using the strengthened spin condition \(\text{LM8}\); cf. §2.4. The local model, in this case, is smooth. Cases (b) and (c) are called the “exotic good reduction type” or the “exotic smooth” case.
(d) In the case when $I = \{m - 1, m\}$ and $n = 2m$ is even, a moduli description is given by [Yu19] using the strengthened spin condition LM8; cf. §2.4. A regular semi-stable splitting model is also studied in [Yu19].

(e) For all other cases, the local model admits a moduli description using the strengthened spin condition LM8; cf. §2.4, which is the main theorem of this paper.

**Remark 1.4.1.** We also have moduli descriptions for limited cases when $F_v/Q_p$ is ramified and $F/F^+$ splits or is inert; see [PR03]. A conjectural moduli description can also be found in loc. cit.

1.5. **The structure of the paper.** The organization of the paper is as follows. In Section 2, we review different moduli functors related to the ramified unitary local models with the split hermitian form, including the naive, wedge, spin, and strengthened spin models. We will also define the local model and discuss its relations with those moduli functors. In Section 3, we focus on the strongly non-special case. We will translate the axioms defining the strengthened spin models into commutative algebra relations. In Section 4, we simplify those relations and study the geometric properties of the strengthened spin model for the strongly non-special parahoric subgroup, in particular, we deduce Theorem 1.3.4 (ii)-(iv). In Section 5, we review the theory of affine flag varieties, and use them to prove Theorem 1.2.1. In Section 6, we give moduli descriptions for any ramified unitary local models with signature $(n - 1, 1)$, whose hermitians form are not necessarily split. We will also discuss a moduli description of the irreducible components of the special fibers local model in the strongly non-special case. Then, we will give a global application to Shimura varieties. In Section 7, we give the proof of some propositions left in Section 3. In Section 8, we give the defining equations of the local model at an affine neighborhood of the worst point over the ring of integers.

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2. **The moduli problems**

In this section, we review the definitions of $M_1^{loc}$, $M_1^{naive}$, $M_1^{\wedge}$, and $M_1^{spin}$ from [PR09], and the strengthened spin model $M_I$ from [Sm15].

2.1. **Basic setup.**
2.1.1. Notation. Except in §5, $F/F_0$ will be a ramified quadratic extension of local fields with rings of integers $\mathcal{O}_F$ and $\mathcal{O}_{F_0}$ respectively, which can be $p$-adic fields or function fields over a finite field. Suppose they have uniformizers $\pi$ and $\pi_0$, respectively, such that $\pi^2 = \pi_0$. Denote their residue field as $k$, whose characteristic is not equal to 2.

We will work with a fixed integer $n \geq 3$. For $i \in \{1, \cdots, n\}$, we write

$$i^\vee := n + 1 - i.$$ 

For $i \in \{1, \cdots, 2n\}$, we write

$$i^\ast := 2n + 1 - i.$$ 

For $S \subset \{1, \cdots, 2n\}$, we write

$$S^\ast = \{i^\ast \mid i \in S\} \text{ and } S^\perp = \{1, \cdots, 2n\} \setminus S^\ast.$$ 

We also define

$$\sum S := \sum_{i \in S} i.$$ 

For a real number $a$, we write $\lfloor a \rfloor$ for the greatest integer $\leq a$, and $\lceil a \rceil$ for the least integer $\geq a$. We write $a, \cdots, b, \cdots, c$ for the list $a, \cdots, c$ with $b$ omitted.

Suppose $X$ is a scheme defined over a discrete valuation ring $\mathcal{O}$; we will denote $X^\eta$ (resp., $X^s$) as the generic (resp., special) fiber of the scheme.

2.1.2. Linear algebra setup of the splitting hermitian form. Consider the vector space $F^n$ with the standard $F$-basis $e_1, \cdots, e_n$. Consider the split $F/F_0$-Hermitian form

$$\phi : F^n \times F^n \to F, \quad \phi(ae_i, be_j) = \bar{a}b\delta_{ij}^\vee, \quad a, b \in F,$$

where $a \mapsto \bar{a}$ is the nontrivial element of $\text{Gal}(F/F_0)$. Attached to $\phi$ are the respective “alternating” and “symmetric” $F_0$-bilinear forms $F^n \times F^n \to F_0$ given by

$$\langle x, y \rangle := \frac{1}{2} \text{tr}_{F/F_0}(\pi^{-1} \phi(x, y)) \quad \text{and} \quad (x, y) := \frac{1}{2} \text{tr}_{F/F_0}(\phi(x, y)).$$

For each integer $i = bn + c$ with $0 \leq c < n$, define the standard $\mathcal{O}_F$-lattices

$$\Lambda_i := \sum_{j=1}^c \pi^{-b-1} \mathcal{O}_F e_j + \sum_{j=c+1}^n \pi^{-b} \mathcal{O}_F e_j \subset F^n.$$ 

(2.1.1)

For all $i$, the $\langle , \rangle$-dual of $\Lambda_i$ in $F^n$ is $\Lambda_{-i}$, by which we mean that

$$\{x \in F^n \mid \langle \Lambda_i, x \rangle \subset \mathcal{O}_{F_0} \} = \Lambda_{-i}.$$ 

We have a perfect $\mathcal{O}_{F_0}$-bilinear pair:

$$\Lambda_i \times \Lambda_{-i} \xrightarrow{\langle , \rangle} \mathcal{O}_{F_0}.$$ 

Similarly, $\Lambda_{n-i}$ is the $\langle , \rangle$-dual of $\Lambda_i$ in $F^n$. The $\Lambda_i$’s forms a complete, periodic, self-dual lattice chain

$$\cdots \subset \Lambda_{-2} \subset \Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \cdots.$$
Let $I \subset \{0, \ldots, m\}$ be a nonempty subset satisfying (1.2.1), and let $r + s = n$ be a partition. Let
\[ E = F \text{ if } r \neq s \text{ and } E = F_0 \text{ if } r = s. \]
This will correspond to the reflex field of the unitary similitude Shimura variety. We will always assume $n = r + s \geq 3$, and $(r, s) = (n - 1, 1)$ for our result.

The unitary similitude group we are interested in is defined as a reductive group over $F_0$, such that given any $F_0$-algebra $R$, we have
\[
\text{GU}(F^n, \phi)(R) = \{ g \in \text{GL}_F(F^n \otimes F_0 R) \mid \langle gx, gy \rangle = c(g) \langle x, y \rangle, c(g) \in R^\times \}. 
\]
We define the group $P_I$ as a smooth group scheme over $O_{F_0}$, which represents the moduli problem that associates each $O_{F_0}$-algebra $R$ to the set of group $P_I(R) = \{ g \in \text{GU}(F^n, \phi)(R) \otimes O_{F_0} R, \forall i \in I \}$. And $P_I^0$ is defined as the neutral connected component of $P_I$. We define $P_I := P_I(O_{F_0})$, and $P_I^0 := P_I^0(O_{F_0})$. The later is a (connected) parahoric subgroup of GU(V, \phi).

2.2. Naive model. For the rest of this section, we will define several moduli functors related to local models. Historically, they were the candidates for the ramified unitary local models with split hermitian form. Nowadays, it is common to define the local model of specific groups using purely group theoretical data, but for this paper, the classical setup is essential.

The naive local model $M_I^{\text{naive}}$ is a projective scheme over $\text{Spec } O_E$. It represents the moduli problem that sends each $O_E$-algebra $R$ to the set of all families:
\[ (\mathcal{F}_i \subset \Lambda_i \otimes_{O_{F_0}} R)_{i \in \pm I + n \mathbb{Z}}, \]
such that
LM1. for all $i$, $\mathcal{F}_i$ is an $O_F \otimes O_{F_0}$ $R$-submodule of $\Lambda_i \otimes_{O_{F_0}} R$, and an $R$-direct summand of rank $n$;
LM2. for all $i < j$, the natural arrow $\Lambda_i \otimes_{O_{F_0}} R \to \Lambda_j \otimes_{O_{F_0}} R$ carries $\mathcal{F}_i$ into $\mathcal{F}_j$;
LM3. for all $i$, the isomorphism $\Lambda_i \otimes_{O_{F_0}} R \xrightarrow{\pi \otimes 1} \Lambda_{i-n} \otimes_{O_{F_0}} R$ identifies $\mathcal{F}_i \cong \mathcal{F}_{i-n}$;
LM4. for all $i$, the perfect $R$-bilinear pairing
\[
(\Lambda_i \otimes_{O_{F_0}} R) \times (\Lambda_{i} \otimes_{O_{F_0}} R) \xrightarrow{(\cdot, \cdot) \otimes R} R
\]
identifies $\mathcal{F}_i^\perp$ with $\mathcal{F}_{-i}$ inside $\Lambda_{-i} \otimes_{O_{F_0}} R$; and
LM5. (Kottwitz condition) for all $i$, the element $\pi \otimes 1 \in O_F \otimes O_{F_0} R$ acts on $\mathcal{F}_i$ as an $R$-linear endomorphism with characteristic polynomial
\[
\det(T \cdot \text{id} - \pi \otimes 1 \mid \mathcal{F}_i) = (T - \pi)^s(T + \pi)^r \in R[T]. 
\]
When $r = s$, the polynomial on the right-hand side of LM5 is $(T^2 - \pi_0)^s$.

**Remark 2.2.1.** The axiom LM4 is equivalent to requiring the perfect $R$-bilinear pairing

$$(\Lambda_i \otimes_{\mathcal{O}_{F_0}} R) \times (\Lambda_{n-i} \otimes_{\mathcal{O}_{F_0}} R) \xrightarrow{\cdot} R$$

identifies $\mathcal{F}_i$ with $\mathcal{F}_{n-i}$ inside $\Lambda_{n-i} \otimes_{\mathcal{O}_{F_0}} R$.

2.3. **Wedge and spin conditions.**

2.3.1. **Wedge condition.** The wedge condition on an $R$-point $(\mathcal{F}_i)_i$ of $M^\text{naive}$ is that LM6: if $r \neq s$, then for all $i$,

$$\bigwedge^{s+1}_R (\pi \otimes 1 + 1 \otimes \pi | \mathcal{F}_i) = 0 \quad \text{and} \quad \bigwedge^{r+1}_R (\pi \otimes 1 - 1 \otimes \pi | \mathcal{F}_i) = 0.$$

(There is no condition when $r = s$.)

The wedge local model $M^\wedge_I$ is the closed subscheme in $M^\text{naive}$ cut out by the wedge condition. Note that the splitting of the hermitian form is unnecessary for defining the wedge condition.

2.3.2. **Spin condition.** Next, we turn to the spin condition, which involves the symmetric form $(\ , \ )$ and requires more notation. Let

$$V := F^n \otimes_{F_0} F,$$

regarded as an $F$-vector space of dimension $2n$ via the action of $F$ on the right tensor factor. Let

$$W := \bigwedge^n_F V.$$

When $n$ is even, the form $(\ , \ )$ splits over $F^n$; when $n$ is odd, $(\ , \ )$ splits after tensoring with $F$. In both cases there is an $F$-basis $f_1, \ldots, f_{2n}$ of $V$ such that $(f_i, f_j) = \delta_{ij}$. Hence there is a canonical decomposition of $W$ as an $\text{SO}(\cdot, \cdot)(F) \simeq \text{SO}_{2n}(F)$-representation:

$$W = W_1 \oplus W_{-1}.$$ Intrinsically, $W_1$ and $W_{-1}$ have the property that for any totally isotropic $n$-dimensional subspace $\mathcal{F} \subset V$, the line $\bigwedge^p_F \mathcal{F} \subset W$ is contained in $W_1$ or $W_{-1}$. In this way, they distinguish the two connected components of the orthogonal Grassmannian $\text{OGr}(n, V)$ over Spec $F$; cf. [PR09, §8.2.1]. Concretely, $W_1$ and $W_{-1}$ can be described as follows. For $S = \{i_1 < \cdots < i_n\} \subset \{1, \ldots, 2n\}$ of cardinality $n$, let

$$f_S := f_{i_1} \wedge \cdots \wedge f_{i_n} \in W.$$

Let $\sigma_S$ be the permutation on $\{1, \ldots, 2n\}$ sending

$$\{1, \ldots, n\} \xrightarrow{\sigma_S} S$$

in increasing order, and

$$\{n + 1, \ldots, 2n\} \xrightarrow{\sigma_S} \{1, \ldots, 2n\} \setminus S.$$
in increasing order. For varying \( S \) of cardinality \( n \), the \( f_s \)'s form a basis of \( W \), and we define an \( F \)-linear operator \( a \) on \( W \) such that:

\[
a(f_s) := \text{sgn}(\sigma_S) f_{S\perp}.
\]

Then, when \( f_1, \ldots, f_{2n} \) is a split basis for \((\ , \ )\),

\[
W_{\pm 1} = \text{span}_F \{ f_s \pm \text{sgn}(\sigma_S) f_{S\perp} \mid \# S = n \}
\]

is the \( \pm 1 \)-eigenspace for \( a \). Any other split basis is carried onto \( f_1, \ldots, f_{2n} \) by an element \( g \) in the orthogonal group. If \( \det g = 1 \) then \( W_1 \) and \( W_{-1} \) are both \( g \)-stable, whereas if \( \det g = -1 \) then \( W_1 \) and \( W_{-1} \) are interchanged by \( g \). In this way, \( W_1 \) and \( W_{-1} \) are independent of choices up to labeling.

For the rest of the paper, we pin down a particular choice of \( W_1 \) and \( W_{-1} \) as in [PR09, Section 7.2] and [Sm11, Sm14]. If \( n = 2m \) is even, then

\[
\begin{aligned}
-\pi^{-1} e_1 \otimes 1, \ldots, -\pi^{-1} e_m \otimes 1, e_{m+1} \otimes 1 - \pi e_{m+1} \otimes \pi^{-1}, e_{m+2} \otimes 1, \ldots, e_n \otimes 1,
\end{aligned}
\]

is a split ordered \( F_0 \)-basis for \((\ , \ )\) in \( F^n \), and we take \( f_1, \ldots, f_{2n} \) to be the image of this basis in \( V \).

If \( n = 2m + 1 \) is odd, then we take \( f_1, \ldots, f_{2n} \) to be the following split ordered basis

\[
\begin{aligned}
-\pi^{-1} e_1 \otimes 1, \ldots, -\pi^{-1} e_m \otimes 1, e_{m+1} \otimes 1 - \pi e_{m+1} \otimes \pi^{-1}, e_{m+2} \otimes 1, \ldots, e_n \otimes 1,
\end{aligned}
\]

\[
\begin{aligned}
e_1 \otimes 1, \ldots, e_m \otimes 1, \frac{e_{m+1} \otimes 1 + \pi e_{m+1} \otimes \pi^{-1}}{2}, \pi e_{m+2} \otimes 1, \ldots, e_n \otimes 1.
\end{aligned}
\]

For \( \Lambda \) an \( \mathcal{O}_{F_0} \)-lattice in \( F^n \), we define a natural \( \mathcal{O}_F \)-lattice in \( W \):

\[
W(\Lambda) := \bigwedge_{\mathcal{O}_F}^{n} (\Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F),
\]

and we define two sublattices:

\[
W(\Lambda)_{\pm 1} := W_{\pm 1} \cap W(\Lambda).
\]

We now formulate the spin condition. If \( R \) is an \( \mathcal{O}_F \)-algebra, then the spin condition on an \( R \)-point \((\mathcal{F}_i \subset \Lambda_i \otimes_{\mathcal{O}_{F_0}} R)_i \) of \( M_{I}^{\text{naive}} \) is:

LM7. for all \( i \), the line \( \bigwedge_{\mathcal{O}_{F_0}}^r \mathcal{F}_i \subset W(\Lambda_i) \otimes_{\mathcal{O}_F} R \) is contained in

\[
\text{im}[W(\Lambda_i)(-1)^s \otimes_{\mathcal{O}_F} R \to W(\Lambda_i) \otimes_{\mathcal{O}_F} R].
\]

This defines the spin condition when \( r \neq s \). When \( r = s \), \( W_{\pm 1} \) is defined over \( F_0 \) since \((\ , \ )\) is already split before extending scalars \( F_0 \to F \), and the spin condition on \( M_{I,\mathcal{O}_F}^{\text{naive}} \) descends to \( M_{I}^{\text{naive}} \) over \( \text{Spec} \mathcal{O}_{F_0} \). In all cases, the spin local model \( M_{I}^{\text{spin}} \) is the closed subscheme of \( M_{I}^{\wedge} \) where the spin condition is satisfied.

**Remark 2.3.3.** Our definition of \( a \) above is the same as in [Sm11, Sm14]. As noted in these papers, this agrees only up to sign with the analogous operators denoted \( a_{f_1 \wedge \cdots \wedge f_{2n}} \) in [PR09, Display 7.6], and \( a \) in [Sm11, §2.3]. There is a sign error in the statement of the spin condition in [PR09, §7.2.1] leading to this discrepancy.
We give an efficient way to calculate the sign $\text{sgn}(\sigma_S)$ occurring in the expression (2.3.1) for $W_{\pm 1}$, which will be used in §7.2.

**Lemma 2.3.4** ([Sm15], Lemma 2.4.1).
\[
\text{sgn}(\sigma_S) = (-1)^{\sum S+ \frac{n(n+1)}{2}} = (-1)^{\sum S+\lceil n/2 \rceil}.
\]

### 2.4. Strengthened spin condition.
We now formulate Smithling’s strengthened spin condition [Sm15], which will be used to define the moduli space $M_I$. The idea is to restrict further the intersection in the definition of $W(\Lambda_{i \pm 1})$ to incorporate a version of the Kottwitz condition. We continue with the notation in the previous subsections.

The operator $\pi \otimes 1$ acts $F$-linearly and semi-simply on $V = F^n \otimes_{F_0} F$ with eigenvalues $\pi$ and $-\pi$. Let $V_\pi$ and $V_{-\pi}$ denote their respective eigenspaces. Let
\[
W_{r,s} := \bigwedge^r_F V_{-\pi} \otimes_F \bigwedge^s_F V_\pi.
\]

Then $W_{r,s}$ is naturally a subspace of $W$, and we have the decomposition
\[
W = \bigoplus_{r+s=n} W_{r,s}.
\]

Let
\[
W_{r,s}^{\pm 1} := W_{r,s} \cap W_{\pm 1}.
\]

For any $\mathcal{O}_{F_0}$-lattice $\Lambda$ in $F^n$, let
\[
(2.4.1) \quad W(\Lambda)_{\pm 1}^{r,s} := W_{r,s}^{\pm 1} \cap W(\Lambda) \subset W,
\]

where $W(\Lambda)$ is defined in (2.3.4). Given any $\mathcal{O}_F$-algebra $R$, we define
\[
L_{(-)}^{r,s}(\Lambda)(R) := \text{im} \left[ W(\Lambda)_{(-)}^{r,s} \otimes_{\mathcal{O}_F} R \rightarrow W(\Lambda) \otimes_{\mathcal{O}_F} R \right].
\]

The **strengthened spin condition** on an $R$-point $(\mathcal{F}_i)_i \in M_I^{\text{naive}}(R)$ is now defined as follows: LM8. For all $i$, the line $\bigwedge^n_R \mathcal{F}_i \subset W(\Lambda_i) \otimes_{\mathcal{O}_F} R$ is contained in $L_{(-)}^{r,s}(\Lambda)(R)$.

This defines the condition when $r \neq s$. When $r = s$, the subspaces $W_{r,s}^{\pm 1}$ are Galois-stable, and the condition descends from $M^{\text{naive}}_{I,\mathcal{O}_F}$ to $M^{\text{naive}}_I$. In all cases, we write $M_I$ for the locus in $M_I^{\text{spin}}$ where the condition is satisfied.

**Remark 2.4.1.** The spin and strengthened spin conditions can be defined for any hermitian spaces, since we always take the base change $V = F^n \otimes_{F_0} F$ first, over which the hermitian and symmetry forms will split.

**Remark 2.4.2.** By [Sm15, Lemma 5.1], the strengthened spin condition LM8 implies the Kottwitz condition LM5. We will show that the strengthened spin condition LM8 also implies the wedge condition LM6 in Corollary 5.2.2.
2.5. The Local model, topological flatness and the coherence conjecture. We recall the definition of the ramified unitary local model in [PR08, §11]. It agrees with the Pappas-Zhu model, which justifies the name “local model”; cf. Proposition 6.1.3.

**Definition 2.5.1.** The local model $M_{I}^{\text{loc}}$ is defined to be the scheme-theoretic closure of the open embedding $M_{I,\eta}^{\text{naive}} \hookrightarrow M_{I}^{\text{naive}}$.

Recall that, when the base ring is a Dedekind domain, a scheme is flat if and only if it agrees with the scheme-theoretic closure of its generic fiber; cf. [GW20, Proposition 14.14]. Therefore, the local model $M_{I}^{\text{loc}}$ is flat over $\mathcal{O}_E$.

There exists a closed embedding from the special fiber of the local model into some affine flag variety: $M_{I,s}^{\text{loc}} \hookrightarrow LGU/L^{+}\mathcal{P}$; cf. [PR08, §11]. The coherence conjecture [PR09, §4] predicts that the image of $M_{I,s}^{\text{loc}}$ equals the union of Schubert varieties, parameterized by the “admissible set”, whose definition is not important in this paper; cf. [PR08, Theorem 11.3] and [Zhu14, Theorem 8.1]. Combine with the theory of Schubert varieties of the affine flag varieties in [PR08], we have

**Theorem 2.5.2.** The special fiber of the local model $M_{I,s}^{\text{loc}}$ is reduced, and each irreducible component is normal, Cohen-Macaulay, and Frobenius-split.

The [PR08] only consider the case when group $G$ split over a tamely ramified extension, which is the reason why we assume $p \geq 3$ throughout the paper. When $p = 2$, the unitary group is not necessarily tamely ramified, and the local model could fail to be reduced; cf. [HPR20, §2.6].

Since the wedge, spin ([PR09, 7.2.2]) and strengthened spin conditions ([Sm15, §2.5]) hold over the generic fiber $M_{I,\eta}^{\text{naive}}$, we have a chain of closed embeddings which are all equalities over the generic fiber:

$$M_{I,s}^{\text{loc}} \subset M_{I} \subset M_{I}^{\text{spin}} \subset M_{I}^{\wedge} \subset M_{I}^{\text{naive}}.$$ 

Smithling [Sm15] conjectured that $M_{I,s}^{\text{loc}}(r,s) = M_{I}(r,s)$ for any $(r,s)$ and $I$. He showed that this is true, at least in the underlying topological space.

**Theorem 2.5.3** ([Sm11, Sm14]). The scheme $M_{I}^{\text{spin}}$ is topologically flat over $\mathcal{O}_F$; in other words, the underlying topological spaces of $M_{I}^{\text{spin}}$ and $M_{I}^{\text{loc}}$ coincide.

The following reduction strategy is well-known among experts.

**Corollary 2.5.4.** To show the closed embedding $\iota : M_{I,s}^{\text{loc}} \hookrightarrow M_{I}$ is an equality, it is equivalent to show $M_{I,s}$ is a reduced scheme.

**Proof.** If the embedding $\iota$ is an equality, since $M_{I,s}^{\text{loc}}$ is reduced by Theorem 2.5.2, then $M_{I,s}$ is reduced.

Conversely, suppose $M_{I,s}$ is reduced. Recall that, for a locally noetherian scheme $X$ defined over a discrete valuation ring, if the special fiber $X_s$ is reduced, and all maximal
points of $X_s$ lift to the scheme-theoretic closure of the generic fiber $X_\eta$, then $X$ is flat; cf. [GW20, Proposition 14.17]. Therefore, we only need to verify that all the closed points of $M_{I,s}$ can lift to $M_{I,s}^{\text{loc}}$. But this is a purely topological argument, hence follows from the topological flatness of the $M_{I,s}$ in Theorem 2.5.3. □

3. Defining equations at the strongly non-special parahoric subgroup

The main goal of this and the next section is to prove Theorem 1.3.2. In this section, we will find the defining equations of the strengthened spin model $M_I$ when the signature is $(n - 1, 1)$ and the index set $I = \{\kappa\}$ is strongly non-special. The self-dual and special cases are generally different from ours, and have been done by others; cf. §1.4, though most of our computations generalize to those cases; cf. Corollary 5.2.2.

3.1. Affine chart of the worst point. Recall that one can identify $M_{I,s}^{\text{loc}}$ with the union of Schubert varieties in the affine flag varieties. Hence, the geometric special fibers of $M_{I,s}$ and $M_I$ topologically contain the same Schubert cells, including the unique closed Schubert cell, the “worst point”.

**Lemma 3.1.1.** The following $k$-point lies in the $M_{\{\kappa\}}$:

$$F_{\kappa} = (\pi \otimes 1)(\Lambda_{\kappa} \otimes k) \subset \Lambda_{\kappa} \otimes k, \quad F_{n-\kappa} = (\pi \otimes 1)(\Lambda_{n-\kappa} \otimes k) \subset \Lambda_{n-\kappa} \otimes k.$$  

This point is closed under the action of $P_{\{\kappa\}}$; therefore, it represents the worst point.

