SIEGEL-WEIL FORMULA

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This note was written when I prepared my seminar talk, any mistakes are due to myself. If you find any typos or have any suggestions, feel free to contact me.

Contents

1. Baby example

Before we go into the deep details, let start with a classical question.

1.1. **Sum of square.**

For the first few primes we easily find that

$$5 = 1^2 + 2^2, \quad 13 = 2^2 + 3^2, \quad 17 = 1^2 + 4^2, \quad 29 = 2^2 + 5^2$$

are sum of squares, while other primes like $3, 7, 11, 19, 23$ are not. We have following fact:

Theorem 1.1.1