**Proof.** Let $g_S := \left(\bigwedge_{i=2}^{n}(e_i \otimes 1 - \pi e_i \otimes \pi^{-1})\right) \wedge \frac{1}{2}(e_1 \otimes 1 + \pi e_1 \otimes \pi^{-1})$. One can check that $2(-1)^{n^2+1}\pi^{n-\kappa}g_S \in W_{-1}^{n-1,1}(\Lambda_{\kappa})$, lifting the line $\bigwedge^n(\pi \otimes 1)(\Lambda_{\kappa} \otimes k) \subset W_{-1}^{n-1,1}(\Lambda_{\kappa} \otimes k)$; cf. Proposition 3.4.1 for calculations. It’s closed under the action of $P_{\{\kappa\}}$ by definition in §2.1.2. The same argument holds for $F_{n-\kappa}$.

The lemma can also be verified using the wedge model $M_{\{\kappa\}}^\wedge$, since it is topologically flat in the strongly non-special case: when $n$ is odd, this follows directly from [Sm11]; when $n$ is even, this follows from [Sm14, Proposition 7.4.7]. □

An open neighborhood of the worst point will intersect every Schubert cell. Therefore, to show the reducedness of $M_{I,s}$, one can restrict to an open affine neighborhood of the worst point.

Notice that $M_I = M_{\{\kappa\}}$ can embed into $\text{Gr}(n, \Lambda_{\kappa} \otimes \mathcal{O}_{F_0}, \mathcal{O}_F)$ as a closed subscheme: by periodicity conditions(LM3), all the algebraic relations in the strengthened spin models can be expressed in terms of $F_{\kappa}$ and $F_{n-\kappa}$; the algebraic relations in $F_{n-\kappa}$ can be expressed in terms of $F_{\kappa}$ by orthogonality conditions (LM4).
Recall that we have chosen the standard basis in §2.1.1:

\[
\Lambda_{\kappa,\mathcal{O}_F} : \pi^{-1} e_1 \otimes 1, \ldots, \pi^{-1} e_\kappa \otimes 1, e_{\kappa+1} \otimes 1, \ldots, e_n \otimes 1;
\]

\[
\Lambda_{n-\kappa,\mathcal{O}_F} : \pi^{-1} e_1 \otimes 1, \ldots, \pi^{-1} e_{n-\kappa} \otimes 1, e_{n-\kappa+1} \otimes 1, \ldots, e_n \otimes 1;
\]

The worst point sits inside the affine chart \( U \) of the Grassmannian given by the standard basis. We will translate the axioms \( LM1-\text{LM7} \) into linear algebra relations, and use them to obtain defining equations of the strengthened spin model in \( U \). We will denote the intersection of \( U \) with the moduli functors \( M_I^\square \) as \( U_I^\square := M_I^\square \cap U \).

A point in the chart \( U \) can be presented by a \( 2n \times n \) matrix \( \begin{pmatrix} X \\ I_n \end{pmatrix} \) with respect to the standard basis, and the worst point corresponds to \( X = 0 \). We further subdivide the matrix into blocks:

\[
(3.1.2) \quad X = \begin{pmatrix}
D & M & C & \kappa \\
F & X & E & \kappa \\
B & L & A & n - 2\kappa \\
\kappa & \kappa & \kappa & \kappa
\end{pmatrix}
\]

Similarly, we can embed \( M_{(\kappa)} \) into \( \text{Gr}(n, \Lambda_{n-\kappa} \otimes \mathcal{O}_F) \). The corresponding standard affine open in \( \text{Gr}(n, \Lambda_{n-\kappa} \otimes \mathcal{O}_F) \) consists of points represented by \( 2n \times n \) matrices \( \begin{pmatrix} Y \\ I_n \end{pmatrix} \).

To simplify the computation, we will also choose the following reordered basis,

\[
(3.1.3) \quad \Lambda_{\kappa,\mathcal{O}_F} : e_{\kappa+1} \otimes 1, \ldots, e_n \otimes 1, \pi^{-1} e_1 \otimes 1, \ldots, \pi^{-1} e_{\kappa+1} \otimes 1, \pi e_{\kappa+1} \otimes 1, \ldots, \pi e_{n-\kappa+1} \otimes 1,
\]

\[
\Lambda_{n-\kappa,\mathcal{O}_F} : e_{\kappa+1} \otimes 1, \ldots, e_n \otimes 1, \pi^{-1} e_1 \otimes 1, \ldots, \pi^{-1} e_{n-\kappa+1} \otimes 1, \pi^{-1} e_{n-\kappa+1} \otimes 1, \ldots, \pi^{-1} e_n \otimes 1;
\]

The points in the open affine charts of \( \text{Gr}(n, \Lambda_{\kappa} \otimes \mathcal{O}_F) \) and \( \text{Gr}(n, \Lambda_{n-\kappa} \otimes \mathcal{O}_F) \) with respect to this basis can be represented by \( \begin{pmatrix} X \\ I_n \end{pmatrix} \), resp., \( \begin{pmatrix} Y \\ I_n \end{pmatrix} \). Since two chosen bases differ by a permutation, their corresponding affine charts in \( \text{Gr}(n, \Lambda_{\kappa} \otimes \mathcal{O}_F) \) are identified. The worst point corresponds to \( X = 0 \), resp., \( Y = 0 \). After reordering the basis, the partition

\footnote{Not the same \( \mathbf{A}, \mathbf{B} \) as in Theorem 1.3.2, see §4.2}
(3.1.2) now becomes

\[
X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = \begin{pmatrix} \kappa & \kappa & n-2\kappa \\ A & B & L \\ C & D & M \\ E & F & X_4 \end{pmatrix}^{n-2\kappa}.
\]

We will subdivide the matrices \( \mathcal{Y} \) and \( Y \) into blocks of the same size as \( X \) and \( X \), respectively. Their entries are determined in (3.2.3). For the rest of the paper, we will mainly focus on \( \text{Gr}(n, \Lambda_{\kappa} \otimes \mathcal{O}_{F_0} \mathcal{O}_F) \).

3.2. **Naive model and Wedge condition.** Conditions \( \text{LM}1-\text{LM}6 \) produce closed relations on \( U \); we will translate all of them except \( \text{LM}5 \) into algebraic relations in \( X \) (cf. Remark 2.4.2). With the chosen reordered basis, the transition maps: \( A_{\kappa} : \Lambda_{\kappa,R} \to \Lambda_{n-\kappa,R} \) and \( A_{n-\kappa} : \Lambda_{n-\kappa,R} \to \Lambda_{n+\kappa,R} \), in \( \text{LM}2 \) are represented by the matrices

\[
(3.2.1) \quad A_{\kappa} = \begin{pmatrix} I_{2\kappa} & \pi_0 \cdot I_{n-2\kappa} \\ \pi_0 \cdot I_{n-2\kappa} & I_{2\kappa} \end{pmatrix}, \quad A_{n-\kappa} = \begin{pmatrix} I_{2\kappa} & \pi_0 \cdot I_{n-2n} \\ I_{2\kappa} & I_{n-2\kappa} \end{pmatrix}.
\]

The symmetric pairing \( (\ , \)_R : \Lambda_{n-\kappa,R} \times \Lambda_{\kappa,R} \to R \) in \( \text{LM}4 \) is represented by the matrix

\[
(3.2.2) \quad M = \begin{pmatrix} J_{2\kappa} & -H_{n-2\kappa} \\ -H_{n-2\kappa} & H_{n-2\kappa} \end{pmatrix},
\]

where

\[
H_l = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad J_{2l} = \begin{pmatrix} -H_l \\ H_l \end{pmatrix}.
\]

We will omit the lower indices of these matrices for simplicity. By \( \text{LM}4 \), we have \( \mathcal{F}_{n-\kappa} = \mathcal{F}_\kappa^\perp \).

Using the represented matrix (3.2.2), we have

\[
(3.2.3) \quad Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} = \begin{pmatrix} -J_{2\kappa} \\ H_{n-2\kappa} \end{pmatrix} X^t \begin{pmatrix} J_{2\kappa} \\ H_{n-2\kappa} \end{pmatrix} = \begin{pmatrix} -JX_1^tJ & -JX_3^tH \\ HX_2^tJ & HX_4^tH \end{pmatrix}.
\]

From \( \text{LM}1 \), we have \( \pi \otimes 1 \)-stability: \( (\pi \otimes 1)\mathcal{F}_\kappa \subset \mathcal{F}_\kappa \), and \( (\pi \otimes 1)\mathcal{F}_{n-\kappa} \subset \mathcal{F}_{n-\kappa} \). For the first one, we get \( (\pi \otimes 1) \begin{pmatrix} X \\ I_n \end{pmatrix} = \begin{pmatrix} \pi_0 I_n \\ X \end{pmatrix} = \begin{pmatrix} X \\ I_n \end{pmatrix} T \), for some \( T \). Hence \( T = X \) and \( XT = \pi_0 I_n \). Therefore, we get \( X^2 = \pi_0 I_0 \), i.e.,

\[
\begin{pmatrix} X_1^2 + X_2X_3 & X_1X_2 + X_2X_4 \\ X_3X_1 + X_4X_3 & X_3X_2 + X_4^2 \end{pmatrix} = \begin{pmatrix} \pi_0 I_{2\kappa} \\ \pi_0 I_{n-2\kappa} \end{pmatrix}.
\]

The \( (\pi \otimes 1) \)-stability on \( \mathcal{F}_{n-\kappa} \) follows from the \( (\pi \otimes 1) \)-stability of \( \mathcal{F}_\kappa \) by (3.2.3).
The condition LM2 requires that the transition map $A_\kappa$ sends $\mathcal{F}_\kappa$ into $\mathcal{F}_{n-\kappa}$. This is equivalent to $(A_\kappa \mathcal{F}_\kappa, \mathcal{F}_\kappa) = 0$, i.e., 
\[
\begin{pmatrix} X \\ I_n \end{pmatrix}^t A_t^\kappa M \begin{pmatrix} X \\ I_n \end{pmatrix} = 0. \]
This gives us
\[
\begin{pmatrix} -JX_1 + X_3^t JX_3 + X_1^t J & -JX_2 + X_3^t JX_4 \\ X_2^t J + X_4^t JX_3 & X_4^t JX_4 - \pi_0 H \end{pmatrix} = 0.
\]
Similarly, the condition that the transition map $A_{n-\kappa}$ sends $\mathcal{F}_{n-\kappa}$ to $\mathcal{F}_{n+\kappa}$ is equivalent to $(\mathcal{F}_\kappa, A_{n-\kappa} \mathcal{F}_{n-\kappa}) = 0$. This gives
\[
\begin{pmatrix} X_1 J X_1^t - \pi_0 J & X_1 J X_3^t - X_2 H \\ X_3 J X_1^t + H X_2^t & X_3 J X_3^t - X_4 H + H X_4^t \end{pmatrix} = 0.
\]
Here we have secretly identified $\Lambda_{n+\kappa}$ with $\Lambda_\kappa$ by LM3.

Finally the wedge condition LM6 for $(r, s) = (n-1, 1)$ says
\[
\wedge^2 (\pi \otimes 1 + 1 \otimes \pi) | \mathcal{F}_\kappa) = 0, \quad \wedge^n (\pi \otimes 1 - 1 \otimes \pi) | \mathcal{F}_\kappa) = 0;
\]
\[
\wedge^2 (\pi \otimes 1 + 1 \otimes \pi) | \mathcal{F}_{n-\kappa}) = 0, \quad \wedge^n (\pi \otimes 1 - 1 \otimes \pi) | \mathcal{F}_{n-\kappa}) = 0.
\]

The wedge condition in $\mathcal{F}_{n-\kappa}$ is implied by the wedge condition in $\mathcal{F}_\kappa$ using (3.2.3). If we focus on the special fiber, as $1 \otimes \pi = 0$, the axiom LM6 then becomes
\[
\bigwedge^2 X = 0.
\]

We summarize what we have found over the special fiber:

**Proposition 3.2.1.** The coordinate ring of the affine chart over special fiber $U_{\{\kappa\}, s} \subset M_{\{\kappa\}, s}^\wedge$ is isomorphic to the polynomial ring $k[X]$ modulo the entries of the following matrices:

**LM1.** $X_1^2 + X_2 X_3, X_1 X_2 + X_2 X_4, X_3 X_1 + X_4 X_3, X_3 X_2 + X_4^2,$

**LM2-1.** $-JX_1 + X_3^t H X_3 + X_1^t J, -JX_2 + X_3^t H X_4, X_2^t J + X_4^t H X_3, X_4^t H X_4$,

**LM2-2.** $X_1 J X_1^t, X_1 J X_3^t - X_2 H, X_3 J X_1^t + H X_2^t, X_3 J X_3^t - X_4 H + H X_4^t$,

**LM6.** $\bigwedge^2 X$.

3.3. **Strengthened Spin condition: set-ups.** Next, we want to translate the strengthened spin conditions into linear algebra relations. While this approach aligns with the discussions in [Sm15, §4.1] and [Yu19, §1.4.2], our case presents additional complexity. In this subsection, we begin by laying out the basic definitions and then outline our method.

3.3.1. **Another basis for $V$.** We choose another basis for $V = F^n \otimes_{F_0} F$. Let $g_1, \ldots, g_{2n}$ to be the following ordered $F$-basis for $V$,

\[
e_1 \otimes 1 - \pi e_1 \otimes \pi^{-1}, \ldots, e_n \otimes 1 - \pi e_n \otimes \pi^{-1}; \frac{e_1 \otimes 1 + \pi e_1 \otimes \pi^{-1}}{2}, \ldots, \frac{e_n \otimes 1 + \pi e_n \otimes \pi^{-1}}{2}.
\]
It splits and separates the eigenspaces $V_{-\pi}$ and $V_{\pi}$. This is also a split orthogonal ordered basis for the symmetric form $(\cdot, \cdot)$, whose transformation matrix to (2.3.3) has determinant 1. Hence, we find a basis for the eigenspaces:

$$W_{\pm 1} = \text{Span}_F\{gs \pm \text{sgn}(\sigma_S)g_{S^\perp} \mid \#S = n\},$$

where $g_S \in W$ is defined to be the wedge product with respect to $g_i$ for the ordered index set $S$.

### 3.3.2. Types and Weights

We recall some definitions and properties in [Sm15, §4.2, §4.3].

**Definition 3.3.3.** We say that a subset $S \subset \{1, \cdots, 2n\}$ has type $(r, s)$ if

$$\#(S \cap \{1, \cdots, n\}) = r \quad \text{and} \quad \#(S \cap \{n + 1, \cdots, 2n\}) = s.$$

Suppose from now on, $r + s = n$, then the elements $g_S$ for varying $S$ of type $(r, s)$ form a basis for $W_{r,s}$.

It is easy to check that $S$ and $S^\perp$ have the same type, and we have

**Lemma 3.3.4** ([Sm15] Lemma 4.2). $W_{\pm 1} = \text{Span}_F\{gs \pm \text{sgn}(\sigma_S)g_{S^\perp} \mid S \text{ is of type } (r, s)\}$.

**Remark 3.3.5.** We will be interested in $S$ of type $(n-1, 1)$. Such an $S$ is of the form

$$S = \{1, \cdots, \hat{i}, \cdots, n, n + i\}$$

for some $i, j \leq n$. By Lemma 2.3.4,

$$\text{sgn}(\sigma_S) = (-1)^{\sum S + \frac{n(n+1)}{2}} = (-1)^{\frac{n(n+1)}{2} + \frac{n(n+1)}{2} - j + n + i} = (-1)^{n+i+j}.$$

Next, to determine a basis for $W(\Lambda_\kappa)^{n-1,1}_{-1}$, we need to determine when a linear combination of elements of the form $g_S - \text{sgn}(\sigma_S)g_{S^\perp}$ is contained in $W(\Lambda_\kappa)$. The following definition will help with the bookkeeping.

**Definition 3.3.6.** Let $S \subset \{1, \cdots, 2n\}$. The weight vector $w_S$ attached to $S$ is the element in $\mathbb{N}^n$ whose $i$th entry is $\#(S \cap \{i, n + i\})$.

**Remark 3.3.7.** If $S$ is of type $(n-1, 1)$, then the weight vector of $S$ will have two different cases. The first case are those $i \neq j$ such that the $i$th entry of $w_S$ is 2, the $j$th entry of $w_S$ is 0, and all the other entries of $w_S$ are 1. In this case, $S = \{1, \cdots, \hat{i}, \cdots, n, n + i\}$ can be recovered from its weight vector. The other case is when $w_S = (1, \cdots, 1)$. In this case all we can say is that $S = \{1, \cdots, \hat{i}, \cdots, n, n + i\}$ for some $i \in \{1, \cdots, n\}$.

The reason we introduce the weight vector is the following lemma:

**Lemma 3.3.8** ([Sm15] Lemma 4.7). For $S$ of cardinality $n$, write

$$g_S = \sum_{S'} c_{S'} e_{S'}, \quad c_{S'} \in F.$$

Then, every $S'$ for which $c_{S'} \neq 0$ has the same weight as $S$. 

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3.3.9. The worst terms. Let $A$ be a finite dimensional $F$-vector space, and let $B$ be an $F$-basis of $A$.

**Definition 3.3.10.** Let $x = \sum_{b \in B} c_b b \in A$, with $c_b \in F$. We say that $c_b b$ is a worst term for $x$ if it is nonzero, and
\[
\text{ord}_\pi(c_b) \leq \text{ord}_\pi(c_{b'}) \text{ for all } b' \in B \text{ such that } c_{b'} \neq 0.
\]

We define
\[
\text{WT}_B(x) = \text{WT}(x) = \sum_{b \in B, c_b b \text{ is a worst term for } x} c_b b.
\]

Let $\Lambda$ be the $O_F$-span of $B$ in $A$. A nonzero element $x \in A$ is contained in $\Lambda$ if and only if one (hence any) of its worst term is. When this is the case, and when $R$ is a $k$-algebra, the image of $x$ under the map
\[
\Lambda \to \Lambda \otimes O_F R
\]

is the same as the image of $\text{WT}_B(x)$.

3.3.11. **Strategy for computing the strengthened spin condition.** The computation of the strengthened spin condition will consist of three parts:

(i) Computing $g_S$ in terms of $e_{S'}$,

(ii) Computing $g_S \pm \text{sgn}(\sigma_S) g_{S\perp}$ and their linear combinations in terms of $e_{S'}$,

(iii) Based on those computations, translate the strengthened spin condition into equations.

Working out the algebraic relationships for the strengthened spin condition over $O_F$ is trickier because taking the $n$-th-wedge product creates lots of terms with higher order. This is the reason we focus on the special fiber and introduce the “worst term”.

However, since our final goal is to compute the worst term of $\sum_S g_S - \text{sgn}(\sigma_S) g_{S\perp}$, as in Corollary 3.4.7, finding the worst term of $g_S$ is not enough. Therefore, we also need to compute some higher order terms of $g_S$. Based on the practical computation, we find that the “second-worst” term is enough.

Note that we only compute the algebraic relations from the strengthened spin condition of $\mathcal{F}_k$. This turns out to be sufficient for giving the local model. This is not clear at the first glance from the moduli description.

Once we find the defining equations over the special fiber, we can find defining equations over the ring of integers based on a similar computation. This is done in Section 8.

3.4. **Strengthened Spin condition: Computation.** The next proposition is the computation of the worst term of $g_S$, similar to [Sm15, Lemma 4.9] and [Yu19, Lemma 1]. To save space, we use the following non-standard notation:

\[
e_{[i,n+j]} := e_{\{i,n+1,\ldots,n+j,\ldots,2n\}} = e_i \wedge e_{n+1} \wedge \cdots \wedge e_{n+j} \wedge \cdots \wedge e_{2n}.
\]

**Proposition 3.4.1.** For the following $S$, we have:
(i) When \( S = \{1, \ldots, \hat{i}, \ldots, n, n+i\} \) for some \( 1 \leq i \leq \kappa, \)

\[
g_{S} = \frac{1}{2}(-1)^{\kappa+i-\pi-(n-\kappa)}e_{\{n+1,\ldots,2n\}} + \pi \left(2(-1)^{i+1}e_{[i,\hat{n}+i]} + \sum_{\sigma=1}^{n}(-1)^{\sigma}e_{[\pi,\sigma,\sigma]}\right) + o(\pi^{-(n-\kappa-1)}).
\]

(ii) When \( S = \{1, \ldots, \hat{i}, \ldots, n, n+i\} \) for some \( \kappa + 1 \leq i \leq n, \)

\[
g_{S} = \frac{1}{2}(-1)^{\kappa+1+\pi-(n-k)}e_{\{n+1,\ldots,2n\}} + \pi \left(2(-1)^{i+1}e_{[i,\hat{n}+i]} + \sum_{\sigma=1}^{n}(-1)^{\sigma}e_{[\pi,\sigma,\sigma]}\right) + o(\pi^{-(n-\kappa-1)}).
\]

(iii) When \( S = \{1, \ldots, \hat{j}, \ldots, n, n+i\} \) for some \( i, j \leq \kappa, i \neq j, \)

\[
g_{S} = (-1)^{\kappa+1-\pi-(n-\kappa-1)}e_{[i,\hat{n}+j]} + o(\pi^{-(n-\kappa-1)}).
\]

(iv) When \( S = \{1, \ldots, \hat{j}, \ldots, n, n+i\} \) for some \( i \leq \kappa < j, \)

\[
g_{S} = (-1)^{\kappa-\pi-(n-\kappa-2)}e_{[i,\hat{n}+j]} + o(\pi^{-(n-\kappa-2)}).
\]

(v) When \( S = \{1, \ldots, \hat{j}, \ldots, n, n+i\} \) for some \( j \leq \kappa < i, \)

\[
g_{S} = (-1)^{\kappa+1-\pi-(n-\kappa)}e_{[i,\hat{n}+j]} + o(\pi^{-(n-\kappa)}).
\]

(vi) \( S = \{1, \ldots, \hat{j}, \ldots, n, n+i\} \) for some \( \kappa + 1 \leq i, j \leq n, i \neq j, \)

\[
g_{S} = (-1)^{\kappa-\pi-(n-\kappa-1)}e_{[i,\hat{n}+j]} + o(\pi^{-(n-\kappa-1)}).
\]

The proof is long and tedious, contributing little to the understanding of the subsequent sections of the paper; hence we postpone it to §7.1.

Observe that \( (S^\perp)^\perp = S, \) and \( g_{S} - \text{sgn}(\sigma_{S})g_{S^\perp} = \pm(g_{S^\perp} - \text{sgn}(\sigma_{S^\perp})g_{S}). \) We introduce the following definition:

**Definition 3.4.2.** When \( S \) is of type \((n-1,1)\), then \( S \cap \{n+1, \ldots, n\} \) consists of a single element \( i_{S} \). We call \( S \) balanced if \( i_{S} \leq i_{S^\perp}. \)

This notation helps us to find the basis of \( W_{1}^{n-1,1}: \)

**Proposition 3.4.3** ([Sm15], Proposition 4.5). The elements \( g_{S} - \text{sgn}(\sigma_{S})g_{S^\perp} \) for balanced \( S \) form a basis for \( W_{1}^{n-1,1}. \) \( \square \)

Now we come to the main result of this subsection, similar to [Sm15, Lemma 4.10] and [Yu19, Lemma 2].

**Proposition 3.4.4.** For balanced \( S = \{1, \ldots, \hat{j}, \ldots, n, n+i\}, \) exactly one of the following situations holds:

- **Case 1.** Let \( S = \downarrow S \) and \( i \neq j. \)

  (i) When \( i \leq k, \) we have

  \[
g_{S} - \text{sgn}(\sigma_{S})g_{S^\perp} = 2(-1)^{\kappa-\pi-(n-\kappa-2)}e_{[i,\hat{n}+j]} + o(\pi^{-(n-\kappa-2)}).
\]
(ii) When \( \kappa < i \leq n - \kappa, i \neq j \), we have
\[
g_s - \text{sgn}(\sigma_s)g_s \perp = 2(-1)^{\kappa+1}\pi^{-(n-\kappa-1)}e_{[i,n+j]} + o(\pi^{-(n-\kappa-1)}).
\]
(iii) When \( i > n - \kappa \), we have
\[
g_s - \text{sgn}(\sigma_s)g_s \perp = 2(-1)^{\kappa+1}\pi^{-(n-\kappa)}e_{[i,n+j]} + o(\pi^{-(n-\kappa)}).
\]
• Case 2. Let \( i = j \).
(iv) When \( i = j \leq \kappa \), we have
\[
g_s - \text{sgn}(\sigma_s)g_s \perp = (-1)^{\kappa+1}\pi^{-(n-\kappa-1)}\left(e_{[i,n+i]} + (-1)^n e_{[i,n+i]}\right) + o(\pi^{-(n-\kappa-1)}).
\]
(v) When \( \kappa < i = j \leq m \), we have
\[
g_s - \text{sgn}(\sigma_s)g_s \perp = (-1)^{\kappa+i+1}\pi^{-(n-\kappa)}\left[e_{\{n+1, \ldots, 2n\}} + \pi\left((-1)^{i+1}e_{[i,n+i]} + (-1)^{i+1}e_{[\nu,n+\nu]} + \sum_{\sigma=1}^{n}(-1)^{\sigma}e_{[\sigma,n+\sigma]}\right)\right] + o(\pi^{-(n-\kappa-1)}).
\]
(vi) When \( i = j = m + 1 \) (this will only happen when \( n \) is odd), then
\[
g_s - \text{sgn}(\sigma_s)g_s \perp = (-1)^{\kappa+m}\pi^{-(n-\kappa)}\left[e_{\{n+1, \ldots, 2n\}} + \pi\left(-2(-1)^{m+1}e_{[m+1,n+m+1]} + \sum_{\sigma=1}^{n}(-1)^{\sigma}e_{[\sigma,n+\sigma]}\right)\right] + o(\pi^{-(n-\kappa-1)}).
\]
• Case 3. Let \( S \neq S \perp \) and \( i \neq j \).
(vii) When \( i < j^{-} \leq \kappa \), we have
\[
g_s - \text{sgn}(\sigma_s)g_s \perp = (-1)^{\kappa-\pi^{-(n-\kappa-2)}\left(e_{[i,n+i]} - (-1)^{n+i+j}e_{[\nu,n+i]}\right) + o(\pi^{-(n-\kappa-2)}).
\]
(viii) When \( i \leq \kappa < j^{-} < n - \kappa + 1 \), we have
\[
g_s - \text{sgn}(\sigma_s)g_s \perp = (-1)^{n+k+1+i+j}\pi^{-(n-\kappa-1)}e_{[j,n+i]} + o(\pi^{-(n-k-1)}).
\]
(ix) When \( i \leq \kappa, j^{-} \geq n - \kappa + 1 \), we have
\[
g_s - \text{sgn}(\sigma_s)g_s \perp = (-1)^{\kappa+1}\pi^{-(n-\kappa-1)}\left(e_{[i,n+i]} + (-1)^{n+i+j}e_{[j,n+i]}\right) + o(\pi^{-(n-\kappa-1)}).
\]
(x) When \( \kappa < i < j^{-} < n - \kappa + 1 \), we have
\[
g_s - \text{sgn}(\sigma_s)g_s \perp = (-1)^{\kappa-\pi^{-(n-\kappa-1)}\left(e_{[i,n+j]} - (-1)^{n+i+j}e_{[j,n+i]}\right) + o(\pi^{-(n-\kappa-1)}).
\]
(xi) When \( \kappa < i < n - \kappa + 1 \leq j^{-} \), we have
\[
g_s - \text{sgn}(\sigma_s)g_s \perp = (-1)^{\kappa+1}\pi^{-(n-\kappa)}e_{[i,n+j]} + o(\pi^{-(n-\kappa)}).
\]
(xii) When \( n - \kappa + 1 \leq i < j^{-} \), we have
\[
g_s - \text{sgn}(\sigma_s)g_s \perp = (-1)^{\kappa+1}\pi^{-(n-\kappa)}\left(e_{[i,n+j]} - (-1)^{n+i+j}e_{[j,n+i]}\right) + o(\pi^{-(n-\kappa)}).
\]
We postpone the proof to Section 7.2.
3.4.5. The lattice $W(\Lambda_\kappa)^{n-1,1}_{-1}$. Next, we will write the basis of the lattice $W(\Lambda_\kappa)^{n-1,1}_{-1}$ in terms of $e_S$, using Proposition 3.4.4.

As mentioned in Remark 3.3.7, when the weight vector $w_S \neq (1, 1, \ldots, 1)$, the set $S$ is uniquely determined by its weight vector $w_S$. It is not hard to foresee that the most subtle discussions will happen when the weight vector is $(1, 1, \ldots, 1)$. We introduce more notation for this case. We denote $S_i := \{1, \ldots, \widehat{i}, \ldots, n, n + i\}$, and write $a_i$ for $a_{S_i}$. Define

$$M := \left\lfloor \frac{n + 1}{2} \right\rfloor = \begin{cases} m + 1 & \text{when } n \text{ is odd;} \\ m & \text{when } n \text{ is even.} \end{cases}$$

Then $S_1, S_2, \ldots, S_M$ are all the balanced subsets with weight vector $(1, 1, \ldots, 1)$.

The next proposition is parallel to [Sm15, Proposition 4.12] and [Yu19, Proposition 2].

**Proposition 3.4.6.** Let $w \in W^{n-1,1}_{-1}$, and write

$$(3.4.2) \quad w = \sum_{S \text{ balanced}} a_S (g_S - \text{sgn}(\sigma_S)g_{S^\perp}), \quad a_S \in F.$$ 

Then $w$ lies in $W(\Lambda_\kappa)^{n-1,1}_{-1}$ if and only if it satisfies the following:

(i) if $S = \{1, \ldots, \widehat{i}, \ldots, n, n + i\}$ for some $i \leq \kappa$ then $\text{ord}_\pi(a_S) \geq n - \kappa - 2$;

(ii) if $S = \{1, \ldots, \widehat{i}, \ldots, n, n + i\}$ for some $k < i \leq n - \kappa, i \neq i', \text{ then } \text{ord}_\pi(a_S) \geq n - \kappa - 1$;

(iii) if $S = \{1, \ldots, \widehat{i}, \ldots, n, n + i\}$ for some $n - \kappa < i$, then $\text{ord}_\pi(a_S) \geq n - \kappa$;

(iv) if $S = \{1, \ldots, \widehat{j}, \ldots, n, n + i\}$ for some $i < j' \leq \kappa$, then $\text{ord}_\pi(a_S) \geq n - \kappa - 2$;

(v) if $S = \{1, \ldots, \widehat{j}, \ldots, n, n + i\}$ for some $i \leq \kappa \leq j' < n - \kappa + 1$, then $\text{ord}_\pi(a_S) \geq n - \kappa - 1$;

(vi) if $S = \{1, \ldots, \widehat{j}, \ldots, n, n + i\}$ for some $i \leq \kappa, j' \geq n - \kappa + 1$, then $\text{ord}_\pi(a_S) \geq n - \kappa - 1$;

(vii) if $S = \{1, \ldots, \widehat{j}, \ldots, n, n + i\}$ for some $k < i < j' < n - \kappa + 1$, then $\text{ord}_\pi(a_S) \geq n - \kappa - 1$;

(viii) if $S = \{1, \ldots, \widehat{j}, \ldots, n, n + i\}$ for some $k < i < n - \kappa + 1 \neq j'$, then $\text{ord}_\pi(a_S) \geq n - \kappa$;

(ix) if $S = \{1, \ldots, \widehat{j}, \ldots, n, n + i\}$ for some $n - \kappa + 1 \leq i < j'$, then $\text{ord}_\pi(a_S) \geq n - \kappa$;

(x) if $S = \{1, \ldots, \widehat{i}, \ldots, n, n + i\}$ for some $i \leq \kappa$, then $\text{ord}_\pi(a_S) \geq n - \kappa$;

(xi) if $\sum_{i=\kappa+1}^{M} (-1)^i a_i = 0$, then $\text{ord}_\pi(a_i) \geq n - \kappa - 1$ for those $i$;

(xii) if $\sum_{i=\kappa+1}^{M} (-1)^i a_i \neq 0$, and

(a) if $\text{ord}(\sum_{i=\kappa+1}^{M} (-1)^i a_i) = \min_{\kappa+1 \leq i \leq M} \{\text{ord}(a_i)\}$, then $\min_{\kappa+1 \leq i \leq M} \{\text{ord}(a_i)\} \geq n - \kappa$;

(b) if $\text{ord}(\sum_{i=\kappa+1}^{M} (-1)^i a_i) \geq \min_{\kappa+1 \leq i \leq M} + 1$, then $\min_{\kappa+1 \leq i \leq M} \{\text{ord}(a_i)\} \geq n - \kappa - 1$.

**Proof.** By Lemma 3.3.8, the element $w$ lies in $W(\Lambda_\kappa)^{n-1,1}_{-1}$ if and only if, for each possible weight vector $w$ (classified in Remark 3.3.7), we have

$$w_w := \sum_{S \text{ balanced with weight } w} a_S (g_S - \text{sgn}(\sigma_S)g_{S^\perp}) \in W(\Lambda_\kappa)^{n-1,1}_{-1}.$$ 

Thus, we reduce to working with the partial summation $w_w$ of a weight vector. In other words, the weight $w$ component of the sum (3.4.2) in $W(\Lambda_\kappa)^{n-1,1}_{-1}$. 

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When $w \neq (1, 1, \cdots, 1)$, by Remark 3.3.7, there is at most one $S$ of type $(n - 1, 1)$ with weight $w$. These correspond to case 1: (i)-(iii) and case 3: (vii)-(xii) in Proposition 3.4.4. This proves (i)-(ix).  

When $w = (1, 1, \cdots, 1)$, the terms of $S_1, \cdots, S_M$ will appear in the summation, write

$$w_{(1,1,\ldots,1)} = \sum_{i=1}^{M} a_i(g_{s_i} - \text{sgn}(S_i)g_{S_i^+}) = \sum_{i=1}^{\kappa} a_i(g_{s_i} - \text{sgn}(S_i)g_{S_i^+}) + \sum_{i=\kappa+1}^{M} a_i(g_{s_i} - \text{sgn}(S_i)g_{S_i^+}).$$

Recall Proposition 3.4.4, for $1 \leq i \leq \kappa$, each $g_{s_i} - \text{sgn}(S_i)g_{S_i^+}$ has different worst term. But for $\kappa + 1 \leq i \leq M$, the worst terms of $g_{s_i} - \text{sgn}(S_i)g_{S_i^+}$ are all $e_{(n+1,\ldots,2n)}$. We first consider the partial summation of the form $\sum_{i=\kappa+1}^{M} a_i(g_{s_i} - \text{sgn}(S_i)g_{S_i^+})$. It equals

$$(-1)^{\kappa+1} \pi^{(n-\kappa)} \left( \sum_{i=\kappa+1}^{M} (-1)^i a_i \right) \left[ e_{(n+1,\ldots,2n)} + \pi \left( \sum_{\sigma=1}^{n} (-1)^{\sigma} e_{[\sigma,n-\sigma]} \right) \right]$$

$$+ (-1)^{\kappa+1} \pi^{(n-\kappa-1)} \sum_{i=\kappa+1}^{M} (-1)^i a_i \left( (-1)^{i+1} e_{[i,n+i]} + (-1)^{i'+1} e_{[i',n+i']} \right) + \sum_{i=\kappa+1}^{M} o(\pi^{(n-\kappa-1)} a_i).$$

Denote $\Delta := \{ i \mid \kappa + 1 \leq i \leq M, \text{ord}(a_i) \text{ is minimal} \}$.

- If $\sum_{i=\kappa+1}^{M} (-1)^i a_i = 0$, then the summation becomes

$$(-1)^{\kappa+1} \pi^{(n-\kappa-1)} \sum_{i=\kappa+1}^{M} (-1)^i a_i \left( (-1)^{i+1} e_{[i,n+i]} + (-1)^{i'+1} e_{[i',n+i']} \right) + \sum_{i=\kappa+1}^{M} o(\pi^{(n-\kappa-1)} a_i).$$

The worst terms are

$$(-1)^{\kappa+1} \pi^{(n-\kappa-1)} \sum_{i \in \Delta} (-1)^i a_i \left( (-1)^{i+1} e_{[i,n+i]} + (-1)^{i'+1} e_{[i',n+i']} \right).$$

Therefore, the summation is defined over the ring of integers if and only if $\text{ord}_{i \in \Delta}(a_i) \geq n - \kappa - 1$.

- If $\sum_{i=\kappa+1}^{M} (-1)^i a_i \neq 0$, it further subdivides into three cases: when $\text{ord}_{i \in \Delta}(a_i)$, when $\text{ord}_{i \in \Delta}(a_i) = \text{ord}_{i \in \Delta}(a_i) + 1$, and when $\text{ord}_{i \in \Delta}(a_i) \geq \text{ord}_{i \in \Delta}(a_i) + 2$.

Suppose $\text{ord}_{i \in \Delta}(a_i) = \text{ord}_{i \in \Delta}(a_i)$, then the worst terms in the partial summation are

$$(-1)^{\kappa+1} \pi^{(n-\kappa)} \left( \sum_{i=\kappa+1}^{M} (-1)^i a_i \right) e_{(n+1,\ldots,2n)}.$$  

Hence, the partial summation is defined over the ring of integers if and only if $\text{ord}_{i \in \Delta}(a_i) = \text{ord}_{i \in \Delta}(a_i) \geq n - \kappa$.

Suppose $\text{ord}_{i \in \Delta}(a_i) = \text{ord}_{i \in \Delta}(a_i) + 1$. The worst terms in the partial summation are

$$(-1)^{\kappa+1} \pi^{(n-\kappa)} \left( \left( \sum_{i=\kappa+1}^{M} (-1)^i a_i \right) e_{(n+1,\ldots,2n)} + \pi \sum_{i \in \Delta} (-1)^i a_i \left( (-1)^{i+1} e_{[i,n+i]} + (-1)^{i'+1} e_{[i',n+i']} \right) \right).$$
Hence, the partial summation is defined over the ring of integers if and only if $\text{ord}_{i \in \Delta}(a_i) \geq n - \kappa - 1$.

Suppose $\text{ord} \left( \sum_{i=\kappa+1}^{M} (-1)^ia_i \right) \geq \text{ord}_{i \in \Delta}(a_i) + 2$. The worst terms in the partial summation are

$$( -1)^{\kappa+1} \pi (n-\kappa-1) \sum_{i \in \Delta} (-1)^i a_i \left( (-1)^{i+1} e_{[i,n+i]} + (-1)^{i+1} e_{[i',n+i']} \right).$$

Hence, the partial summation is defined over the ring of integers if and only if $\text{ord}_{i \in \Delta}(a_i) \geq n - \kappa - 1$.

Through the computation of the worst terms of the partial summation $\sum_{i=\kappa+1}^{M} a_i(g_{S_i} - \text{sgn}(S_i)g_{S_i})$, we notice that as long as the worst term lies in the lattice, the coefficients of the terms $e_{[i,n+i]}$ for those $1 \leq i \leq \kappa$ or $1 \leq i' \leq \kappa$ will always lies in the ring of integers.

On the other hand, the worst terms of $g_{S_i} - \text{sgn}(S_i)g_{S_i}$, when $1 \leq i \leq \kappa$, are the terms of $e_{[i,n+i]}$ and $e_{[i',n+i']}$. The partial summation $\sum_{i=1}^{\kappa} a_i(g_{S_i} - \text{sgn}(S_i)g_{S_i})$ lies in $W(\Lambda_\kappa)_{n-1,1}$ if and only if (x) holds.

Furthermore, two partial summations in $w_{(1,1,\ldots,1)} = \sum_{i=1}^{\kappa} + \sum_{i=\kappa+1}^{M}$ have different worst terms, their worst terms computations are independent. Hence, (x),(xi), and (xii) follow. □

Combined with the computation of worst terms and the proof of Proposition 3.4.6, we deduce the following corollary, which is parallel to [Sm15, Corollary 4.14] and [Yu19, Proposition 2]. Recall the definition of $L_{(-1)^{\kappa}}^n(\Lambda)(R)$ in (2.4) and the definition of $e_{[i,n+j]}$ in (3.4.1).

**Corollary 3.4.7.** Let $R$ be a $k$-algebra, then the submodule $L_{-1,1}^{n-1,1}(\Lambda_\kappa)(R) \subset W_{-1,1}^{n-1,1}(R)$ is a free $R$-module with $R$-basis described by:

(i) $e_{[n+1,\ldots,2n]}$;

(ii) $e_{[i,n+i']}$ for $i \neq i'$;

(iii) $e_{[i,n+j]} - (-1)^{n+i+j} e_{[j',n+i']}$ for $i < j' \leq \kappa$, $i \neq j$;

(iv) $e_{[i',n+i]}$ for $i \leq \kappa < j' < n - \kappa + 1$;

(v) $e_{[j',n+i]} + (-1)^{n+i+j} e_{[j',n+i]}$ for $i \leq \kappa$, $j' \leq n - \kappa + 1$, $i \neq j$;

(vi) $e_{[i,n+j]} - (-1)^{n+i+j} e_{[j',n+i]}$ for $\kappa < i < j' < n - \kappa + 1$, $i \neq j$;

(vii) $e_{[i',n+j]}$ for $\kappa < i < n - k + 1 \leq j'$, $i \neq j$;

(viii) $e_{[i,n+j]} - (-1)^{n+i+j} e_{[j',n+i]}$ for $n - k + 1 \leq i < j'$, $i \neq j$;

(ix) $e_{[i,n+i]} - (-1)^{n} e_{[i',n+i]}$ for $i \leq \kappa$;

(x) $e_{[i,n+i]} + (-1)^{n} e_{[i',n+i]}$ for $\kappa < i \leq M$;

(xi) Let $w = \sum_{i=1}^{M} c_i e_{[i,n+i]} \in W(\Lambda_\kappa) \otimes R$, Then $w$ lies in the image if and only if

(a) When $n = 2m$, we have $\sum_{i=\kappa}^{m} (-1)^i c_i = 0$;
(b) When \( n = 2m + 1 \), we have \( \sum_{i=\kappa}^{m} (-1)^i c_i + \frac{1}{2}(-1)^{m+1}c_{m+1} = 0 \). All the \( w \) of the form (a) and (b) generate a free submodule, a basis of which can be completed to a basis of \( L^{n-1,1}_{-1}(A_\kappa)(R) \) by the elements (i)-(x).

**Remark 3.4.8.** If we abandon the condition that \( S \) is balanced and retain the symmetry, we will get \( c_i = c_{i^\vee} \), and then the condition in (xi) can be rewritten as \( \sum_{i=\kappa+1}^{n} (-1)^i c_i = 0 \).

**Proof.** Part (i)-(x) follow immediately from Proposition 3.4.6 and its proof.

For part (xi), we again use Proposition 3.4.6. When \( \kappa + 1 \leq \sigma \leq m \), we have \( \sigma \neq \sigma^\vee \), and

\[
c_{\sigma} = (-1)^{\sigma - (n-\kappa-1)} \sum_{\kappa+1 \leq \iota \leq M \atop \iota \neq \sigma} (-1)^{\kappa + 1 + \iota} a_{i} = (-1)^{\sigma - (n-\kappa-1)} a_{\sigma}.
\]

When \( \sigma = m + 1 \), we have

\[
c_{m+1} = (-1)^{\kappa + m} \pi^{-(n-\kappa-1)} \left( \sum_{i=\kappa+1}^{M} (-1)^i a_{i} \right) + 2(-1)^{\kappa} \pi^{-(n-\kappa-1)} a_{m+1},
\]

\[
= 2 \cdot (-1)^{\kappa + 1} \pi^{-(n-\kappa-1)} a_{m+1}.
\]

Applying the requirement that \( \sum_{i=\kappa+1}^{M} (-1)^i a_{i} = 0 \), we get (xi). \( \square \)

3.5. **Coordinate ring of \( U_{\{\kappa\}} \) over special fiber.** Next, we will translate the strengthened spin condition in Corollary 3.4.7 into algebraic relations in the affine chart \( U \subset \text{Gr}(n, A_\kappa \otimes_{\mathcal{O}_F} \mathcal{O}_F) \). Combined with §3.2, we obtain the coordinate rings of \( U_{\{\kappa\},s} \).

3.5.1. **Translation of the strengthened Spin relations.** In this part, we will return to the standard basis (3.1.1) of \( \mathcal{F}_\kappa \).

Let \( R \) be a \( k \)-algebra. For \( \mathcal{F}_\kappa \in U_{\{\kappa\},s}(R) \), denote \( \wedge_R^n \mathcal{F}_\kappa = \wedge_R^n \left( \frac{X}{I_n} \right) = \sum_S e_S e_S \in W(A_\kappa) \otimes_{\mathcal{O}_F} R \). By Remark 3.3.7, for those \( S \) of type \((1, n-1)\) and weight vector not equal to \((1, \cdots, 1)\), the coefficient \( e_S \) is determined by exactly an entry of \( \mathcal{X} \) (and \( X \)). To be more precise, denote \( \mathcal{X} := (z_{ij}) \) for now, we the coefficient \( c_{\{i, n+1, \cdots, n+j, \cdots, 2n\}} \) of the term \( e_{\{i, n+1, \cdots, n+j, \cdots, 2n\}} \) in \( \wedge_R^n \mathcal{F}_\kappa \) is \((-1)^{j-1} \pi^{-(n-\kappa-1)} z_{ij} \).

**Proposition 3.5.2.** The following \( e_S \)’s do not show up in the terms of the basis of \( L^{n-1,1}_{-1}(A_\kappa)(R) \subset W^{n-1,1}_{-1}(R) \) listed in Corollary 3.4.7:

(i) \( e_S \) for \( \#(S \cap 1, \cdots, n) \geq 2 \);

(ii) \( e_S \) for \( S = \{i, n+1, \cdots, n+j, \cdots, n\} \), where \( \kappa + 1 \leq j \leq n - \kappa \), and \((1 \leq i \leq \kappa \) or \( n - \kappa + 1 \leq i \leq n) \).

In particular, we have (i) \( \wedge^2 X = 0 \) and (ii) \( E = F = 0 \) in \( U_{\{\kappa\},s} \). Recall the notation in (3.1.2).
Proof. This is an immediate consequence of Corollary 3.4.7. For the reader’s convenience, we illustrate Corollary 3.4.7 by the following figure. The entry \((i, j)\) of the diagram corresponds to the coefficients of the terms \(e_{\{i, n+1, \ldots, n+j, \ldots, 2n\}}\). We will note the corresponding part in the figure if it appears in Corollary 3.4.7. Besides, the dotted line in the figure corresponds to (ii), and the dashed line corresponds to (ix).

Now part (ii) follows directly from the figure and implies that \(z_{ij} = 0\) for \(\kappa + 1 \leq j \leq n - \kappa\), and \((1 \leq i \leq \kappa \text{ or } n - \kappa + 1 \leq i \leq n)\), hence \(E = F = 0\). Part (i) is straightforward without the figure since only the terms of type \((n, 0)\) and \((n - 1, 1)\) appear; this gives \(\wedge^2 X = 0\). □

Next, we want to translate Corollary 3.4.7 (iii)(v)(vi)(viii)(ix)(x)(xi) into algebraic relations in \(X\). The following lemma will be helpful in the rest of the subsection, which follows from direct computation.

**Lemma 3.5.3.** Suppose \(M = (m_{ij})\) and \(N = (n_{ij})\) are two \(\kappa \times \kappa\) matrices. Then \(m_{ij} = n_{\kappa+1-j, \kappa+1-i}\) for any \(i, j\) is equivalent to \(M = HN^tH := N^{ad}\). □

3.5.4. By (iii), \(i < j' \leq \kappa\) implies \(i \leq \kappa, j \geq n - \kappa + 1, j' \leq k\) and \(i' \geq n - \kappa + 1.\) Then \(z_{ij} = c_{ij}\) and \(z_{j'i'j'} = c_{ij}\) for \(i = i, j = j - (n - 2\kappa), i' = i' - (n - 2\kappa) = \kappa + 1 - i, j' = j' = \kappa + 1 - j.\) Here we reorder the entries using \(i, j, i'\) and \(j'\) so that \(C = (c_{ij})\) for \(1 \leq i, j \leq \kappa.\) We will use this reordered notation again later for the same purpose. We have

\[
c_{\{i, n+1, \ldots, n+j, \ldots, 2n\}} = (-1)^{n+i+j+1}c_{\{j', n+1, \ldots, n+i', \ldots, 2n\}}.
\]

Hence

\[
(3.5.1)\quad c_{ij} = z_{ij} = z_{j'i'j'} = c_{ij}'.
\]

By Lemma 3.5.3, the relation in (3.5.1) from Corollary 3.4.7(iii) is equivalent to \(C = C^{ad}\). Similarly, (viii) is equivalent to \(B = B^{ad}\).
3.5.5. Next, the basis in Corollary 3.4.7 (v) and (ix) can be reformulated into the basis
\[ e_{\{i,n+1,\ldots,n+j,\ldots,2n\}} + (-1)^{n+i+j} e_{\{j^\nu,n+1,\ldots,n+i^\nu,\ldots,2n\}} \] for \( i \leq \kappa, j^\nu \geq n - \kappa + 1 \).
In this case, we have \( i \leq \kappa, j \leq \kappa, j^\nu \geq n - \kappa + 1 \) and \( i^\nu \geq n - \kappa + 1 \). Therefore, \( z_{ij} = d_{ij} \) and \( z_{j^\nu,i^\nu} = a_{ji} \), where \( i = i, j = j, i = i^\nu - (n - 2\kappa) = k + 1 - i \) and \( j = j^\nu = k + 1 - j \). We have
\[ c_{\{i,n+1,\ldots,n+j,\ldots,2n\}} = (-1)^{n+i+j} c_{\{j^\nu,n+1,\ldots,n+i^\nu,\ldots,2n\}}. \]
Hence
\[ d_{ij} = z_{ij} = -z_{j^\nu,i^\nu} = -a_{ji}. \]
By Lemma 3.5.3, we have \( D = -A^{ad} \).

3.5.6. The basis in Corollary 3.4.7 (vi) and (viii) can be rewritten as the basis of the form
\[ e_{\{i,n+1,\ldots,n+j,\ldots,2n\}} - (-1)^{n+i+j} e_{\{j^\nu,n+1,\ldots,n+i^\nu,\ldots,2n\}}, \] for \( \kappa < i < j^\nu < n - \kappa + 1 \) and \( i + j^\nu \leq n + 1 \).
In this case, we have \( \kappa < i < n - \kappa + 1, \kappa < j < n - \kappa + 1, \kappa < j^\nu < n - \kappa + 1 \) and \( \kappa < i^\nu < n - \kappa + 1 \), therefore \( z_{ij} = x^{(4)}_{ij} \) and \( z_{j^\nu,i^\nu} = x^{(4)}_{ji} \) for \( i = i - \kappa, j = j - \kappa \), \( \bar{i} = i^\nu - i = (n - 2\kappa + 1) - i \) and \( \bar{j} = j^\nu - \kappa = (n - 2\kappa + 1) - j \). Then we obtain
\[ c_{\{i,n+1,\ldots,n+j,\ldots,2n\}} = (-1)^{n+i+j+1} c_{\{j^\nu,n+1,\ldots,n+i^\nu,\ldots,2n\}}. \]
Hence
\[ x^{(4)}_{ij} = z_{ij} = z_{j^\nu,i^\nu} = x^{(4)}_{ji}. \]
By Lemma 3.5.3, Corollary 3.4.7 (vi) and (x) are equivalent to claiming that \( X_4 = X_4^{ad} \).

3.5.7. As for Corollary 3.4.7 (xi) when \( \kappa < i \leq M \), we have \( c_i = c_{\{i,n+1,\ldots,n+i,\ldots,2n\}} = (-1)^{i-1} x_{ii} \). Therefore, when \( n = 2m \), the equality \( \sum_{i=\kappa+1}^m (-1)^i c_i = 0 \) is equivalent to \( \sum_{i=\kappa+1}^m x_{ii} = 0 \), and equivalent to \( \sum_{i=\kappa+1}^{n-\kappa+1} x_{ii} = 0 \) by \( X_4 = X_4^{ad} \). When \( n = 2m + 1 \), then \( \sum_{i=\kappa+1}^m (-1)^i c_i + \frac{1}{2}(-1)^{n+1} c_{n+1} = 0 \), is equivalent to \( \sum_{i=\kappa+1}^{n-\kappa+1} x_{ii} = 0 \) for the same reason.

Combining these two cases, we deduce that the description of (xi) in Corollary 3.4.7 is equivalent to requiring that \( \text{tr}(X_4) = 0 \).

To summarize, what we have shown is

**Theorem 3.5.8.** The affine chart over the special fiber \( U_s \subset \text{Gr}(n, \Lambda_\kappa \otimes \mathcal{O}_F / (\pi)) \) with notation in (3.1.4), the strengthened spin condition is equivalent to:

(i) \( \bigwedge^2 X = 0, X_2 = 0 \),
(ii) \( C = C^{ad}, B = B^{ad}, D = -A^{ad} \),
(iii) \( X_4 = X_4^{ad}, \text{tr}(X_4) = 0 \).

**Proof.** Equations in (i) follow from Proposition 3.5.2. Equations in (ii) follow from 3.5.4, 3.5.5. Equations in (iii) follow from 3.5.6 and 3.5.7. \( \square \)
3.5.9. Coordinate ring of $U_{\{\kappa\}}$. Combined with Proposition 3.2.1 and Theorem 3.5.8, we deduce the defining equations of $U_{\{\kappa\}}$.s.

**Theorem 3.5.10.** The special fiber of affine chart $U_{\{\kappa\}}$ defined in §3.1 is isomorphic to the spectrum of the ring $k[X]/I$, where $I$ is the ideal generated by the entries of the following matrices:

- **LM1.** $X_1^2 + X_2X_3, X_1X_2 + X_2X_4, X_3X_1 + X_4X_3, X_3X_2 + X_4^2$,
- **LM2.** $-JX_1 + X_3^2HX_3 + X_1^2J, -JX_2 + X_3^2HX_4, X_2^2J + x_4^2HX_3, X_1^2HX_4$,
- **LM2-1.** $X_1JX_1^t, X_1JX_3^t - X_2H, X_3JX_1^t + HX_2^t, X_3JX_3^t - X_4H + HX_4^t$,
- **LM8.** $\Lambda^2 X, X_2$,
- **LM8-1.** $C - C_{\text{ad}}, B - B_{\text{ad}}, D + A_{\text{ad}}, X_4 = -X_4^\text{ad}, \text{tr}(X_4)$.

4. Reducedness for the strongly non-special parahoric subgroup

In this section, we will first simplify the ideal $I$ of $U_{\{\kappa\}}$ in Theorem 3.5.10, then show it is reduced.

4.1. Simplification. In this subsection, we will simplify the coordinate ring of $U_{\{\kappa\}}$. It takes several steps to achieve the final goal.

- **Step 1.** In the first step, as $X_2 = 0$ modulo $I$, we immediately get that the ideal $I$ is generated by the entries of the following matrices:
  - (i) $X_1^2, X_3X_1 + X_4X_3, X_4^2$,
  - (ii) $-JX_1 + X_3^2HX_3 + X_1^2J, X_3^2HX_4, X_4^2HX_3, X_4^2HX_4$,
  - (iii) $X_1JX_1^t, X_1JX_3^t - X_2H, X_3JX_1^t + HX_2^t, X_3JX_3^t - X_4H + HX_4^t$,
  - (iv) $\Lambda^2 X, X_2$,
  - (v) $C - C_{\text{ad}}, B - B_{\text{ad}}, D + A_{\text{ad}}, X_4 = -X_4^\text{ad}, \text{tr}(X_4)$.

  We will show that the matrices in (i),(ii), and (iii) can be further simplified, giving the following proposition.

**Proposition 4.1.1.** The ideal $I$ is equal to the ideal generated by the entries of the following matrices:

- $-JX_1 + X_3^2J, X_3^2HX_3, X_3X_1$.
- $\Lambda^2 X, X_2$.
- $C - C_{\text{ad}}, B - B_{\text{ad}}, D + A_{\text{ad}}, X_4 = -X_4^\text{ad}, \text{tr}(X_4)$.

For the rest of Step 1, we will give a proof of the Proposition 4.1.1. The following Lemma will turn out to be useful.

**Lemma 4.1.2.** Given a matrix $M = \begin{pmatrix} X & Y \end{pmatrix}$ where $X$ is a square matrix (hence $XY$ is defined), then $\text{tr}(X) = 0$ and $\Lambda^2 M = 0$ implies $XY = 0$. 

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Proposition 4.1.5. The ideal $I$ is equal to the ideal generated by the entries of the following matrices:

\[
\begin{pmatrix} X, X_2, X_4 - X_4^{ad}, \text{tr}(X_4), X_1 + \frac{1}{2} JX_3^t HX_3 \end{pmatrix}.
\]

Proof. Recall that $X_3 = \begin{pmatrix} E & F \end{pmatrix}$, where $E, F$ are matrices of the size $(n - 2\kappa) \times \kappa$, then

\[
-\frac{1}{2} JX_3^t HX_3 = -\frac{1}{2} \begin{pmatrix} E^t \\ F^t \end{pmatrix} H \begin{pmatrix} E & F \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} F^{ad} E & F^{ad} E \\ -E^{ad} E & -E^{ad} F \end{pmatrix}.
\]
Therefore, the equality $X_1 = -\frac{1}{2}JX_3H X_3$ implies

$$A = -\frac{1}{2}E^{ad}E, \quad B = -\frac{1}{2}F^{ad}F, \quad C = \frac{1}{2}E^{ad}E, \quad D = \frac{1}{2}E^{ad}F.$$  

Since $(X^{ad}Y)^{ad} = Y^{ad}X$, we deduce $C = C^{ad}, B = B^{ad}, D = -A^{ad}$ from $X_1 = -\frac{1}{2}JX_3H X_3$.

Next, we claim that $X_3X_1 = 0$ will follow from $\wedge^2 X_3 = 0$ modulo the ideal generated by the entries of (4.1.1). In fact, we have

$$X_3X_1 = \begin{pmatrix} E & F \\ \end{pmatrix} \begin{pmatrix} F^{ad}E & F^{ad}F \\ -E^{ad}E & -E^{ad}F \end{pmatrix},$$

$$= \begin{pmatrix} EF^{ad}E - FE^{ad}E & E F^{ad}F - F E^{ad}F \end{pmatrix},$$

$$= \begin{pmatrix} EHF^t H E - FHE^t H E & EHF^t H F - FHE^t H F \end{pmatrix},$$

$$= \begin{pmatrix} (EHF^t - FHE^t) H E & (EHF^t - FHE^t) H F \end{pmatrix}.$$  

The proposition now follows from the Lemma 4.1.6 below.

Lemma 4.1.6. $\wedge^2 X_3 = 0$ implies $EHF^t - FHE^t = 0$.

Proof. Denote $E = (e_{ij})$ and $F = (f_{ij})$. Both are matrices of size $(n - 2k) \times k$. We have $HF^t = H(f_{ji}) = (f_{k+1-j,i})$. Similarly $HE^t = H(e_{ji}) = (e_{k+1-j,i})$. Hence the $(i, j)$-entry of $EHF^t - FHE^t$ is

$$\sum_{s=1}^{k} e_{is}f_{j, k+1-s} - \sum_{t=1}^{k} e_{js}f_{i, k+1-s},$$

which is zero since $\wedge^2 X_3 = 0$.

Remark 4.1.7. Before we move on, we want to point out that the relations $X_2 = 0$ and $\wedge^2 X = 0$ have some important consequences. Consider 2-minors in $X$ of the form $st$ where $s$ is an entry of $X_1$ and $t$ is an entry of $X_4$. This is a 2-minor since we have:

$$\begin{pmatrix} X_1 & 0 \\ X_3 & X_4 \end{pmatrix} \xrightarrow{\text{2-minors}} \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} \cdots \begin{pmatrix} * \\ \vdots \\ * \end{pmatrix} \cdots \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} \sim 2\text{-minors } s \cdot t - \star \cdot 0 = st.$$  

Now, it is not hard to guess that Spec $k[X]/I$ will have two irreducible components, corresponding to $X_1 = 0$ and $X_4 = 0$ respectively. Combining with Proposition 4.1.5, these two “components” correspond to the ideals generated by the entries of the following matrices respectively:

- $I_1: X_4, X_2; \wedge^2(X_1, X_3), X_1 + \frac{1}{2}JX_3^tHX_3,$

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\( \mathcal{I}_2: X_1, X_2, \bigwedge^2(X_3, X_4), X_4 - X_4^{\text{ad}}, \text{tr}(X_4), X_3^t H X_3. \)

We will verify in the Proposition 4.3.2 that these two ideals do produce reduced and irreducible components. Here we will show that this is true at least topologically.

**Lemma 4.1.8.** *Keep the notations in Remark 4.1.7, we have \( \sqrt{\mathcal{I}} = \sqrt{\mathcal{I}_1} \cap \sqrt{\mathcal{I}_2}. \)*

**Proof.** Let \( K \subset k[X] \) be the ideal generated by the entries of

\[
\bigwedge^2(X_3, X_4), \bigwedge(X_1, X_3), X_2, X_4 - X_4^{\text{ad}}, \text{tr}(X_4), X_1 + \frac{1}{2} J X_3^t H X_3.
\]

Let \( I, J \) be the ideals of \( k[X] \) generated by the entries of \( X_1 \) and \( X_4 \) respectively. Then we have \( \mathcal{I}_1 = I + K \) and \( \mathcal{I}_2 = J + K \), and the whole ideal \( \mathcal{I} \) is generated by \( IJ + K \). Consider the inclusion

\[
IJ + K^2 \subset (I + K)(J + K) \subset IJ + K.
\]

By taking the radical, we get \( \sqrt{\mathcal{I}} = \sqrt{\mathcal{I}_1} \cap \sqrt{\mathcal{I}_2} \). \( \square \)

\* Step 3. In the final step, we will show that

**Theorem 4.1.9.** *There is an isomorphism of \( k \)-algebras*

\[
\frac{k[X]}{\bigwedge^2 X, X_2, X_4 - X_4^{\text{ad}}, \text{tr}(X_4), X_1 + \frac{1}{2} J X_3^t H X_3} \cong \frac{k[X_3, X_4]}{\bigwedge^2(X_3, X_4), X_4 - X_4^{\text{ad}}, \text{tr}(X_4)}.\]

*The map is defined by replacing the entries of \( X_1 \) by \( -\frac{1}{2} J X_3^t H X_3 \) and the entries of \( X_2 \) by 0.***

**Proof.** We only need to show that the entries of \( \bigwedge^2 X \) lies in the ideal \( \mathcal{I}' \subset k[X] \) generated by the entries of

\[
\bigwedge^2(X_3, X_4), X_2, X_4 - X_4^{\text{ad}}, \text{tr}(X_4), X_1 + \frac{1}{2} J X_3^t H X_3.
\]

Notice that the entries of \( \bigwedge^2 X \) that do not lie in \( \bigwedge^2(X_3, X_4) \) fall into the following two cases:

(i) The monomials \( xy \) where \( x \) is an entry of \( X_1 \) and \( y \) is an entry of \( X_4 \), see Remark 4.1.7;

(ii) the entries of \( \bigwedge^2(X_1, X_3) \).

In the case of (i), to show \( xy \in \mathcal{I}' \), we replace \( x \) by elements in \( X_3 \) via the equality

\[
X_1 = -\frac{1}{2} J X_3^t H X_3. \quad \text{Recall that } X_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \quad \text{we will only consider the case when} \quad x = a_{ij} \text{ for some } i, j, \text{ where } a_{ij} \text{ is an entry of the matrix } A = (a_{ij}). \quad \text{The other cases when} \quad x \text{ is an entry of } B, C, \text{ or } D \text{ will follow by the same computations.}
\]

From \( X_1 = -\frac{1}{2} J X_3^t H X_3 \) and \( F^{\text{ad}} = H F^t H \), we have

\[
A = -\frac{1}{2} F^{\text{ad}} E = \left( -\frac{1}{2} \sum_{s=1}^{n-2\kappa} f_{n-2\kappa+1-s,\kappa+1-i} e_{sj} \right).
\]
Denote the entries \( X_4 := Y = (y_{ij}) \) for now, then we can rewrite \( xy \) as

\[
xy = y_{pq} \sum_{s=1}^{n-2\kappa} f_{n-2\kappa+1-s,\kappa+1-i} e_{sj},
\]

for some \( p, q, i, j \). Recall that we assume \( x = a_{ij} \). Applying \( \bigwedge^2(X_3, X_4) = 0 \) and \( X_4 = X_4^{ad} \), we have

\[
xy = \sum_{s=1}^{n-2\kappa} f_{n-2\kappa+1-s,\kappa+1-i} y_{sq} e_{pj},
\]

\[
= e_{pj} \sum_{s=1}^{n-2\kappa} f_{n-2\kappa+1-s,\kappa+1-i} y_{sq},
\]

\[
= e_{pj} \sum_{s=1}^{n-2\kappa} f_{n-2\kappa+1-s,\kappa+1-i} y_{n-2\kappa+1-q,n-2\kappa+1-s}, \quad (X_4 = X_4^{ad})
\]

\[
= e_{pj} \sum_{s=1}^{n-2\kappa} f_{n-2\kappa+1-q,\kappa+1-i} y_{n-2\kappa+1-s,n-2\kappa+1-s},
\]

\[
= e_{pj} f_{n-2\kappa+1-q,\kappa+1-i} \sum_{s=n}^{n-2\kappa} y_{n-2\kappa+1-s,n-2\kappa+1-s}.
\]

For the last equality, we use

\[
f_{n-2\kappa+1-s,\kappa+1-i} y_{n-2\kappa+1-q,n-2\kappa+1-s} - f_{n-2\kappa+1-q,\kappa+1-i} y_{n-2\kappa+1-s,n-2\kappa+1-s} = 0,
\]

which is deduced from \( \text{tr}(X_4) = 0 \). Hence \( xy = 0 \) follows from \( \bigwedge^2(X_3, X_4) = 0, X_4 = X_4^{ad}, \text{tr}(X_4) = 0 \) and \( X_1 = -\frac{1}{2} J X_3^3 H X_3 \), i.e., \( xy \in T' \).

In the case of (ii), we want to show that the entries of \( \bigwedge^2(X_1, X_3) \) lie in the ideal \( T' \). Replace \( X_1 \) using \( X_1 = -\frac{1}{2} J X_3^3 H X_3 \), and notice that

\[
\begin{pmatrix}
-\frac{1}{2} J & I_{n-2k}
\end{pmatrix}
\begin{pmatrix}
X_3^3 H X_3 & X_3
\end{pmatrix}
= \begin{pmatrix}
0 & X_3
\end{pmatrix}
\begin{pmatrix}
I & J X_3^3 H & I
\end{pmatrix}.
\]

Therefore, the ideal generated by the entries of \( \bigwedge^2(X_1, X_3) = \bigwedge^2 \left( -\frac{1}{2} J X_3^3 H X_3 \right) \) equals the ideal generated by the entries of \( \bigwedge^2 \left( 0 \ X_3 \right) \), which lies in \( T' \). \( \square \)

**Remark 4.1.10.** Using the same computation, we can show that the ideals \( I_1 \) and \( I_2 \) of the “irreducible components” defined in Remark 4.1.7 can be simplified into the ideals of \( k[X_3, X_4] \), generated by the entries of following matrices:

\[
I_1 : \bigwedge^2 X_3, X_4, \text{ and } I_2 : \bigwedge^2 (X_3, X_4), X_4 - X_4^{ad}, \text{tr}(X_4), X_3^4 H X_3.
\]

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4.2. Reducedness of the strengthened spin model for a strongly non-special parahoric subgroup. For the simplicity of notation, we substitute

\[ A = X_4 H, \quad B = X_3. \]

Note that this differs from the \( A, B \) we defined in (3.1.4). The goal of this and next subsections is to prove the following theorem:

**Theorem 4.2.1.** Suppose \( n \geq 5 \) and \( I = \{ \kappa \} \) is strongly non-special. Consider the affine scheme \( \text{Spec} R_s = \text{Spec} \frac{k[A,B]}{\Lambda^2(A,B)A - A^t, \text{tr}(AH)} \). Define closed subschemes of \( \text{Spec} R_s \) as

\[
\begin{align*}
\text{Spec} R_{s,1} &:= \text{Spec} \frac{k[A,B]}{A, \Lambda^2(B)}, \\
\text{Spec} R_{s,2} &:= \text{Spec} \frac{k[A,B]}{\Lambda^2(A,B), A - A^t, \text{tr}(AH), B'BH},
\end{align*}
\]

and their scheme-theoretic intersection

\[ \text{Spec} R_{s,12} := \text{Spec} \frac{k[A,B]}{A, \Lambda^2(B), B'BH}. \]

We also define the worst point \( \{ * \} \) to be the point defined by \( A = 0, B = 0 \). Then

(i) The affine scheme \( \text{Spec} R_s \) is reduced and Cohen-Macaulay of dimension \( n - 1 \), with the singular locus \( \text{Spec} R_{s,12} \). When \( n = 4\kappa + 2 \), it is Gorenstein (converse is not necessarily true).

(ii) The affine schemes \( \text{Spec} R_{s,1} \) and \( \text{Spec} R_{s,2} \) are the irreducible components of \( \text{Spec} R_s \). They are normal and Cohen-Macaulay of dimension \( n - 1 \). The singular locus of them are both the worst point.

(iii) The affine scheme \( \text{Spec} R_{s,12} \) is the intersection of two irreducible components. It is normal and Cohen-Macaulay of dimension \( n - 2 \), with singular locus the worst point.

In this subsection, we will verify Theorem 4.2.1(i). We will use the theory of symmetric determinant varieties studied by several papers of A. Conca. We summarize the results we will use here:

**Theorem 4.2.2.** Suppose \( n \geq 3 \) and \( n - 2\kappa \geq 2 \). Let \( R_0 = \frac{k[A,B]}{\Lambda^2(A,B)A - A^t} \), where \( A \) and \( B \) are matrices of indeterminates of size \((n - 2\kappa) \times (n - 2\kappa)\), resp., \((n - 2\kappa) \times 2\kappa\). Then the ring \( R_0 \) is normal and Cohen-Macaulay of dimension \( n \). It is Gorenstein if and only if \( n = 4\kappa + 2 \).

**Proof.** Since \( n - 2\kappa \geq 2 \), the dimension of \( \text{Spec} R_0 \) is given by [Co94-2, Page 52]. Note that there is a typo in Conca’s formula: the correct dimension formula should be

\[ \dim R_t(Z) = (n + m + 1 - s + \frac{t}{2})(t - 1). \]

Applying \( m = s = n - 2\kappa, n = n \) and \( t = 2 \), we get the dimension for our case.

By [Co94-2, Proposition 2.5], the ring \( R_0 \) is a Cohen-Macaulay domain. By [Co94-1, Theorem 2.4], \( R_0 \) is normal. By the same theorem, it is Gorenstein if and only if \( 2(n - 2\kappa) = n + 2 \), i.e., \( n = 4\kappa + 2 \). \( \Box \)
Corollary 4.2.3. The ring $R_s$ is Cohen-Macaulay. When $n = 4\kappa + 2$, it is Gorenstein.

Proof. Recall that, a local ring $R$ is Cohen-Macaulay (resp. Gorenstein) if and only if $R/(t)$ is Cohen-Macaulay (resp. Gorenstein) for some non-zero-divisor $t \in \mathfrak{m} \subset R$ in the maximal ideal; cf. [Stacks, 02JN] (resp. [Stacks, 0BJJ]).

The ring $R_0 = \frac{k[A,B]}{\Lambda^2(A,B), A - A^t, tr(AH)}$ is integral by Theorem 4.2.2, hence the element $tr(AH)$ is not a zero-divisor. Therefore, the scheme $\text{Spec } R_s = \text{Spec } \frac{R_0}{tr(AH)}$ is Cohen-Macaulay. It is Gorenstein if and only if $\text{Spec } R_0$ is Gorenstein; by Theorem 4.2.2, if and only if $n = 4\kappa + 2$.

Denote

$$I = \left( \bigwedge^2(A, B), A - A^t, tr(AH) \right) \subset k[A, B].$$

And recall

$$I_1 = \left(A, \bigwedge^2 B\right), \quad I_2 = \left(\bigwedge^2(A, B), A - A^t, tr(AH), B^t HB\right).$$

Recall Lemma 4.1.8 that $\sqrt{I} = \sqrt{I_1} \cap \sqrt{I_2}$.

Proposition 4.2.4. (i) The spectra of $R_{s,1} = R_s/I_1$ and $R_{s,2} = R_s/I_2$ are irreducible of dimension $n - 1$. They are smooth outside the worst point.

(ii) The spectrum of $R_{s,12} = R_s/(I_1 + I_2)$ is irreducible of dimension $n - 2$. It is smooth outside the worst point.

Proof. (i) This is straightforward for $\text{Spec } R_{s,1}$ by [BU98, (2.11)(1.1)]. As for $\text{Spec } R_{s,2} = \text{Spec } k[A, B]/I_2 = \text{Spec } \frac{k[A, B]}{\Lambda^2(A, B), A - A^t, tr(AH), B^t HB}$, notice that the ideal $I_2$ is homogeneous; hence to show that $\text{Spec } R_{s,2}$ is irreducible of dimension $n - 1$, it is equivalent to show the projective scheme

$$\text{Proj } \frac{k[A, B]}{\Lambda^2(A, B), A - A^t, tr(AH), B^t HB}$$

is irreducible of dimension $n - 2$.

Denote the entries of $\begin{pmatrix} A & B \end{pmatrix}$ to be $x_{ij}$ with $1 \leq i \leq n - 2\kappa$ and $1 \leq j \leq n$. To show the irreducibility, by [Stacks, 01OM], we only need to show that the principal opens $D_+(x_{ij})$ in this projective variety are irreducible, and the intersection of any two principal opens is nonempty.

Consider $D_+(x_{pq})$ for some $p$ and $q$. We have a 2-minor $x_{ij}x_{pq} = x_{iq}x_{pj}$, which implies $\frac{x_{ij}}{x_{pq}} = \frac{x_{iq}}{x_{pq}} \cdot \frac{x_{pj}}{x_{pq}}$. Note that this is true even when $i = p$ or $j = q$. Hence, if we denote

$$T_i := \frac{x_{iq}}{x_{pq}}, \quad S_j := \frac{x_{pj}}{x_{pq}},$$

then we have $\frac{x_{ij}}{x_{pq}} = T_i S_j$.

Therefore, the principal opens $D_+(x_{ij})$ is isomorphic to the spectrum of a factor ring of polynomial ring $k[T_i, S_j], 1 \leq i \leq n - 2\kappa, 1 \leq j \leq n$ (note that from construction, we always have $T_p = S_q = 1$). Our goal is to rewrite the ideal $I_2$ in terms of $S_i, T_j$. 

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The identity $A = A^t$ reads $T_i S_j = T_j S_i$ for $1 \leq i, j \leq n - 2\kappa$. In particular, we have

$$S_i = T_i \cdot S_p.$$  

(4.2.2) It is not hard to show that these relations are all we can get from the equality $A = A^t$.

Next, we have

$$\text{tr}(A H) = x_{1,n-2\kappa} + x_{2,n-2\kappa-1} + \cdots + x_{n-2\kappa,1} = T_1 S_{n-2\kappa} + T_2 S_{n-2\kappa-1} + \cdots + T_n S_{n-2\kappa} S_1.$$  

(4.2.3) Finally, denote $B = (b_{ij})$, then $b_{ij} = \frac{x_{i,n-2\kappa-1+i}}{x_{pq}} = T_i S_{n-2\kappa+j}$. Moreover

$$B^t H B = \left(\sum_{\alpha=1}^{n-2\kappa} b_{n-2\kappa+1-\alpha,i} b_{\alpha,j}\right)_{2\kappa \times 2\kappa}.$$  

But now

$$\sum_{\alpha=1}^{n-2\kappa} b_{n-2\kappa+1-\alpha,i} b_{\alpha,j} = \sum_{\alpha=1}^{n-2\kappa} T_{n-2\kappa+1-\alpha} S_{n-2\kappa+i} T_\alpha S_{n-2\kappa+j},$$  

(4.2.4) $$= (S_{n-2\kappa+i} S_{n-2\kappa+j}) \cdot \left(\sum_{\alpha=1}^{n-2\kappa} T_{n-2\kappa+1-\alpha} T_\alpha\right).$$  

The next simplification subdivides into the cases when $1 \leq q \leq n - 2\kappa$ and $n-2\kappa+1 \leq q \leq n$.

★ Let $1 \leq q \leq n - 2\kappa$. Since $S_p$ is invertible by (4.2.2), the equation (4.2.3) is equivalent to

$$T_1 T_{n-2\kappa} + T_2 T_{n-\kappa} + \cdots + T_{n-2\kappa} T_1 = 0.$$  

Hence the entries of $B^t H B$ in (4.2.4) vanishes by (4.2.3). Therefore, the affine coordinate ring of the principal open $D_+(x_{pq})$ is isomorphic to the factor ring

$$\overline{k[T_i, S_j]}_{T_p - 1, S_q - 1, T_1 - T_i S_p, T_1 T_{n-2\kappa} + T_2 T_{n-2\kappa-1} + \cdots + T_{n-2\kappa} T_1}.$$  

Since we assume $\kappa < m - 1$ when $n$ is even and $\kappa \leq m - 1$ when $n$ is odd, we have $n-2\kappa > 2$, and the element $\sum_{i=1}^{n-2\kappa} T_i T_{n-2\kappa+1-i}$ is irreducible. Therefore, any element of the form $T_i S_j$ is not a zero-divisor. This shows that $D_+(x_{pq}) \cap D_+(x_{ij}) \neq \emptyset$ for any $i, j$.

If we replace $S_i$ by $T_i S_p$, the affine ring of $D_+(x_{pq})$ is isomorphic to

$$\overline{k[T_1, \cdots, T_{n-2\kappa}, S_{n-2\kappa+1}, \cdots, S_n, \frac{1}{T_q}]}_{T_p - 1, T_1 T_{n-2\kappa} + \cdots + T_{n-2\kappa} T_1}.$$  

The further simplification subdivides into the following cases:

- When $p \neq q$, and $2p \neq n - 2\kappa$, the factor ring is isomorphic to

$$\overline{k[T_1, \cdots, T_p, \cdots, T_{n-2\kappa+1-p}, \cdots, T_{n-2\kappa}, S_{n-2\kappa+1}, \cdots, S_n, \frac{1}{T_q}]}.$$  

Hence, the principal open $D_+(x_{pq})$ is an open subset of the affine space of dimension $n - 2$, and in particular, is irreducible.
• When \( p \neq q \) and \( 2p = n - 2\kappa \), the factor ring is isomorphic to
\[
k[T_1, \ldots, \widehat{T_p}, \ldots, T_{n-2\kappa}, S_{n-2\kappa+1}, \ldots, S_n, \frac{1}{T_1}]
\frac{2(T_1T_{n-2\kappa} + \cdots + T_{p-1}T_{n-2\kappa+2-p}) - 1}{2(T_1T_{n-2\kappa} + \cdots + T_{p-1}T_{n-2\kappa+2-p}) - 1}.
\]
Hence, the principal open \( D_+(x_{pq}) \) is a smooth irreducible variety of dimension \( n - 2 \).

• When \( p = q \) and \( 2p \neq n - 2\kappa \), the factor ring is isomorphic to
\[
k[T_1, \ldots, \widehat{T_p}, \ldots, T_{n-2\kappa+1-p}, \ldots, T_{n-2\kappa}, S_{n-2\kappa+1}, \ldots, S_n].
\]
Hence, the principal open \( D_+(x_{pq}) \) is an affine space of dimension \( n - 2 \), and in particular, is irreducible.

• When \( p = q \) and \( 2p = n - 2\kappa \), the factor ring is isomorphic to
\[
k[T_1, \ldots, T_{n-2\kappa}, S_{n-2\kappa+1}, \ldots, S_n]
\frac{2(T_1T_{n-2\kappa} + \cdots + T_{p-1}T_{n-2\kappa+2-p}) - 1}{2(T_1T_{n-2\kappa} + \cdots + T_{p-1}T_{n-2\kappa+2-p}) - 1}.
\]
Hence, the principal open \( D_+(x_{pq}) \) is isomorphic to a smooth irreducible hypersurface of dimension \( n - 2 \) in affine space.

★ Let \( n - 2\kappa + 1 \leq q \leq n \). We have \( S_q = S_{(n-2\kappa+(q-(n-2\kappa))} = 1 \), hence the \( (q + 2\kappa - n, q + 2\kappa - n) \)'s entry of (4.2.4) gives
\[
(4.2.5) \quad T_1T_{n-2\kappa} + \cdots + T_{n-2\kappa}T_1 = 0.
\]
Now (4.2.5) implies (4.2.3) and (4.2.4). Therefore, the affine coordinate ring of the principal open \( D_+(x_{pq}) \) is isomorphic to the factor ring:
\[
k[T_j, S_i]
\frac{T_p - 1, S_q - 1, S_i - T_i S_p, T_1T_{n-2\kappa} + T_2T_{n-2\kappa-1} + \cdots + T_{n-2\kappa}T_1}{T_p - 1, T_1T_{n-2\kappa} + T_2T_{n-2\kappa-1} + \cdots + T_{n-2\kappa}T_1}.
\]
Note that when \( i = p \), \( S_i - T_i S_p \) also holds. As \( n - 2\kappa > 2 \), any element of the form \( T_i S_j \) is not a zero-divisor. This shows that \( D_+(x_{pq}) \cap D_+(x_{ij}) \neq \emptyset \) for any \( i, j \). By replacing \( S_i \) with \( T_i S_p \), the affine ring is isomorphic to
\[
k[T_1, \ldots, T_{n-2\kappa}, S_{n-2\kappa+1}, \ldots, \widehat{S_q}, \ldots, S_n, S_p]
\frac{T_p - 1, T_1T_{n-2\kappa} + T_2T_{n-2\kappa-1} + \cdots + T_{n-2\kappa}T_1}{T_p - 1, T_1T_{n-2\kappa} + T_2T_{n-2\kappa-1} + \cdots + T_{n-2\kappa}T_1}.
\]
Similarly, the simplification subdivides into the following cases in bullets:

• When \( 2p \neq n - 2\kappa \), the factor ring is isomorphic to
\[
k[T_1, \ldots, \widehat{T_p}, \ldots, T_{n-2\kappa+1-p}, \ldots, T_{n-2\kappa}, S_{n-2\kappa+1}, \ldots, \widehat{S_q}, \ldots, S_n, S_p].
\]
Hence, the principal open \( D_+(x_{pq}) \) is an irreducible affine space of dimension \( n - 2 \).

• When \( 2p = n - 2\kappa \), the factor ring is isomorphic to
\[
k[T_1, \ldots, \widehat{T_p}, \ldots, T_{n-2\kappa}, S_{n-2\kappa+1}, \ldots, \widehat{S_q}, \ldots, S_n, S_p]
\frac{2(T_1T_{n-2\kappa} + \cdots + T_{p-1}T_{n-2\kappa+2-p}) - 1}{2(T_1T_{n-2\kappa} + \cdots + T_{p-1}T_{n-2\kappa+2-p}) - 1}.
\]
Hence, the principal open \( D_+(x_{pq}) \) is isomorphic to a smooth irreducible hypersurface of dimension \( n - 2 \) in affine space.
For any cases, the principal opens \( D_+(x_{pq}) \) are irreducible of dimension \( n - 2 \). Therefore, \( \text{Spec } R_{s,2} \) is irreducible of dimension \( n - 1 \).

Next, consider the open subsets \( D(x_{pq}) \subset \text{Spec } R_{s,2} \) for \( 1 \leq p \leq n - 2\kappa \) and \( 1 \leq q \leq n \), the complement of the union \( \bigcup_{p,q} D(x_{pq}) \) corresponds to the worst point. Using the same computation as we did for \( D_+(x_{pq}) \), we can show the affine opens \( D(x_{pq}) \) are smooth. In fact, we can obtain the coordinate ring of \( D(x_{pq}) \) from \( D_+(x_{pq}) \) by adding variables “\( x_{pq}^{-1} \)” and “\( 1 \)”. Hence the singular locus of \( \text{Spec } R_{s,2} \) lies in the conic point \( A = 0, B = 0 \). It is singular by Jacobian criterion. This finish the proof of part (i).

(ii) The same argument in part (i) also works for \( \text{Spec } R_{s,12} \).

\[ \square \]

**Corollary 4.2.5.** The affine scheme \( \text{Spec } R_s \) is smooth outside \( \text{Spec } R_{s,12} \).

**Proof.** Using the same argument as Proposition 4.2.4, we can show \( \text{Spec } R_s - \{ \ast \} \) is reduced. To be more precise, we want to compute the affine rings of \( D(x_{pq}) \) for all possible \( p \) and \( q \). Using the substitution in (4.2.2) and (4.2.3), when \( 1 \leq q \leq n - 2\kappa \), the \( D(x_{pq}) \) is isomorphic to

\[
\text{Spec } \frac{k[T_1, \cdots , T_{n-2\kappa}, S_p, S_{n-2\kappa+1}, \cdots , S_n, x_{pq}, 1_{x_{pq}}]}{T_p - 1, S_pT_q - 1, T_1T_{n-2\kappa} + T_2T_{n-2\kappa-1} + \cdots + T_{n-2\kappa}T_1};
\]

when \( n - 2\kappa + 1 \leq q \leq n \), the \( D(x_{pq}) \) is isomorphic to

\[
\text{Spec } \frac{k[T_1, \cdots , T_{n-2\kappa}, S_p, S_{n-2\kappa+1}, \cdots , S_n, x_{pq}, 1_{x_{pq}}]}{T_p - 1, S_p(T_1T_{n-2\kappa} + T_2T_{n-2\kappa-1} + \cdots + T_{n-2\kappa}T_1)}.
\]

By (4.2.2) and (4.2.4), over the affine chart \( D(x_{pq}) \), \( A = 0 \) is equivalent to \( S_p = 0 \), and \( B^t H B = 0 \) is equivalent to \( T_1T_{n-2\kappa} + T_2T_{n-2\kappa-1} + \cdots + T_{n-2\kappa}T_1 \). From the isomorphism, we can see a point in \( D(x_{pq}) \) is singular if and only if \( n - 2\kappa + 1 \leq q \leq n \), \( S_p = 0 \), and \( T_1T_{n-2\kappa} + T_2T_{n-2\kappa-1} + \cdots + T_{n-2\kappa}T_1 \), i.e., lies in the locus \( D(x_{pq}) \cap \text{Spec } R_{s,12} \). Therefore, \( \text{Spec } R_s - \{ \ast \} \) is reduced, and \( \text{Spec } R_s - \{ \ast \} \) is smooth. \( \square \)

**Corollary 4.2.6.** The affine scheme \( \text{Spec } R_s \) is reduced.

**Proof.** Recall Serre’s criterion, by which a ring \( R \) is reduced if and only if it satisfies \( (R_0) \) and \( (S_1) \); cf. [Stacks, 031R]. In our situation, \( R_s \) is Cohen-Macaulay by Corollary 4.2.3, hence satisfies \( (S_1) \). Therefore, we only need to show that \( R_s \) satisfies the axiom \( (R_0) \): it is regular in codimension 0. This is equivalent to showing that the singular locus has codimension \( \geq 1 \). This follows from Corollary 4.2.5. \( \square \)

4.3. Geometric structure of the local model with the strongly non-special parahoric subgroup. In this subsection, we will prove the rest part of Theorem 4.2.1.

**Proposition 4.3.1.** The affine scheme \( \text{Spec } R_{s,12} \) is integral.
Proof. Since Spec $R_{s,12}$ is irreducible, we only need to show that it is reduced. We will again use Serre’s criterion [Stacks, 031R]. Spec $R_{s,12}$ satisfy the $(R_0)$ by Proposition 4.2.4 (ii). Therefore, we only need to verify it satisfy the $(S_1)$ that for any prime ideal $p \in \text{Spec } R_{s,12}$, we have $\text{depth}_p(R_{s,12}) \geq \min\{1, \dim(R_{s,12})_p\}$. Since Spec $R_{s,12}$ is smooth outside the worst point, we only need to consider the case when $p \subset (A,B)$. We will use Gröbner basis to show that, $\overline{b_{11}} \in R_{s,12} \cong \frac{k[B]}{\text{lcm}(B,B^2)})$ is a non-zero-divisor.

To save the notation, we let $s := n - 2k, t := 2k$, so that $B$ is a $s \times t$ matrix. We choose the following lexicographic monomial order in $B$: $b_{ij} < b_{pq}$ if $i < p$ or $i = p$ and $j < q$. This is a diagonal monomial order, in the sense that the “diagonal” $b_{ij}b_{jq}$ of a 2-minor $[ij \mid pq] := b_{ij}b_{jq} - b_{iq}b_{jp}$ ($i < j, p < q$) is the initial monomial. By [BCRV22, Theorem 4.1.1], the 2-minors $[ij \mid pq]$ form a Gröbner basis of $(\Lambda^2 B) \subset k[B]$ and the initial ideal $\text{in}(\Lambda^2 B)$ is generated by the monomials $b_{ip}b_{jq}, i < j, p < q$.

The $(\alpha, \beta)$-entry ($\alpha \leq \beta$) of the matrix $B'BH$ is $\sum_{\delta=1}^{t} b_{t+1-\delta,\alpha}b_{\delta,\beta}$. After the reduction modulo the 2-minors, when $t = 2r$ is even, the remainder term $f_{\alpha\beta}$ is

$$2 \sum_{\delta=1}^{r} b_{t+1-\delta,\alpha}b_{\delta,\beta};$$

when $t = 2r + 1$ is odd, the remainder term $f_{\alpha\beta}$ is

$$2 \left( \sum_{\delta=1}^{r} b_{t+1-\delta,\alpha}b_{\delta,\beta} \right) + b_{t+1,\alpha}b_{t+1,\beta}.$$
Observe that
\[ S(f, g) - b_{j\beta}f_{aq} = \left( \sum_{\delta=2}^{r} 2b_{t+1-\delta,\alpha}(b_{jq}b_{\delta\beta} - b_{j\beta}b_{\delta q}) \right) + b_{r+1,\alpha}(b_{jq}b_{r+1,\beta} - b_{j\beta}b_{r+1,q}), \]
where \( \text{in}(b_{j\beta}f_{aq}) = \text{in}(S(f, g)) \), \( \text{in}(b_{t+1-\delta,\alpha}(b_{jq}b_{\delta\beta} - b_{j\beta}b_{\delta q})) < \text{in}(S(f, g)) \). Therefore, \( S(f, g) \) has remainder term 0 modulo the chosen generators. A similar computation works for the case when \( j = t, q = \alpha \).

When \( f = f_{ij}, g = f_{pq} \), such that \( \text{in}(f) \) and \( \text{in}(g) \) has one common factor. Since \( \text{in}(f_{ij}) = 2b_{ti}b_{1j}, \text{in}(f_{pq}) = 2b_{tp}b_{1q} \), from the assumption we have \( i = p \) or \( j = q \). When \( i = p \), we have
\[ S(f_{ij}, f_{pq}) = 2 \left( \sum_{\delta=2}^{r} b_{t+1-\delta,p}(b_{1q}b_{\delta j} - b_{1j}b_{\delta q}) \right) + b_{r+1,p}(b_{1q}b_{r+1,j} - b_{1j}b_{r+1,q}). \]
Therefore, \( S(f, g) \) has remainder term 0 modulo the chosen generators. A similar computation works for the case when \( j = q \).

Therefore, the elements \( \{ij \mid pq : i < j, p < q \} \cup \{f_{\alpha\beta} : \alpha \leq \beta \} \) form a Gröbner basis of the ideal \( \langle \Lambda^2 \mathcal{B}, \mathcal{B}^t \mathcal{H} \mathcal{B} \rangle \). In particular, the initial ideal \( \text{in}(I) \) is generated some monomials of the form \( b_{ij}b_{pq} \) with \( (i, j) \neq (p, q) \). Now, if \( \overline{b_{11}} \in \frac{k[\mathcal{B}]}{\Lambda^2 \mathcal{B}, \mathcal{B}^t \mathcal{H} \mathcal{B}} \) is a zero-divisor, then since \( \text{Spec} \ R_{s,12} \) is irreducible, we have \( \overline{b_{11}N} = 0 \) for some integer \( N > 1 \). Hence, \( b_{11}^N = \text{in}(b_{11}^N) \in \langle \Lambda^2 \mathcal{B}, \mathcal{B}^t \mathcal{H} \mathcal{B} \rangle \). This is impossible since the later is generated by square-free quadratic monomials. \( \square \)

**Proposition 4.3.2.** The affine schemes \( \text{Spec} \ R_{s,1} \) and \( \text{Spec} \ R_{s,2} \) are integral.

**Proof.** The affine scheme \( \text{Spec} \ R_{s,1} \) is integral by the theory of determinant varieties [BU98, (2.11)].

To show \( \text{Spec} \ R_{s,2} \) is integral, we retain the notation in Lemma 4.1.8. By Remark 4.1.10, we have:
\[ R_s \simeq \frac{k[X]}{IJ + K}, \quad R_{s,1} \simeq \frac{k[X]}{I + K}, \quad R_{s,2} \simeq \frac{k[X]}{J + K}, \quad R_{s,12} \simeq \frac{k[X]}{I + J + K}. \]
Here, the isomorphisms are described in Theorem 4.1.9. Furthermore, \( IJ + K \) is radical by Corollary 4.2.6; \( I + K \) is prime as we just showed; \( I + J + K \) is prime by Proposition 4.3.1; \( \sqrt{J + K} \) is prime by Proposition 4.2.4(i). We want to show that \( J + K \) is a prime ideal.

Given \( f^n \in J + K \subset I + J + K \), we have \( f \in I + J + K \). Write \( f = \alpha + \beta \), where \( \alpha \in I \) and \( \beta \in J + K \). Then \( f^n = \alpha^n + \beta^n \gamma \) for some \( \gamma \in k[X] \), hence \( \alpha^n \in J + K \). Now we have \( \alpha^{n+1} \in I(J + K) \subset IJ + K \). Therefore, \( \alpha \in IJ + K \subset J + K \), and \( f = \alpha + \beta \in J + K \). Therefore, \( J + K = \sqrt{J + K} \) is a prime ideal. \( \square \)

**Corollary 4.3.3.** (i) The affine schemes \( \text{Spec} \ R_{s,1} \) and \( \text{Spec} \ R_{s,2} \) are the irreducible components of \( \text{Spec} \ R_s \). They are normal and Cohen-Macaulay, with singular locus \( \{\ast\} \).

(ii) The affine scheme \( \text{Spec} \ R_{s,12} \) is the intersection of the irreducible components of \( \text{Spec} \ R_s \). It is normal and Cohen-Macaulay, with singular locus \( \{\ast\} \).
Proof. (i) Recall Lemma 4.1.7 that \((\text{Spec } R_s)^\text{red} = (\text{Spec } R_{s,1})^\text{red} \cup (\text{Spec } R_{s,2})^\text{red}\). We can remove the “red” by Proposition 4.3.2 and Corollary 4.2.6. Therefore, these two closed subschemes are the irreducible components of \(\text{Spec } R_s\). By Theorem 2.5.2 they are Cohen-Macaulay. They are normal since the singular locus is of dimension 0 defined by \(A = 0, B = 0\).

(ii) Recall that, for a notherian scheme \(Z\) with two irreducible components \(X\) and \(Y\) which are Cohen-Macaulay, if the intersection \(X \cap Y\) is of codimension 1 in both \(X\) and \(Y\), then \(X \cap Y\) is Cohen-Macaulay if and only if \(Z = X \cup Y\) is Cohen-Macaulay; cf. [Gö01, Lemma 4.22]. Since \(\text{Spec } R_s\) is Cohen-Macaulay by Theorem 4.2.1, \(\text{Spec } R_{s,1}\) and \(\text{Spec } R_{s,2}\) are Cohen-Macaulay by (i), applying [Gö01, Lemma 4.22], we have \(\text{Spec } R_{s,12}\) is Cohen-Macaulay.

Now, we have finished the proof of Theorem 4.2.1: (i) follows from Corollary 4.2.5 and Corollary 4.2.6. (ii)(iii) follows from Corollary 4.3.3.

Remark 4.3.4. Since the intersection of the irreducible components \(\text{Spec } R_{s,12}\) is singular with the singular point defined by \(A = 0, B = 0\), in particular, it is not regular. Therefore, the local model does not admit semi-stable reduction.

Corollary 4.3.5. The local model \(M^{\text{loc}}_{\{\kappa\}}(n-1, 1)\) for strongly non-special \(\{\kappa\}\) over the ring of integers is normal and Cohen-Macaulay. It is Gorenstein when \(n = 4\kappa + 2\).

Proof. Recall that a flat scheme of finite type \(Y\), over a discrete valuation ring with normal generic fiber and reduced special fiber, is normal, cf. [PZ13, Proposition 9.2]. For us, since the generic fiber of the local model is smooth, the normality follows. In particular, \(\pi \in \mathcal{O}_F\) is not a zero-divisor. Therefore, by [Stacks, 02JN,0BJJ], the local model \(M^{\text{loc}}_I\) is Cohen-Macaulay (resp. Gorenstein) if and only if its special fiber \(M^{\text{loc}}_{I,s}\) is Cohen-Macaulay (resp. Gorenstein).

Remarks 4.3.6. We can also recover several known results from Theorem 4.1.9.

(i) When \(\kappa = 0\), the matrix \(X_3\) has size 0. Hence, the ring becomes \(k[X_4]/\Lambda^2(X_4)X_4 - X_4^{\text{ad}}\text{tr}(X_4)^{\dag}\), this recovers Pappas’s result [Pa00] (but note that Pappas use a different choice of hermitian form).

(ii) When \(\kappa = m\) and \(n = 2m + 1\), the matrices \(X_3\), resp., \(X_4\) are of size \(1 \times (n - 1)\) and \(1 \times 1\). The equation \(\text{tr}(X_4) = 0\) implies \(X_4 = 0\), and the ring \(R = k[X_3,X_4]/\Lambda^2(X_3,X_4)X_4 - X_4^{\text{ad}}\text{tr}(X_4)^{\dag}\) is isomorphic to \(k[X_3]\), this recovers Smithling’s computation [Sm15, 3.10, 3.11].
(iii) When $\kappa = m - 1$ and $n = 2m$, the equations $\text{tr}(X_4) = 0$ and $X_4 = X_4^{ad}$ are equivalent to $X_4 = \begin{pmatrix} \ast \end{pmatrix}$, hence the ring is isomorphic to the ring

$$
\Lambda^2 \left( X_3 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \cong \frac{k[X_3, x_1, x_2]}{x_1 x_2, \Lambda^2 X_3, \left( \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \right)} X_3
$$

This recovers the case studied by Yu [Yu19, 1.4.22, 1.4.23, 1.4.24].

(iv) We do not expect to recover the defining equations in the $\pi$-modular case [RSZ17] ($I = \{m\}$ for $n = 2m$) since then both $X_3, X_4$ have size 0. This should not be surprising since in this (and only in this case) the worst point does not lie in the local model; cf. [PR09, Remark 7.4].

5. Reducedness for the general parahoric subgroup

5.1. Recollection of the (partial) affine flag varieties. In this subsection, we quickly review the theory of (partial) affine flag varieties of the ramified unitary group, and its relation to the local model.

5.1.1. Local Dynkin diagram of quasi-split unitary groups. For the rest of the section, we will use the local Dynkin diagram instead of the convention (1.2.1) used in [PR08, 4.a]. When $n = 2m + 1$, the local Dynkin diagram takes the form

$$
\begin{array}{ccccccc}
& & & & \ast & \ast & \ast \\
\end{array}
$$

We numerate the vertices from left to right as $0, 1, \cdots, m$.

When $n = 2m$, the local Dynkin diagram takes the form

$$
\begin{array}{ccccccc}
& & & & \ast & \ast & \ast
\end{array}
$$

A special point $m'$ is related to a lattice given by

$$
\Lambda_{m'} := \sum_{i=1}^{m-2} \pi^{-1} O_F e_i + \pi^{-1} O_F e_{m-1} + \pi^{-1} O_F e_{m+1} + \sum_{j=m+2}^{n} O_F e_j,
$$

using the notation in §2.1.2. The other vertices from left to right are indexed by $0, 1, \cdots, m - 2, m$.

The parahoric subgroups associated with $m'$ and $m$ are isomorphic under diagram automorphism. The parahoric subgroup which corresponds to the index set $\{m - 1, m\}$ in §2.1.2 equals the one constructed from the vertices $\{m', m\}$. When we use the local Dynkin diagram, the associated lattices do not form a lattice chain as defined in [RZ96, Definition 3.1]; cf. §6.1.4. But this is not a concern for this section. For the rest of the section, the index set $\mathcal{I}$ will always refer to a non-empty subset of vertices of the local Dynkin diagram. We will continue to call the associated periodic lattice as a “lattice chain” indexed by $\mathcal{I}$. Besides, different moduli functors, $M^I_{\mathcal{I}}$, are still valid in this context.
5.1.2. Unitary affine flag varieties. Let $F_0 = k((t))$ be the field of Laurent power series with coefficients in $k$, and $F = k((u))$ with uniformizer $u$ such that $u^2 = t$. We define the split hermitian form as in §2.1.1 and associate with it the unitary similitude group $G = \text{GU}(F^n, \phi)$.

We briefly list some facts about the unitary affine flag variety. The precise descriptions are in [PR09, Section 3.2]. For any non-empty subset $I$ of vertices of the local Dynkin diagram, Pappas and Rapoport define a moduli functor $\mathcal{F}_{\ell_I}$, which is represented by an ind-scheme, called unitary affine flag variety, which parametrizes the space of periodic self-dual “lattice chains” with some restrictions.

The functor $\mathcal{F}_{\ell_I}$ does not necessarily define the same affine flag variety as described in [PR08]. It corresponds to the ind-scheme $L_G/L^+P_I$.

We are going to apply the Frobenius-splitting properties of the Schubert varieties in the $\mathcal{F}_{\ell_I}$, this follows from those in $L_G/L^+P\sigma^I$, cf. [PR09, §3.3].

Using the moduli description, we can embed the special fiber of the naive model $M_{I,s}^{\text{naive}}$ into the unitary affine flag variety $\mathcal{F}_{\ell_I}$; cf. [PR09, §3.2]:

$$M_{I,s}^{\text{loc}} \subset M_{I,s} \subset M_{I,s}^{\text{naive}} \hookrightarrow \mathcal{F}_{\ell_I}.$$ 

This embedding is equivariant under the action of $L^+P_I$; therefore, the reduced loci are the unions of affine Schubert varieties. This embedding is naturally functorial from the constructions; in other words, given $I' \subset I$, we have the commutative diagram:

$$
\begin{array}{ccc}
M_{I,s}^{\text{loc}} & \hookrightarrow & M_{I,s} \\
\downarrow & & \downarrow \\
M_{I',s}^{\text{loc}} & \hookrightarrow & M_{I',s}.
\end{array}
$$

The coherence conjecture implies that $M_{I,s}^{\text{loc}}$ is, indeed, the union of Schubert varieties parameterized by the admissible set $\text{Adm}^I(\mu)$, where $\mu$ is the minuscule cocharacter of signature $(n-1,1)$; cf. [PR08, Theorem 11.3] and [Zhu14, Theorem 8.1]. In particular, $M_{I,s}^{\text{loc}}$ is reduced; cf. Theorem 2.5.2.

5.2. Flatness in the case of general parahoric level structure. Fix $\mathcal{I}$ a non-empty subset of vertices of the local Dynkin diagram. Consider the natural embedding

$$M_{I,s}^{\text{loc}} \subset M_{I,s} \hookrightarrow \mathcal{F}_{\ell_I}.$$ 

For any $\tau \in \mathcal{I}$, we define $\widetilde{M}_{I,\tau}$ to be the pull-back along the affine flag varieties

$$
\begin{array}{ccc}
\widetilde{M}_{I,\tau} & \longrightarrow & \mathcal{F}_{\ell_I} \\
\downarrow & \downarrow & \\
M_{\{\tau\},s} & \longrightarrow & \mathcal{F}_{\{\tau\}}.
\end{array}
$$
**Proposition 5.2.1.** Identify the special fiber of the strengthened spin model \( M_{I,s} \) with its image in \( \mathcal{F}_{\ell_I} \), it equals the scheme-theoretic intersection \( \bigcap_{i \in I} \overline{M_{I,(i)}} \subset \mathcal{F}_{\ell_I} \).

**Proof.** The case when \( m' \in I \) will be discussed in the end of the proof. Assume now \( m' \notin I \).

In the pull-back diagram (5.2.1), \( M_{(\tau),s}^{(\tau)}, \mathcal{F}_{\ell(\tau)} \), and \( \mathcal{F}_{\ell_I} \) have moduli descriptions. Using them we describe the pull-back \( f_{M_{I,\tau}} \) as a moduli functor which to a \( k \)-algebra \( R \) associates the set of \( R[[u]] \)-lattice chains indexed by \( \pm I + n\mathbb{Z} \):

\[
\cdots \subset L_{i_1} \subset L_{i_2} \subset \cdots \subset L_{i_N} \subset u^{-1}L_{i_1} = L_{i_1+n} \subset \cdots
\]

in \( \mathcal{F}_{\ell_I} \), such that the sub-chain \( \{L_j\}_{j \in \pm I + n\mathbb{Z}} \) satisfies the following properties:

(i) There is a natural inclusion of lattice in \( R((u))^n \):

\[
t \lambda_j \otimes_{k[[t]]} R[[t]] \subset L_j \subset \lambda_j \otimes_{k[[t]]} R[[t]].
\]

(ii) Through the identification

\[
\lambda_j \otimes_{k[[t]]} R[[t]] \otimes_{R[[t]]} R \simeq \Lambda_j \otimes_{\mathcal{O}_{F_0}} R,
\]

the lattices \( L_j \) is isomorphic to a chain of filtrations \( \{\mathcal{F}_j \subset \Lambda_j \otimes_{\mathcal{O}_{F_0}} R\}_{j \in \pm I + n\mathbb{Z}} \), satisfies the strengthened spin model axioms.

Note that the embedding defined in [PR09, §3.2] has a scalar by \( u \), hence the lattice chain inclusion in their paper is \( u\lambda_\bullet \subset L_\bullet \subset u^{-1}\lambda_\bullet \).

Using the moduli description of \( \widetilde{M}_{I,\tau} \), one can see that the scheme-theoretic intersection \( \bigcap_{i \in I} \overline{M_{I,i}} \) represents the lattice chains of (5.2.2) in \( \mathcal{F}_{\ell_I} \) such that all of them satisfy the properties (i) and (ii). This is exactly the moduli functor of the strengthened spin model \( M_{I,s} \).

Finally, when \( m' \in I \). If \( m \notin I \), then the same proof follows, by replacing \( m' \) to \( m \) using the diagram automorphism. When \( \{m, m'\} \subset I \), the lattice chain will be of the form:

\[
\cdots \subset L_\bullet \subset L_{m'} \subset \cdots \subset L_m \subset L_{m'} \subset \cdots
\]

Except that, the same argument follows.

\( \square \)

We are ready to prove Theorem 1.2.1.

**Proof of Theorem 1.2.1.** When \( \{m-1, m\} \subset I \), by [PR08, Remark 4.2] and the proof of [RSZ18, Proposition 9.12], the strengthened spin model \( M_I \) is isomorphic to \( \overline{M_{I \cup \{m'\} \setminus \{m-1\}}} \) with local Dynkin diagram index set. Therefore, we only need to prove the same theorem with the index set \( I \) in the formalism of the local Dynkin diagram.

By Theorem 4.2.1 for strongly non-special case, and §1.4 for the other cases, the special fiber of the strengthened spin model \( M_{\{j\},s} \) is reduced. Therefore, \( M_{\{j\},s}^\text{loc} = M_{\{j\},s} \hookrightarrow \mathcal{F}_{\ell_I} \).
is a union of Schubert varieties. Hence pull-backs \( \tilde{M}_{I \setminus \{i\}} \subset \mathcal{F}_{\ell_I} \) are also unions of Schubert varieties. Since the intersection \( M_I = \bigcap_{i \in I} \tilde{M}_{I \setminus \{i\}} \subset \mathcal{F}_{\ell_I} \) is an intersection of unions of Schubert varieties, it is Frobenius split by \([Gö01, \text{Corollary 2.7}]\); hence reduced. □

**Corollary 5.2.2.** For the ramified unitary local model of signature \((n – 1, 1)\), the strengthened spin condition \(\text{LM8} \) implies the wedge condition \(\text{LM6} \).

*Proof.* Since the strengthened spin models are flat, we only need to verify this over the special fiber of the strengthened spin models. It further boils down to the maximal parahoric cases by Proposition 5.2.1.

Notice that the computations in Proposition 3.4.4 are valid for any indices \( 0 \leq \kappa \leq m \). Therefore, the similar basis in the Corollary 3.4.7 can be listed for self-dual, \( \pi \)-modular, and almost \( \pi \)-modular case. When \( \kappa \) is self-dual or almost \( \pi \)-modular, we get the same basis as in the Corollary 3.4.7 (in almost \( \pi \)-modular case, we recover \([Sm15, \text{Corollary 4.14}]\)). In particular, the elements \( e_S \) occurring in the basis are all for \( S \) of type \((1, n – 1)\) and \((0, n)\). Therefore, when taking the affine chart of the worst point, all the \( 2 \times 2 \)-minors of \( X \) vanishes, and the wedge condition follows; cf. Proposition 3.5.2. When \( \kappa \) is \( \pi \)-modular, Proposition 3.4.4 Case 2 will only have situation (iv). Therefore there is no element \( e_{\{n+1, \ldots, 2n\}} \) in the basis. As a consequence, the worst point does not lie in the local model; cf. \([PR09, \text{Remark 7.4}]\). However, the other elements in Corollary 3.4.7 will still form a basis. In this case, consider the open chart of the “best point”, cf. \([PR09, \S 5.3]\). Since the elements \( e_S \) occurring in the basis are all for \( S \) of type \((1, n – 1)\), the wedge condition again follows from the strengthened spin condition. □

From Proposition 5.2.1 one can deduce

**Corollary 5.2.3.** The local model is equal to the following scheme-theoretical intersection

\[ M_{I,s}^{\text{loc}} = \bigcap_{i \in I} \pi_{I \setminus \{i\}}^{-1}(M_{\{i\},s}^{\text{loc}}), \]

where \( \pi_{I \setminus \{i\}} : M_{I,s}^{\text{loc}} \to M_{\{i\},s}^{\text{loc}} \) is the natural projection.

Since the special fiber of the local model \( M_{I,s}^{\text{loc}} \) is identified with the union of Schubert varieties indexed by \( \text{Adm}^T(\mu) \), we deduce the vertex-wise conjecture:

**Corollary 5.2.4.** \( \text{Adm}^T(\mu) = \bigcap_{i \in I} \pi_{I \setminus \{i\}}^{-1}(\text{Adm}^{\{i\}}(\mu)) \). □

This proves the vertex-wise conjecture in \([PR09, \S 4.2]\) for the ramified unitary group of signature \((n – 1, 1)\). Haines and He have proven the conjecture for a general group using purely combinatorial methods, cf. \([HH17]\).

Note that one of the motivations of the vertex-wise conjecture is proving the Corollary 5.2.3; cf. \([PR09, \text{Proposition 4.5}]\) and \([HH17, \text{Remark 8.3}]\).
6. Consequences and applications

6.1. Moduli description of the general unitary local model. In this subsection, we will give a moduli description for the Pappas-Zhu local model of any ramified unitary group with signature \((n - 1, 1)\). In particular, this will include the case when the hermitian form is not split (hence, it may produce a non-quasi-split unitary group). Moreover, even in the split case, such description (intrinsically) does not depend on the choice of a basis. Therefore, it should be more flexible in the applications. Note that the strategy applied here is well-known and has been used by many experts; however, we will provide more detail due to the lack of references.

6.1.1. Hermitian forms over a local field. Suppose \(F/F_0\) is a quadratic extension of \(p\)-adic fields and \(V\) is a hermitian space of dimension \(\dim_F V \geq 3\), equipped with a hermitian form \(\phi : V \times V \to F\). We will keep this notation throughout.

For each dimension \(n\), there are two isomorphic classes of the hermitian spaces: \(V^+\) and \(V^-\), classified by the Hasse invariant. The space \(V^+=F^n\) equips the split hermitian form given by \(h_n(xe_i, ye_j) = \bar{xy} \delta_{i,n+1-j}\). The induced unitary similitude group \(GU(V^+, h_n)\) is quasi-split. The other space \(V^-=F^{n-2} \oplus F^2\) is equipped with a non-split hermitian form \(h^{-n} = h^{n-2} \oplus h_\alpha\), where \(h_{n-2}\) is the split hermitian form in \(F^{n-2}\) and \(h_\alpha((x_1, x_2), (y_1, y_2)) = \bar{x_1}y_1 - \alpha \bar{x_2}y_2\) for \(\alpha\) some representative element of the nontrivial element in \(F^\times_0/\text{Norm}(F^\times) \simeq \mathbb{Z}/2\mathbb{Z}\).

When \(n\) is even, the non-split hermitian space \((V^-, h^{-n})\) produces a non-quasi-split unitary similitude group \(GU(V^-, h^{-n})\). When \(n\) is odd, \(GU(V^+, h_n) \simeq GU(V^-, h^{-n})\) are both quasi-split. In both cases, the hermitian form splits over some finite unramified extension. Therefore, the corresponding unitary group will become quasi-split after passing to some finite unramified extension.

6.1.2. Pappas-Zhu local model. Assume from now on that \(F/F_0\) is a ramified extension, with uniformizers \(\pi \in F\) and \(\pi_0 \in F_0\), respectively, such that \(\pi^2 = \pi_0\) (we assume \(p \neq 2\) in the paper).

Let \((H, \phi)\) be any hermitian space over \(F/F_0\), with the corresponding unitary similitude group \(G := GU(H, \phi)\). Fix a local model triple \((G, \{\mu\}, K)\) (cf. [HPR20, §2.1]), where \(\{\mu\}\) is a geometric conjugacy class of cocharacter of \(G\), corresponding to signature \((n - 1, 1)\) (the reflex field will be \(F\) because \(n \geq 3\)); cf. [PR09, §2.4]. Besides, \(K \subset G(F_0)\) is a parahoric subgroup, with the associated smooth group scheme \(K\) defined over \(O_{F_0}\). Attached to the local model triple, we obtain the Pappas-Zhu model \(M_{K}^{lc}(G, \{\mu\})\) defined over \(O_F\), equipped with a \(K_{O_F}\)-action; cf. [HPR20, Theorem 2.5].

Let \(L_0/F_0\) be some finite unramified extension, such that the base change of the hermitian form \(\phi_{L_0}\) splits. Define \(L := FL_0\). By choosing a splitting basis, we obtain an isomorphism

\[ (6.1.1) \quad G_{L_0} = GU(H, \phi)_{L_0} \simeq GU(V^+, h_n). \]
Through the isomorphism, up to conjugacy, the parahoric subgroup $K(\mathcal{O}_L) = K_{L_0} \subset G(L_0)$ becomes $P_I^0$ for some index $I$ as in §2.1. We call $K$ strongly non-special if the corresponding $P_I^0$ is. Therefore, we obtain an isomorphism between the local model triples

$$(6.1.2) \quad (G, \{\mu\}, K_{L_0}) := (G_{L_0}, \{\mu_{L_0}\}, K_{L_0}) \simeq (\text{GU}(V^+, h_n), \{\mu\}, P_I^0).$$

By [HPR20, Theorem 2.5, Proposition 2.7], it induces an isomorphism between Pappas-Zhu models

$$M_{K}^{\text{loc}}(G, \{\mu\}) \otimes_{\mathcal{O}_F} \mathcal{O}_L \xrightarrow{\sim} M_{K_{L_0}}^{\text{loc}}(G_{L_0}, \{\mu_{L_0}\}) \xrightarrow{\sim} M_{P_I^0}^{\text{loc}}(\text{GU}(V^+, h_n), \{\mu\}).$$

These isomorphisms are equivariant under the group actions of $K_{L_0}$ and $P_I^0$, respectively.

The following proposition is well-known among experts.

**Proposition 6.1.3.** The Pappas-Zhu model $M_{P_I^0}^{\text{loc}}(\text{GU}(V^+, h_n), \{\mu\})$ associated to the local model triple $(\text{GU}(V^+, h_n), \{\mu\}, P_I^0)$ has a natural closed embedding to the naive model $M_{I}^\text{naive}$. Under the closed embedding, it is identified with the local model $M_{I}^{\text{loc}}$ defined in §2.5.

**Proof.** By the discussion in [PZ13, 8.2.4] (see also [HPR20, Lemma 4.1]), the Pappas-Zhu model can embed into the naive model $M_{P_I^0}^{\text{loc}}(\text{GU}(V^+, h_n), \{\mu\}) \hookrightarrow M_{I}^\text{naive}$. Since both Pappas-Zhu model and the local model $M_{I}^{\text{loc}}$ are defined as the scheme-theoretic closure over the generic fiber, the identification follows. \qed

6.1.4. **Moduli description.** For a parahoric subgroup $K \subset G(F_0)$, we choose a polarized self-dual chain of lattices $L$ in $H$ in the sense of [RZ96, Definition 3.14], whose connected component of the stabilizer is $K$; see [RZ96] or [PZ13, §5] for a summary.

Note that when $n$ is even, and the corresponding facet of the parahoric subgroup contains two special vertices $m'$ and $m$ (cf. §5.1.1), the naive filtration in the Bruhat-Tits building is of the form:

$$\cdots \subset \Lambda_m \subset \Lambda \subset \Lambda \subset \Lambda \subset \cdots,$$

This is not a chain of lattices in the sense of [RZ96, Definition 3.14]. But one can modify it into a chain of lattices by replacing $\Lambda_m$ by $\Lambda_{m-1}$; cf. [PR08, 4.b].

The base change $\mathcal{L}_{L_0}$ of the lattice chain to $\mathcal{O}_L$ in $H_{L_0}$ corresponds to the parahoric subgroup $K_{L_0} \subset G(L_0)$. When $(H, \phi) = (V^+, h_n)$, up to conjugation, the lattice chain is exactly the one we defined in §2.1.1.

Among those data, we can define a moduli functor $\mathcal{M}_{\text{naive}}(G, \mathcal{L})$ over $\text{Spec} \mathcal{O}_F$ as in [RZ96, Definition 3.27], which we will call the naive model.

We are now going to formulate the strengthened spin condition. Recall Remark 2.4.1, since the symmetric form on $V = H \otimes_{F_0} F$ always splits, we have a natural splitting $W = \Lambda^N V = W_1 \oplus W_{-1}$. The sign $\pm 1$ here is chosen via the isomorphism to the pair
(V_F^+, h_n). As mentioned in §2.3.2, the splitting is intrinsic once we fix a framing basis. Besides, the splitting V = V_\pi \oplus V_\pi always exists; we define W^{r,s} := \Lambda^r V_\pi \otimes \Lambda^s V_\pi. For any lattice \Lambda \subset V, we define W(\Lambda)^{r,s} := W^{r,s} \cap W(\Lambda) \subset W, where W(\Lambda) := \Lambda^{\otimes} (\Lambda \otimes \mathcal{O}_F \otimes \mathcal{O}_F). For any \mathcal{O}_F\text{-}algebra R, we define

\[ L^{r,s}_{(-1)^r}(\Lambda)(R) := \text{im}[W(\Lambda)^{r,s} \otimes \mathcal{O}_F R \to W\Lambda(i) \otimes \mathcal{O}_F R]. \]

We are ready to give a moduli description of the Pappas-Zhu model \( M_{K}^{\text{loc}}(G, \{\mu\}) \). We formulate the following moduli functor:

**Definition 6.1.5.** Let G and \( \mathcal{L} \) be as above. We define a moduli functor \( \mathcal{M}_{\mathcal{L}}(G, \mu) \) over \( \text{Spec} \mathcal{O}_F \), such that given any \( \mathcal{O}_F\text{-}algebra R, \) we associate to it the following data:

- a functor \( \Lambda \mapsto \mathcal{F}_\Lambda \) from \( \mathcal{L} \) (regarded as a category in which the morphisms are inclusions of lattices in V) to the category of \( \mathcal{O}_F \otimes \mathcal{O}_F \text{-} R\text{-}modules; and
- a natural transformation of functors \( j_\Lambda : \mathcal{F}_\Lambda \to \Lambda \otimes R \),

such that the following axioms hold:

**LM1&2.** For all \( \Lambda \in \mathcal{L} \), \( \mathcal{F}_\Lambda \) is a \( \mathcal{O}_F \otimes \mathcal{O}_F \text{-} R\text{-}submodule of \( \Lambda \otimes \mathcal{O}_F \text{-} R \), which is an \( R\text{-}direct \) summand, and \( j_\Lambda \) is the inclusion \( \mathcal{F}_\Lambda \subset \Lambda \otimes \mathcal{O}_F \text{-} R \).

**LM3.** For all \( \Lambda \in \mathcal{L} \), the composition

\[ \mathcal{F}_\Lambda \subset \Lambda \otimes \mathcal{O}_F \text{-} R \xrightarrow{\pi} \pi \Lambda \otimes \mathcal{O}_F \text{-} R \]

identifies \( \mathcal{F}_\Lambda \) with \( \mathcal{F}_{\pi \Lambda} \).

**LM4.** For all \( \Lambda \in \mathcal{L} \), under the perfect pairing \( (\Lambda \otimes \mathcal{O}_F \text{-} R) \times (\hat{\Lambda} \otimes \mathcal{O}_F \text{-} R) \to R \) induced by \( \langle , \rangle \), where \( \hat{\Lambda} \) denotes the dual lattice of \( \Lambda \), the submodules \( \mathcal{F}_\Lambda \) and \( \mathcal{F}_{\pi \Lambda} \) pair to 0.

**LM8.** For all \( \Lambda \in \mathcal{L} \), the line \( \bigwedge_{R} \mathcal{F}_\Lambda \subset W(\Lambda) \otimes \mathcal{O}_F \text{-} R \) is contained in \( L^{n-1,1}_{(-1)}(\Lambda)(R) \).

Note that the naive model \( \mathcal{M}_{\mathcal{L}}^{\text{naive}}(G, \mu) \) is defined by replacing axiom LM8 by Kottwitz condition (determinant condition).

**Remark 6.1.6.** From the definition, when the hermitian form is split and \( \mathcal{L} \) is the standard lattice chain defined in §2.1, we can identify the \( \mathcal{M}_{\mathcal{L}}^{\text{naive}}(G, \mu) \) and \( \mathcal{M}_{\mathcal{L}}(G, \mu) \) with the naive model and strengthened spin model defined in §2.2 and §2.4 respectively.

**Theorem 6.1.7.** The moduli functor \( \mathcal{M}_{\mathcal{L}}(G, \mu) \) represents the Pappas-Zhu model \( M_{K}^{\text{loc}}(G, \{\mu\}) \).

**Proof.** By [HPR20, Lemma 4.1], the Pappas-Zhu model \( M_{K}^{\text{loc}}(G, \{\mu\}) \) has a natural closed embedding into the naive model \( \mathcal{M}_{\mathcal{L}}^{\text{naive}}(G, \mu) \):

\[ M_{K}^{\text{loc}}(G, \{\mu\}) \hookrightarrow \mathcal{M}_{\mathcal{L}}^{\text{naive}}(G, \mu) \hookrightarrow \mathcal{M}_{\mathcal{L}}(G, \mu). \]

As a consequence, we only need to identify \( \mathcal{M}_{\mathcal{L}}(G, \mu) \) and \( M_{K}^{\text{loc}}(G, \{\mu\}) \) as closed subschemes of \( \mathcal{M}_{\mathcal{L}}^{\text{naive}}(G, \mu) \).

Choose \( L_0/F_0 \) to be some finite unramified extension that splits the hermitian form, hence inducing the isomorphism of the local model triple and corresponding Pappas-Zhu
model as in (6.1.1) and (6.1.2). Fix the splitting basis as in §2.1, we have the following diagram

\[
\begin{array}{ccc}
\mathcal{M}_K^{\text{loc}}(G, \{\mu\})_{\mathcal{O}_L} & \xleftarrow{\cong} & \mathcal{M}_E^{\text{naive}}(G, \mu)_{\mathcal{O}_L} \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{M}_P^{\text{loc}}(G, h_n, \{\mu\})_{\mathcal{O}_L} & \xleftarrow{\cong} & M_I.
\end{array}
\]

The right-hand square commutes from the moduli description. The left-hand square commutes since the Pappas-Zhu model can be defined as the scheme-theoretic closure of the generic fiber of the naive model (see [HPR20, Lemma 4.1]).

By Proposition 6.1.3, the bottom two closed subschemes of \(M_I^{\text{naive}}\) are equal. Therefore, through the commutative diagram, we identify the closed subscheme \(\mathcal{M}_K^{\text{loc}}(G, \{\mu\})_{\mathcal{O}_L}\) with \(\mathcal{M}_L(G, \mu)_{\mathcal{O}_L}\). Descending from \(\mathcal{O}_L\) to \(\mathcal{O}_F\), we get an identification between \(\mathcal{M}_K^{\text{loc}}(G, \{\mu\})\) and \(\mathcal{M}_L(G, \mu)\).

\[\square\]

6.2. Moduli description of the irreducible components over the special fiber. In this subsection, we will give a moduli description for the irreducible components of the special fiber of the local model, in the case of the strongly non-special parahoric subgroup.

Consider the moduli functor \(\mathcal{M} := \mathcal{M}_L(G, \mu)\) defined in (6.1.2). One can choose a lattice \(\Lambda \in \mathcal{L}\) such that

\[(6.2.1)\quad \Lambda^\vee \subset \Lambda \subset \pi^{-1}\Lambda^\vee.\]

We can complete it into a self-dual chain of lattices. When the hermitian space splits, such lattice \(\Lambda\) will correspond to \(\Lambda_\kappa\) in §2.1.

The inclusions of the lattices induce two transition maps in the moduli functor:

\[(6.2.2)\quad \mathcal{F}_\Lambda^\Lambda \xrightarrow{\lambda^\Lambda} \mathcal{F}_\Lambda \xrightarrow{\lambda} \mathcal{F}_{\pi^{-1}\Lambda^\vee}.\]

By periodicity, one can restrict the moduli functor \(\mathcal{M}\) into the sub-lattices chain (6.2.1) without losing any information. We will use this convention in this subsection for simplicity.

**Definition 6.2.1.** The projective scheme \(\mathcal{M}_{s,1}\) is defined to be a subscheme of \(\mathcal{M}_s\). It represents the moduli problem, which sends each \(\mathcal{O}_F \otimes \mathcal{O}_{F_0}\) \(k\)-algebra \(R\) to the set of all families

\[\mathcal{M}_{s,1}(R) := \left\{ (\mathcal{F}_\Lambda^\Lambda \xrightarrow{\lambda^\Lambda} \mathcal{F}_\Lambda \xrightarrow{\lambda} \mathcal{F}_{\pi^{-1}\Lambda^\vee}) \in \mathcal{M}(S) \mid \lambda(\mathcal{F}_\Lambda) \subset (\pi \otimes 1)\Lambda_S^\vee \right\}.\]

Similarly, we define \(\mathcal{M}_{s,2}\) as the projective scheme representing the moduli problem, which sends each \(\mathcal{O}_F \otimes \mathcal{O}_{F_0}\) \(k\)-algebra \(R\) to the set of all families

\[\mathcal{M}_{s,2}(R) := \left\{ (\mathcal{F}_\Lambda^\Lambda \xrightarrow{\lambda^\Lambda} \mathcal{F}_\Lambda \xrightarrow{\lambda} \mathcal{F}_{\pi^{-1}\Lambda^\vee}) \in \mathcal{M}(S) \mid \lambda^\Lambda(\mathcal{F}_\Lambda) \subset (\pi \otimes 1)\Lambda_S \right\}.\]

We define \(\mathcal{M}_{s,12} := \mathcal{M}_{s,1} \times_{\mathcal{M}_s} \mathcal{M}_{s,2}\) as their scheme-theoretic intersection.
Theorem 6.2.2. The geometric special fiber $\mathcal{M}_{s, k}$ of the local model $\mathcal{M} = \mathcal{M}_{\mathcal{L}}(G, \mu)$ has two irreducible components $\mathcal{M}_{s, 1, k}$ and $\mathcal{M}_{s, 2, k}$, with intersection $\mathcal{M}_{s, 12, k}$. When the hermitian space splits, all statements hold without base change from $k$ to $\bar{k}$.

Proof. By passing to some unramified extension, we may assume $\mathcal{M} = \mathcal{M}_{\text{loc}}^{\mathcal{L}}(\kappa)$ for some chosen $\kappa$ in Theorem 1.3.2. By conjugation, the lattices we chose in (6.2.1) can be identified with the lattices in (2.1.1):

\[ \Lambda \simeq \Lambda_\kappa, \Lambda^\vee \simeq \Lambda_{-\kappa} \text{ and } \pi^{-1}\Lambda^\vee \simeq \Lambda_{n-\kappa}. \]

By passing to the open affine chart of the worst point as in §3.1, with the reordered basis (3.1.3), we only need to identify two irreducible components: $\text{Spec } R_{s, 1}$ and $\text{Spec } R_{s, 2}$, with $\mathcal{M}_{s, 1}$ and $\mathcal{M}_{s, 2}$ respectively.

Choose a $R$-point $\mathcal{F}_\kappa = \begin{pmatrix} X \\ I_n \end{pmatrix} \in U_{\text{loc}}^{\mathcal{L}}(\kappa, s)$. By (3.2.1), we have

\[ \lambda(\mathcal{F}_\kappa) = \begin{pmatrix} I_{2\kappa} & 0 \\ I_{2\kappa} & I_{n-2\kappa} \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ 0 & 0 \end{pmatrix}. \]

Recall Theorem 3.5.8 that the strengthened spin condition implies $X_2 = 0$. Moreover, the worst point $(\pi \otimes 1)\Lambda_R^\vee = (\pi \otimes 1)\Lambda_{n-\kappa,R}$ is represented by $\begin{pmatrix} 0 \\ I_n \end{pmatrix}$. Therefore, $\lambda(\mathcal{F}_\kappa) \subset (\pi \otimes 1)\Lambda_{n-\kappa,R}$ is equivalent to $X_1 = 0$.

Similarly, choose a $R$-point $\mathcal{F}_{-\kappa} = \begin{pmatrix} Y \\ I_n \end{pmatrix}$; we have

\[ \lambda(\mathcal{F}_{-\kappa}) = \begin{pmatrix} I_{2\kappa} & 0 \\ I_{n-2\kappa} & I_{2\kappa} \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ Y_1 & Y_2 \end{pmatrix}. \]

Recall (3.2.3) that we have the identification

\[ Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} = \begin{pmatrix} -JX_1^tJ & -JX_3^tH \\ JH_{X_2}^tJ & JH_{X_4}^tH \end{pmatrix}. \]

Therefore, by Theorem 3.5.8, $\lambda^\vee(\mathcal{F}_{-\kappa}) \subset (1 \otimes \pi)\Lambda_{\kappa,R}$ is equivalent to $X_4 = 0$. Now the identification follows, since over the open affine chart of the worst point, the irreducible components of $\mathcal{M}_s$ are cut by $X_1 = 0$ and $X_4 = 0$ respectively; cf. Remark 4.1.7 and Corollary 4.3.3. \qed

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6.3. Application to Shimura varieties. Next, we give a moduli description for the arithmetic model of the unitary similitude Shimura varieties. We will only consider the hermitian form defined over an imaginary quadratic field for simplicity. All statements, except possibly the Hasse principle, will remain true when we generalize to a CM extension \( F/F_0 \), but will become much more complicated. See [LZ21, Section 11] for examples of general treatments.

Let \( \mathbb{F}/\mathbb{Q} \) be an imaginary quadratic extension. Consider an \( \mathbb{F} \)-vector space \( V \) of dimension \( n \), equipped with a skew-hermitian form \( \varphi : V \times V \to \mathbb{Q} \). We can define the unitary similitude group \( G = \text{GU}(V, \varphi) \). By fixing an embedding \( \mathbb{F} \to \mathbb{C} \), we assume the base change \( \varphi_\mathbb{C} \) has the signature \( (1, n-1) \), which induces a conjugacy class of maps \( h : \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to \mathbb{G}_\mathbb{R} \). The pair \((G_\mathbb{R}, h)\) is a (PEL-type) Shimura datum.

Now choose a prime \( p \neq 2 \) ramified in \( \mathbb{F}/\mathbb{Q} \); the base change to \( \mathbb{Q}_p \) gives a ramified extension \( \mathbb{F}/\mathbb{Q}_p \). Let \( G := G_{\mathbb{Q}_p} \) be the unitary similitude group associated with the induced hermitian form \( \phi = \varphi_{\mathbb{Q}_p} \). Choose a parahoric subgroup \( K_p \subset G(F) \), which can always be expressed as the connected component of the stabilizer of some self-dual chain of lattices \( L \) in \( V := V_F \). Up to conjugation, those lattice chains are in one-to-one correspondence with non-empty subsets of the local Dynkin diagram of \( G_F \). We call \( L \) non-connected if \( n = 2m \) and the corresponding subset of the local Dynkin diagram does not contain any special vertices; otherwise, we call it connected. Define \( K_p' := \text{Stab}_{G(F)}(L) \) and \( K_p = \text{Stab}^0_{G(F)}(L) \). We have \( K_p = K_p' \) if and only if the lattice is connected. Choose some open compact subgroup \( K^p \subset G(\mathbb{A}_F^p) \); define \( K \) := \( K_pK^p \) and \( K' := K_p'K^p \) as compact open subgroups of \( G(\mathbb{A}_\mathbb{Q}) \).

We define a Deligne-Mumford stack \( \mathcal{A}_{K'} \) over \( \text{Spec} \mathcal{O}_F \) modifying the definition in [RZ96]. Let \( R \) be a \( \mathcal{O}_F \)-algebra; we associate to it the groupoid of the data \((\mathcal{A}, \iota, \bar{\lambda}, \bar{\eta})\) so that

(i) \( \mathcal{A} = \{ A_\Lambda \}_{\Lambda \in \mathcal{L}} \) an \( \mathcal{L} \)-set of abelian schemes over \( \text{Spec} R \), compatibly endowed with an action of \( \mathcal{O}_F \):

\[
\iota : \mathcal{O}_F \to \text{End} \, A_\Lambda \otimes \mathbb{Z}_p.
\]

(ii) \( \bar{\lambda} \) a \( \mathbb{Q} \)-homogeneous principal polarization of the \( \mathcal{L} \)-set \( A \).

(iii) A \( K^p \)-level structure

\[
\bar{\eta} : H_1(A, A_f^p) \simeq V \otimes A_f^p \mod K^p,
\]

that respects the bilinear forms on both sides up to a constant in \((A_f^p)^\times\).

In addition, consider the de Rham homology with Hodge filtration

\[
\text{Fil}^1(A_\Lambda) \hookrightarrow \mathbb{D}(A_\Lambda).
\]

We can always find some étale cover \( \text{Spec} R'/\text{Spec} R \) such that the base change of the chain of de Rham homologies \( \{\mathbb{D}(A_\Lambda)\}_{\Lambda \in \mathcal{L}} \) can be identified with the chain lattices: \( \mathbb{D}(A_\Lambda) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \simeq \Lambda_{S'} \); cf. [RZ96, Appendix to Chapter 3]. Now we can consider \( \text{Fil}^1(A_\Lambda)_{S'} \) as a
subbundle of \( \Lambda_S \). We further impose the strengthened spin condition that
\[
\bigwedge^n_R (\text{Fil}_1 (A_\Lambda)_{R'}) \subset L_{-1}^{n-1,1} (\Lambda)(R').
\]
The strengthened spin condition we impose here is independent of the choice of \( \text{étale cover} \ Spec R' / Spec R \), thanks to the following lemma.

Lemma 6.3.1 ([RSZ18], Lemma 7.1). For any \( O_F \)-algebra \( R \) and lattice \( \Lambda \in \mathcal{L} \), the submodule \( L_{r,s}^{n-1} (\Lambda)(R') \subset \bigwedge^n \Lambda \otimes_{O_F} R \) is stable under the natural action of \( \text{Aut}(\mathcal{L})(R) \) on \( \bigwedge^n \Lambda \otimes_{O_F} R \).

The main theorem is the following:

Theorem 6.3.2. The moduli functor \( A_{K'} \) is representable by a flat and normal Deligne-Mumford stack over \( \text{Spec}(O_F) \). When \( \mathcal{L} \) is connected, it defines an integral model of \( \text{Sh}_{K'} (G, \mu) \). When \( \mathcal{L} \) is non-connected, there exists an \( \text{étale double cover} \ A_K \to A_{K'} \), together with an open immersion \( \text{Sh}_{K'} (G, h) \otimes F \to A_K \), such that the following diagram is cartesian:
\[
\begin{array}{ccc}
\text{Sh}_{K'} (G, h) \otimes F & \hookrightarrow & A_K \\
\downarrow & & \downarrow \\
\text{Sh}_{K'} (G, h) \otimes F & \hookrightarrow & A_{K'}.
\end{array}
\]

Proof. By Lemma 6.3.1, the strengthened spin condition can be passed through the local model diagram and corresponds to the strengthened spin model. The moduli functor \( A_{K'} \) lies in \( A_{K'}^{\text{naive}} \) defined in [RZ96], cut by the strengthened spin condition, which is a closed condition. Therefore, \( A_{K'} \) is representable by a Deligne-Mumford stack over \( \text{Spec}(O_F) \). By Theorem 6.1.7, and local model diagram, the DM stack \( A_{K'} \) is flat and normal. The \( \text{étale double cover} \) discussion in the end of the theorem follows from [PR09, Section 1.2, Proposition 1.1], due to the difference between \( P_I \) and \( P_{0_I} \). \( \square \)

Remark 6.3.3. As pointed out in [PR09, §1.3], the Shimura varieties \( \text{Sh}_{K'} (G, h) \) and \( \text{Sh}_{K'} (G, h) \) have isomorphic geometric connected components.

7. PROOFS OF SOME RESULTS IN SECTION 3

In this section, we give the proofs left in §3. Recall that \( \Lambda_\kappa \) is generated by
\[
\pi^{-1} e_1 \otimes 1, \cdots, \pi^{-1} e_\kappa \otimes 1, e_{\kappa+1} \otimes 1, \cdots, e_n \otimes 1; e_1 \otimes 1, \cdots, e_\kappa \otimes 1, e_{\kappa+1} \otimes 1, \cdots, e_n \otimes 1.
\]
We will boldface the worst terms in each term throughout the proof. Below are the terms of \( g_i \) and their worst terms:
\[
\begin{align*}
e_1 \otimes 1 - \pi e_1 \otimes \pi^{-1}, & \cdots, e_\kappa \otimes 1 - \pi e_\kappa \otimes \pi^{-1}; e_{\kappa+1} \otimes 1 - \pi e_{\kappa+1} \otimes \pi^{-1}, \cdots, e_n \otimes 1 - \pi e_n \otimes \pi^{-1}; \\
\frac{1}{2} e_1 \otimes 1 + \frac{1}{2} \pi e_1 \otimes \pi^{-1}, & \cdots, \frac{1}{2} e_\kappa \otimes 1 + \frac{1}{2} \pi e_\kappa \otimes \pi^{-1}; \frac{1}{2} e_{\kappa+1} \otimes 1 + \frac{1}{2} \pi e_{\kappa+1} \otimes \pi^{-1}, \cdots, \frac{1}{2} e_n \otimes 1 + \frac{1}{2} \pi e_n \otimes \pi^{-1}.
\end{align*}
\]
Recall the notation convention \( e_{[i,n+j]} \) in (3.4.1).
7.1. **Proof of Proposition 3.4.1.** In this subsection, we will give the proof of Proposition 3.4.1. The following lemma will be useful.

**Lemma 7.1.1** ([Sm15], equation before Lemma 4.9).

\[ g_i \wedge g_{n+i} = (e_i \otimes 1) \wedge (\pi e_i \otimes \pi^{-1}) = (e_i \otimes 1) \wedge (\pi^{-1} e_i \otimes \pi). \]

\[ \square \]

**Proof of Proposition 3.4.1.**

(i) When \( S = \{1, \ldots, i, \ldots, n, n+i\} \) for some \( 1 \leq i \leq \kappa \), \( g_S = (e_1 \otimes 1 - \pi^{-1} e_1 \otimes \pi) \wedge \cdots \wedge (e_i \otimes 1 - \pi^{-1} e_i \otimes \pi) \wedge \cdots \wedge (e_k \otimes 1 - \pi^{-1} e_k \otimes \pi) \wedge (e_{n+1} \otimes 1 - \pi e_{n+1} \otimes \pi^{-1}) \wedge \cdots \wedge (e_n \otimes 1 - \pi e_n \otimes \pi^{-1}) \wedge (e_{n+i} \otimes 1 + \frac{1}{2} \pi^{-1} e_i \otimes \pi). \)

A quick simplification shows that

\[ g_S = \frac{1}{2} (-1)^{\kappa+i} \pi^{-(n-\kappa)} e_{\{n+1, \ldots, 2n\}} \]

\[ + \frac{1}{2} (-1)^{\kappa+i} \pi^{-(n-\kappa-1)} \left[ \sum_{\sigma=1}^{i-1} (-1)^{\sigma} e_{[\sigma, n+\sigma]} + \sum_{\sigma=i+1}^{\kappa} (-1)^{\sigma} e_{[\sigma, n+\sigma]} + \sum_{\sigma=\kappa+1}^{n} (-1)^{\sigma} e_{[\sigma, n+\sigma]} \right] \]

\[ + \frac{1}{2} (-1)^{\kappa+1} \pi^{-(n-\kappa-1)} e_{[i, n+i]} + o(\pi^{-(n-k-1)}), \]

\[ = \frac{1}{2} (-1)^{\kappa+i} \pi^{-(n-\kappa)} \left[ (-1)^i e_{\{n+1, \ldots, 2n\}} + \pi \left( \sum_{\sigma \neq i} (-1)^{\sigma} e_{[\sigma, n+\sigma]} \right) + (-1)^{i+1} \pi e_{[i, n+i]} \right] + o(\pi^{-(n-k-1)}), \]

\[ = \frac{1}{2} (-1)^{\kappa+i} \pi^{-(n-\kappa)} \left[ e_{\{n+1, \ldots, 2n\}} + \pi \left( 2(-1)^{i+1} e_{[i, n+i]} + \sum_{\sigma=1}^{n} (-1)^{\sigma} e_{[\sigma, n+\sigma]} \right) \right] + o(\pi^{-(n-k-1)}). \]

(ii) When \( S = \{1, \ldots, i, \ldots, n, n+i\} \) for some \( \kappa+1 \leq i \leq n \), the proof will be the same as the case when \( 1 \leq i \leq \kappa \).

(iii) When \( S = \{1, \ldots, j, \ldots, n, n+i\} \) for some \( i, j \leq \kappa \), \( i \neq j \),

\[ g_S = (e_1 \otimes 1 - \pi^{-1} e_1 \otimes \pi) \wedge \cdots \wedge (e_j \otimes 1 - \pi^{-1} e_j \otimes \pi) \wedge \cdots \wedge (e_k \otimes 1 - \pi^{-1} e_k \otimes \pi) \]

\[ \wedge (e_{n+1} \otimes 1 - \pi e_{n+1} \otimes \pi^{-1}) \wedge \cdots \wedge (e_n \otimes 1 - \pi e_n \otimes \pi^{-1}) \wedge (e_{n+i} \otimes 1 + \frac{1}{2} \pi^{-1} e_i \otimes \pi), \]

\[ = (e_1 \otimes 1) \wedge \cdots \wedge (e_j \otimes 1) \wedge \cdots \wedge (e_k \otimes 1) \wedge (e_{n+1} \otimes 1 - \pi e_{n+1} \otimes \pi^{-1}) \wedge \cdots \wedge (e_n \otimes 1 - \pi e_n \otimes \pi^{-1}) \]

\[ \wedge \cdots \wedge (e_{n+i} \otimes 1 + \frac{1}{2} \pi^{-1} e_i \otimes \pi), \]

\[ = (-1)^{n-1} (-1)^{n-\kappa} \cdot \pi^{-(n-\kappa-1)} e_{[i, n+i]} + o(\pi^{-(n-k-1)}), \]

\[ = (-1)^{\kappa+1} \pi^{-(n-\kappa-1)} e_{[i, n+i]} + o(\pi^{-(n-k-1)}). \]
(iv) When \( S = \{1, \ldots, \hat{j}, \ldots, n, n + i\} \) for some \( i \leq \kappa < j \),
\[
g_S = (e_1 \otimes 1) \land \cdots \land (e_j \otimes 1) \land (\pi \pi^{-1}) \land \cdots \land (\pi \pi^{-1}) \\
\land \cdots \land (\pi \pi^{-1}) \land (\pi^{-1} e_i \land \pi) + o(\pi^{-(n-\kappa-2)}),
\]
\[
= (-1)^{n-1} \cdot (-1)^{n-\kappa-1} \cdot \pi^{(n-\kappa)} e_{[i,n+j]} + o(\pi^{-(n-\kappa-2)}),
\]
\[
= (-1)^{\kappa} \pi^{-(n-\kappa-2)} e_{[i,n+j]} + o(\pi^{-(n-\kappa-2)}).
\]

(v) When \( S = \{1, \ldots, \hat{j}, \ldots, n, n + i\} \) for some \( j \leq \kappa < i \),
\[
g_S = (e_1 \otimes 1) \land \cdots \land (e_j \otimes 1) \land (\pi \pi^{-1}) \land \cdots \land (\pi \pi^{-1}) \\
\land \cdots \land (\pi \pi^{-1}) \land (\pi^{-1} e_i \land \pi) + o(\pi^{-(n-\kappa)}),
\]
\[
= (-1)^{n-1} \cdot (-1)^{n-\kappa} \cdot \pi^{-(n-\kappa)} e_{[i,n+j]} + o(\pi^{-(n-\kappa)}),
\]
\[
= (-1)^{\kappa+1} \pi^{-(n-\kappa)} e_{[i,n+j]} + o(\pi^{-(n-\kappa)}).
\]

(vi) When \( S = \{1, \ldots, \hat{j}, \ldots, n, n + i\} \) for some \( i, j > \kappa \),
\[
g_S = (e_1 \otimes 1) \land \cdots \land (e_\kappa \otimes 1) \land (\pi \pi^{-1}) \land \cdots \land (\pi \pi^{-1}) \\
\land \cdots \land (\pi \pi^{-1}) \land (\pi^{-1} e_i \land \pi) + o(\pi^{-(n-\kappa-1)}),
\]
\[
= (-1)^{n-1} \cdot (-1)^{n-\kappa-1} \cdot \pi^{(n-\kappa-1)} e_{[i,n+j]} + o(\pi^{-(n-\kappa-1)}),
\]
\[
= (-1)^{\kappa} \pi^{-(n-\kappa-1)} e_{[i,n+j]} + o(\pi^{-(n-\kappa-1)}).
\]

\( \square \)

### 7.2. Proof of Proposition 3.4.4.

Recall Remark 3.3.5, if \( S = \{1, 2, \ldots, \hat{j}, \ldots, n, n + i\} \), then
\[
\text{sgn}(\sigma_S) = (-1)^{1+\cdots+n-j+n+i+\frac{n(n+1)}{2}} = (-1)^{n+i+j}.
\]
Also, it is straightforward to see

**Lemma 7.2.1.** \( g_S \) and \( g_{S'} \) have the same worst term. \( \square \)

**Proof of Proposition 3.4.4.**

\( \star \) **Case 1.** In this case, the conditions \( S = S' \) and \( i \neq j \) are equivalent to \( j = i' = n+1-i \) and \( i \neq j \). We have
\[
\text{sgn}(\sigma_S) = (-1)^{n+i+j} = (-1)^{2n+1} = -1.
\]
Therefore, \( \text{WT}(g_S - \text{sgn}(\sigma_S)g_{S'}) = 2\text{WT}(g_S) \), and (i)(ii)(iii) follow directly from Proposition 3.4.1.
Case 2. When \( i = j \), we have \( \text{sgn}(\sigma_S) = (-1)^{n+i+j} = (-1)^n \).

(iv) When \( i = j \leq \kappa \), we have \( \bar{i}^\vee = \bar{j}^\vee > n - \kappa \), therefore

\[
\begin{align*}
g_S &= \frac{1}{2}(-1)^{\kappa+i} \pi^{-(n-\kappa)} \left[ e_{\{n+1,\ldots,2n\}} + \pi \left( 2(-1)^{i+1} e_{\{i,n+i\}} + \sum_{\sigma=1}^n (-1)^{\sigma} e_{\{\sigma,n+\sigma\}} \right) \right] + o(\pi^{-(n-\kappa-1)}) ; \\
g_S^\perp &= \frac{1}{2}(-1)^{\kappa+i+1} \pi^{-(n-\kappa)} \left[ e_{\{n+1,\ldots,2n\}} + \pi \left( 2(-1)^{i+1} e_{\{\bar{i}^\vee,n+i\}} + \sum_{\sigma=1}^n (-1)^{\sigma} e_{\{\sigma,n+\sigma\}} \right) \right] + o(\pi^{-(n-\kappa-1)}).
\end{align*}
\]

Since \((-1)^n(-1)^{\kappa+i+1} = (-1)^{\kappa+i}\), the first terms cancel. Therefore, we have

\[
g_S - \text{sgn}(\sigma_S)g_S^\perp = \frac{1}{2}(-1)^{\kappa+i+1} \pi^{-(n-\kappa-1)} \left( 2(-1)^{i+1} e_{\{i,n+i\}} - 2(-1)^{\bar{i}^\vee+1} e_{\{\bar{i}^\vee,n+i\}} \right),
\]

(v) When \( \kappa + 1 \leq i = j \leq m \), we have \( \bar{i}^\vee = \bar{j}^\vee \geq m + 1 > \kappa \). Therefore,

\[
g_S - \text{sgn}(\sigma_S)g_S^\perp = \left( \frac{1}{2}(-1)^{\kappa+i+1} \right) \pi^{-(n-\kappa)} e_{\{n+1,\ldots,2n\}}
\]

\[
+ \frac{1}{2}(-1)^{\kappa+i+1} \pi^{-(n-\kappa-1)} \left( 2(-1)^{i+1} e_{\{i,n+i\}} + \sum_{\sigma=1}^n (-1)^{\sigma} e_{\{\sigma,n+\sigma\}} \right)
\]

\[
- \frac{1}{2}(-1)^n(-1)^{\kappa+i+1} \pi^{-(n-\kappa)} \left( 2(-1)^{\bar{i}^\vee+1} e_{\{\bar{i}^\vee,n+i\}} + \sum_{\sigma=1}^n (-1)^{\sigma} e_{\{\sigma,n+\sigma\}} \right) + o(\pi^{-(n-\kappa-1)}),
\]

(vi) When \( i = j = m + 1 \) (note that this will only happen when \( n \) is odd). This directly follows from the case (v).

Case 3. When \( S \neq S^\perp, i \neq j \) and \( i < \bar{j}^\vee \), for the same reason as in case 1, we need only to care about the worst term.

(vii) When \( i < \bar{j}^\vee \leq \kappa \), we have \( i < k < j, \bar{j}^\vee \leq \kappa < i^\vee \). In this case, we have

\[
\text{WT}(g_S) = (-1)^{\kappa} \pi^{-(n-\kappa-2)} e_{\{i,n+j\}}, \quad \text{WT}(g_S^\perp) = (-1)^{\kappa} \pi^{-(n-\kappa-2)} e_{\{\bar{j}^\vee,n+i\}}.
\]

Hence

\[
\text{WT}(g_S - \text{sgn}(\sigma_S)g_S^\perp) = (-1)^{\kappa} \pi^{-(n-\kappa-2)} \left( e_{\{i,n+j\}} - (-1)^{n+i+j} e_{\{\bar{j}^\vee,n+i\}} \right),
\]

(viii) When \( i \leq \kappa < \bar{j}^\vee < \kappa^\vee \), we have \( i \leq \kappa < j, \kappa < i^\vee \neq \bar{j}^\vee \). In this case, we have

\[
\text{WT}(g_S) = (-1)^{\kappa} \pi^{-(n-\kappa-2)} e_{\{i,n+j\}}, \quad \text{WT}(g_S^\perp) = (-1)^{\kappa} \pi^{-(n-\kappa-1)} e_{\{\bar{j}^\vee,n+i\}}.
\]
Hence

\[ \text{WT}(g_s - \text{sgn}(\sigma_s)g_{s\perp}) = (-1)^{n+k+1+i+j} \pi^{-(n-k-1)} e_{[\nu, n+i+j]} \].

(xi) When \( i \leq \kappa \) and \( \kappa^\nu \leq j^\nu \), we have \( i \neq j, \kappa < j^\nu \neq j^\nu \). In this case, we have

\[ \text{WT}(g_s) = (-1)^{\kappa+1} \pi^{-(n-k-1)} e_{[\nu, n+i+j]}, \quad \text{WT}(g_{s\perp}) = (-1)^{\kappa} \pi^{-(n-k-1)} e_{[\nu, n+i+j]}. \]

Hence

\[ \text{WT}(g_s - \text{sgn}(\sigma_s)g_{s\perp}) = (-1)^{\kappa+1} \pi^{-(n-k-1)} (e_{[\nu, n+i+j]} - e_{[\nu, n+i+j]}). \]

(x) When \( \kappa < i < j^\nu < \kappa^\nu \), we have \( k < i \neq j, \kappa < i^\nu \neq j^\nu \). In this case, we have

\[ \text{WT}(g_s) = (-1)^{\kappa^\nu-\kappa-1} e_{[\nu, n+i]}, \quad \text{WT}(g_{s\perp}) = (-1)^{\kappa} \pi^{-(n-k-1)} e_{[\nu, n+i+j]}. \]

Hence

\[ \text{WT}(g_s - \text{sgn}(\sigma_s)g_{s\perp}) = (-1)^{\kappa^\nu-\kappa-1} (e_{[\nu, n+i]} - (-1)^{n+i+j} e_{[\nu, n+i+j]}). \]

(xi) When \( k < i < \kappa^\nu \leq j^\nu \), we have \( j \leq \kappa < i, \kappa < i^\nu \neq j^\nu \). In this case, we have

\[ \text{WT}(g_s) = (-1)^{\kappa+1} \pi^{-(n-k-1)} e_{[\nu, n+i+j]}, \quad \text{WT}(g_{s\perp}) = (-1)^{\kappa} \pi^{-(n-k-1)} e_{[\nu, n+i+j]}. \]

Hence

\[ \text{WT}(g_s - \text{sgn}(\sigma_s)g_{s\perp}) = (-1)^{\kappa+1} \pi^{-(n-k-1)} e_{[\nu, n+i+j]} \cdot \]

(xii) When \( \kappa^\nu \leq i < j^\nu \), we have \( j < \kappa < i \) and \( i^\nu \leq \kappa < j^\nu \). In this case, we have

\[ \text{WT}(g_s) = (-1)^{\kappa+1} \pi^{-(n-k-1)} e_{[\nu, n+i+j]}, \quad \text{WT}(g_{s\perp}) = (-1)^{\kappa+1} \pi^{-(n-k-1)} e_{[\nu, n+i+j]}. \]

Hence

\[ \text{WT}(g_s - \text{sgn}(\sigma_s)g_{s\perp}) = (-1)^{\kappa+1} \pi^{-(n-k-1)} (e_{[\nu, n+i]} - (-1)^{n+i+j} e_{[\nu, n+i+j]}). \]

\( \square \)

8. Integral equations of the local model

In this section, we explicitly express the affine ring of \( U_{\{\kappa\}}^{\text{loc}} \subset M_{\{\kappa\}}^{\text{loc}} \) defined in §3.1.

**Theorem 8.0.1.** Over the worst point of the local model, we write

\[
X = \begin{pmatrix} \frac{X_1}{X_3} & X_2 & X_4 \end{pmatrix} = \begin{pmatrix} A & B & L \\ C & D & M \\ E & F & X_4 \end{pmatrix}
\]

Then the affine chart \( U_{\{\kappa\}}^{\text{loc}} \) over the worst point is isomorphic to the factor ring of \( \Theta_F[\mathbb{X}] \) modulo the entries of the following matrices:

\( \text{LM1.} \quad X_1^2 + X_2X_3 = \pi_0 I, \quad X_1X_2 + X_2X_4 = 0, \quad X_3X_1 + X_4X_3 = 0, \quad X_3X_2 + X_2^2 = \pi_0 I, \)

\( \text{LM2.} \quad -JX_1 + X_3^2H X_3 + X_1^4J = 0, \quad -JX_2 + X_3^2H X_4 = 0, \quad X_4 J + X_4^2 H X_3 = 0, \quad X_1^4 H X_4 - \pi_0 H n_{-2k} = 0, \)

\( \text{LM2.} \quad X_1 J X_1^4 - \pi_0 J = 0, \quad X_1 J X_3^4 - X_2 H = 0, \quad X_3 J X_1^4 + H X_2 = 0, \quad X_3 J X_3^4 - X_4 H + H X_4^4 = 0, \)

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\[ \text{LM6. } \wedge^2(X + \pi \id) = 0, \wedge^n(X - \pi \id) = 0, \]

\[ \text{LM8. } B = B^\text{ad}, C = C^\text{ad}, D = -2\pi I - A^\text{ad}, M = \pi E^\text{ad}, L = -\pi F^\text{ad}, X_4 = X_4^\text{ad}, \text{tr}(X_4) = -(n - 2\kappa - 2)\pi. \]

Hence after simplification, the affine chart \( U^\text{loc}_{\{\kappa\}} \) is isomorphic to the scheme

\[ \text{Spec } \frac{\mathcal{O}_F[X_3, X_4]}{\wedge^2(X_3, X_4 + \pi \id), X_4 - X_4^\text{ad}, \text{tr}(X_4) + (n - 2k - 2)\pi}. \]

8.0.2. The equations defined above produce a closed subscheme \( U' \) of \( U^\text{naive}_{\{\kappa\}} \). We want to show \( U^\text{loc}_{\{\kappa\}} \subset U' \subset U^\text{naive}_{\{\kappa\}} \). Once this is true, since \( U^\text{loc}_{\{\kappa\}} \subset U^\text{naive}_{\{\kappa\}} \) is equality over the generic fiber, and \( U' \) has the same defining equations as \( U^\text{loc}_{\{\kappa\}} \) over the special fiber, by \([GW20, \text{Proposition 14.17}]\), we have \( U^\text{loc}_{\{\kappa\}} = U' \).

Therefore, we only need to show the relations in \( \text{LM8} \) hold over \( U \). The proof is the same as for Theorem 3.5.8, except that we need to keep track of more terms in the integral equations.

Recall that using the standard basis, we have \( \mathcal{F}_\kappa = \text{Span} \left( X_i \right) \), where \( X \) is defined in (3.1.2). Write

\[ \wedge^n \mathcal{F}_\kappa = \sum_S c_{S_0} e_S = \sum_{T \text{ balanced}} a_T (g_T - \text{sgn}(\sigma_T) g_T^\perp). \]

Where \( S, T \) are subsets of \( \{1, \cdots , 2n\} \) of size \( n \). All those relations will come from the comparison of coefficients of \( a_T \) and \( c_S \) with \( S \) of the type \( (n,0) \) and \( (n-1,1) \), as we did over the special fiber in \$3.5 \$. We will find the expansion of \( g_T - \text{sgn}(\sigma_T) g_T^\perp \) into summation of \( c_{S_0} e_S \) with type \( (n,0) \) and \( (n-1,1) \). We write \( "O" \) for the terms of the other types.

8.0.3. Recall that we define \( c_{ij} \) := \( c_{\{i,n+1, \ldots , n+j, \ldots , 2n\}} \). In particular, we have

\[ c_{ij} e_{\{i,n+1, \ldots , n+j, \ldots , 2n\}} = e_{n+1} \wedge \cdots \wedge e_{n+j-1} \wedge (x_{ij} e_i) \wedge e_{n+j+1} \wedge \cdots e_{2n} = (-1)^{j-1} x_{ij} e_{\{i,n+j\}}. \]

Hence, \( x_{ij} = (-1)^{j-1} c_{ij} \).

It is straightforward from Proposition 3.4.1 or the proof of Proposition 3.4.4 that we have

\star \text{ Case 1. } S = S^\perp, i \neq j, \iff i + j = n + 1, i \neq j.

(i) When \( i \leq k \), we have

\[ gs - \text{sgn}(\sigma_S) g_S^\perp = 2(-1)^{\kappa} \pi^{-(n-\kappa-2)} e_{\{i,n+j\}} + O. \]

(ii) When \( \kappa < i \leq n - \kappa, i \neq j \), we have

\[ gs - \text{sgn}(\sigma_S) g_S^\perp = 2(-1)^{\kappa+1} \pi^{-(n-\kappa-1)} e_{\{i,n+j\}} + O. \]

(iii) When \( i > n - \kappa \Rightarrow j \leq \kappa, i \neq j \), we have

\[ gs - \text{sgn}(\sigma_S) g_S^\perp = 2(-1)^{\kappa+1} \pi^{-(n-\kappa)} e_{\{i,n+j\}} + O. \]

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**Case 2.** $i = j$.

(iv) When $i = j \leq \kappa$, we have

$$gs - \text{sgn}(\sigma_S)g_{S^\perp} = (-1)^{\kappa+1} \pi^{-(n-\kappa-1)} e_{[i,n\mathbf{+}i]} + (-1)^n e_{[i^\vee,n+i^\vee]} + O.$$ 

(v) When $\kappa < i = j \leq m$, we have

$$gs_i - \text{sgn}(\sigma_{S_i})g_{S_i^\perp} = (-1)^{\kappa+i+1} \pi^{-(n-\kappa)}
\left[ e_{\{n+1,\ldots,2n\}} + \pi \left( (-1)^{j+1} e_{[i,n\mathbf{+}i]} + (-1)^{j+i+1} e_{[i^\vee,n+i^\vee]} + \sum_{\sigma=1}^{n} (-1)^{\sigma} e_{[\sigma,n+\sigma]} \right) \right] + O.$$ 

(vi) When $i = j = m + 1$ (this will only happen when $n$ is odd), we have

$$gs - \text{sgn}(\sigma_S)g_{S^\perp} = (-1)^{\kappa+m} \pi^{-(n-\kappa)}
\left[ e_{\{n+1,\ldots,2n\}} + \pi \left( 2(-1)^m e_{[m+1,n+m+1]} + \sum_{\sigma=1}^{n} (-1)^{\sigma} e_{[\sigma,n+\sigma]} \right) \right] + O.$$ 

**Case 3.** $S$ balanced, $S \neq S^\perp, i \neq j$.

(vii) When $i < j^\vee \leq \kappa$, we have

$$gs - \text{sgn}(\sigma_S)g_{S^\perp} = (-1)^n \pi^{-(n-\kappa-2)} e_{[i,n\mathbf{+}j]} - (-1)^{n+i+j} e_{[j^\vee,n+i^\vee]} + O.$$ 

(viii) When $i \leq \kappa < j^\vee < n - \kappa + 1$, we have

$$gs - \text{sgn}(\sigma_S)g_{S^\perp} = (-1)^{\kappa} \pi^{-(n-\kappa-2)} e_{[i,n\mathbf{+}j]} + (-1)^{n+i+1+i+j} \pi^{-(n-\kappa-1)} e_{[j^\vee,n+i^\vee]} + O.$$ 

(ix) When $i \leq \kappa, j^\vee \geq n - \kappa + 1$, we have

$$gs - \text{sgn}(\sigma_S)g_{S^\perp} = (-1)^{\kappa+1} \pi^{-(n-\kappa-1)} e_{[i,n\mathbf{+}j]} + (-1)^{n+i+j} e_{[j^\vee,n+i^\vee]} + O.$$ 

(x) When $\kappa < i < j^\vee < n - \kappa + 1$, we have

$$gs - \text{sgn}(\sigma_S)g_{S^\perp} = (-1)^n \pi^{-(n-\kappa-1)} e_{[i,n\mathbf{+}j]} - (-1)^{n+i+j} e_{[j^\vee,n+i^\vee]} + O.$$ 

(xi) When $\kappa < i < n - \kappa + 1 \leq j^\vee$, we have

$$gs - \text{sgn}(\sigma_S)g_{S^\perp} = (-1)^{\kappa+1} e_{[\kappa,n\mathbf{+}j]} + (-1)^{n+i+1+i+j} \pi^{-(n-\kappa-1)} e_{[j^\vee,n+i^\vee]} + O.$$ 

(xii) When $n - \kappa + 1 \leq i < j^\vee$, we have

$$gs - \text{sgn}(\sigma_S)g_{S^\perp} = (-1)^{\kappa+1} \pi^{-(n-\kappa)} e_{[i,n\mathbf{+}j]} - (-1)^{n+i+j} e_{[j^\vee,n+i^\vee]} + O.$$ 

Note that only case (viii) and (xi) are different from Proposition 3.4.4.
8.0.4. Next, we compare all the coefficients that appear here.

- For $1 \leq i \leq \kappa, n - \kappa + 1 \leq j \leq n$, we have
  \[ c_{ij} = (-1)^{\kappa} \pi^{-(n-\kappa-2)} a_{ij}, \quad c_{j^\vee i^\vee} = (-1)^{\kappa+n+1+j^\vee+i^\vee} \pi^{-(n-\kappa-2)} a_{j^\vee i^\vee}. \]
  Hence
  \[ x_{ij} = (-1)^{j-1}(-1)^{\kappa} \pi^{-(n-\kappa-2)} a_{ij}, \quad x_{j^\vee i^\vee} = (-1)^{i^\vee-1}(-1)^{\kappa+n+1+i^\vee+j^\vee} \pi^{-(n-\kappa-2)} a_{ij}. \]
  Therefore, we get $x_{ij} = x_{j^\vee i^\vee}$, this proves $B = B^{ad}$. Similarly we have $C = C^{ad}$.

- For $1 \leq i \leq \kappa, \kappa+1 \leq j \leq n - \kappa$, we have
  \[ c_{ij} = (-1)^{\kappa} \pi^{-(n-\kappa-2)} a_{ij}, \quad c_{j^\vee i^\vee} = (-1)^{n+k+1+i+j} \pi^{-(n-\kappa-1)} a_{ij}. \]
  Hence
  \[ x_{ij} = (-1)^{j-1}(-1)^{\kappa} \pi^{-(n-\kappa-2)} a_{ij}, \quad x_{j^\vee i^\vee} = (-1)^{i^\vee-1}(-1)^{n+k+1+i+j} \pi^{-(n-\kappa-1)} a_{ij}. \]
  Therefore, we get $x_{ij} = \pi x_{j^\vee i^\vee}$, this proves $F = \pi L^{ad}$.

- For $\kappa+1 \leq i \leq n - \kappa, 1 \leq j \leq \kappa$, we have
  \[ c_{ij} = (-1)^{\kappa+1} \pi^{-(n-\kappa)} a_{ij}, \quad c_{j^\vee i^\vee} = (-1)^{i^\vee-1}(-1)^{n+k+1+i+j} \pi^{-(n-\kappa-1)} a_{ij}. \]
  Hence
  \[ x_{ij} = (-1)^{j-1}(-1)^{\kappa+1} \pi^{-(n-\kappa)} a_{ij}, \quad x_{j^\vee i^\vee} = (-1)^{i^\vee-1}(-1)^{n+k+1+i+j} \pi^{-(n-\kappa-1)} a_{ij}. \]
  Therefore, we get $\pi x_{ij} = -x_{j^\vee i^\vee}$, this proves $\pi M = -E^{ad}$.

- Let $k + 1 \leq i, j \leq n - \kappa$.
  (i) When $i \neq j$ and $i + j < n + 1$, we have
    \[ c_{ij} = (-1)^{\kappa} \pi^{-(n-\kappa-1)} a_{ij}, \quad c_{j^\vee i^\vee} = (-1)^{n+1+k+i+j} \pi^{-(n-\kappa-1)} a_{j^\vee i^\vee}. \]
    Hence
    \[ x_{ij} = (-1)^{j-1}(-1)^{\kappa} \pi^{-(n-\kappa-1)} a_{ij}, \quad x_{j^\vee i^\vee} = (-1)^{i^\vee-1}(-1)^{n+1+k+i+j} \pi^{-(n-\kappa-1)} a_{ij}. \]
    Therefore $x_{ij} = x_{j^\vee i^\vee}$.
  (ii) When $i = j$. For $i \neq m + 1$, we have
    \[ c_{ii} = (-1)^{i} \pi^{-(n-\kappa-1)} \sum_{\kappa \leq i \leq n-\kappa, \sigma \neq 1} (-1)^{\kappa+\sigma+1} a_{\sigma \sigma}, \quad c_{i^\vee i^\vee} = (-1)^{i^\vee} \pi^{-(n-\kappa-1)} \sum_{\kappa \leq i \leq n-\kappa, \sigma \neq 1} (-1)^{\kappa+\sigma+1} a_{\sigma \sigma}. \]
    For $i = m + 1$, we have
    \[ c_{ii} = (-1)^{i} \pi^{-(n-\kappa-1)} \sum_{\kappa \leq i \leq n-\kappa, \sigma \neq 1} (-1)^{\kappa+\sigma+1} a_{\sigma \sigma}, \quad c_{i^\vee i^\vee} = (-1)^{i^\vee} \pi^{-(n-\kappa-1)} \sum_{\kappa \leq i \leq n-\kappa, \sigma \neq 1} (-1)^{\kappa+\sigma+1} a_{\sigma \sigma}. \]
    For $i = m + 1$, we have
    \[ c_{m+1,m+1} = (-1)^{\kappa} \pi^{-(n-\kappa-1)} a_{m+1,m+1} + (-1)^{m+1} \pi^{-(n-\kappa-1)} \sum_{\sigma=\kappa+1}^{M} (-1)^{\kappa+\sigma+1} a_{\sigma \sigma}. \]
Therefore, when $i \neq m + 1$,

$$x_{ii} = (-1)^{n-\kappa} \sum_{\sigma \neq i} (-1)^{\sigma} a_{\sigma \sigma}, \quad x_{i \vee i \vee} = (-1)^{n-\kappa} \sum_{\sigma \neq i} (-1)^{\sigma} a_{\sigma \sigma}.$$ 

$$x_{m+1,m+1} = (-1)^{n-\kappa} \sum_{\sigma \neq m+1} (-1)^{\sigma} a_{\sigma \sigma} + (-1)^{m+\kappa} \pi^{-(n-\kappa)} a_{m+1,m+1}.$$ 

Hence, we deduce that $X_4 = X_{4}^{ad}$.

- Next since $c_{\{n+1,\ldots,2n\}} = 1$, we get

$$\sum_{\sigma=\kappa+1}^{M} a_{\sigma} (-1)^{\kappa+\sigma+1} \pi^{-(n-\kappa)} = 1.$$ 

Hence,

$$\sum_{\sigma=\kappa+1}^{M} (-1)^{\sigma} a_{\sigma} = (-1)^{\kappa+1} \pi^{n-\kappa}.$$ 

(i) When $n = 2m$,

$$\text{tr}(X_4) = 2(-1)^{n-\kappa} \sum_{i=\kappa+1}^{m} \left( \sum_{\sigma \neq i} (-1)^{\sigma} a_{\sigma} \right),$$

$$= (-1)^{n-\kappa} \sum_{\sigma \neq \kappa+1}^{M} (-1)^{\sigma} a_{\sigma},$$

$$= -(n - 2\kappa - 2) \pi.$$ 

(ii) When $n = 2m + 1$,

$$\text{tr}(X_4) = x_{m+1,m+1} + 2 \sum_{i=\kappa+1}^{m} x_{ii},$$

$$= (-1)^{m+\kappa} \pi^{-(n-\kappa)} a_{m+1} + (-1)^{n-\kappa} \pi^{-(n-\kappa)} \left( \sum_{\sigma \neq m+1} (-1)^{\sigma} a_{\sigma} \right)$$

$$+ 2(-1)^{\kappa} \pi^{-(n-\kappa)} \left( \sum_{i=\kappa+1}^{m} \sum_{\sigma \neq i} (-1)^{\sigma} a_{\sigma} \right),$$

$$= -(n - 2\kappa - 2) \pi.$$ 

These proves $\text{tr}(X_4) = -(n - 2\kappa - 2) \pi$.

- Finally, let $1 \leq i, j \leq \kappa$.

(i) When $i \neq j$, we have

$$c_{ij} = (-1)^{\kappa+1} \pi^{-(n-\kappa)} a_{ij}, \quad c_{j \vee i \vee} = (-1)^{n+1+\kappa+i^\vee+j^\vee} \pi^{-(n-\kappa)} a_{ij}$$
Hence
\[ x_{ij} = (-1)^{j-1}(-1)^{\kappa+1} \pi^{-(n-\kappa-1)} a_{ij}, \quad x_{j^\vee i^\vee} = (-1)^{i^\vee-1}(-1)^{n+\kappa+i^\vee+j^\vee} \pi^{-(n-\kappa-1)} a_{ij}. \]
Therefore, we get \( x_{ij} = -x_{j^\vee i^\vee} \).

(ii) When \( i = j \), we have
\[
\begin{align*}
  c_{ii} &= (-1)^{\kappa+1} \pi^{-(n-\kappa-1)} a_{ii} + (-1)^i \sum_{\sigma=\kappa+1}^M \left( -1 \right)^{\kappa+\sigma+1} \pi^{-(n-\kappa-1)}, \\
  c_{i^\vee i^\vee} &= (-1)^{n+\kappa+i^\vee} \pi^{-(n-\kappa-1)} a_{i^\vee i^\vee} + (-1)^i \sum_{\sigma=\kappa+1}^M \left( -1 \right)^{\kappa+\sigma+1} \pi^{-(n-\kappa-1)}.
\end{align*}
\]
Hence
\[
\begin{align*}
  x_{ii} &= (-1)^{\kappa+i} \pi^{-(n-\kappa-1)} a_{ii} - \sum_{\sigma=\kappa+1}^M \left( -1 \right)^{\kappa+\sigma+1} \pi^{-(n-\kappa-1)} a_{\sigma,\sigma}, \\
  x_{i^\vee i^\vee} &= (-1)^{n+\kappa+i^\vee} \pi^{-(n-\kappa-1)} a_{ii} - \sum_{\sigma=\kappa+1}^M \left( -1 \right)^{\kappa+\sigma+1} \pi^{-(n-\kappa-1)} a_{\sigma,\sigma}.
\end{align*}
\]
Recall that \( \sum_{\sigma=\kappa+1}^M (-1)^\sigma a_\sigma = (-1)^{\kappa+1} \pi^{n-\kappa} \), hence \( x_{ii} + x_{i^\vee i^\vee} = -2\pi \), these give \( D + A^{\text{ad}} = -2\pi I \).

**Remark 8.0.5.** One can also prove Theorem 8.0.1 by verifying those equations hold over the generic fiber. Recall that the local model is defined as the scheme-theoretic closure of the generic fiber. Hence, those equations will automatically hold integrally.

**References**


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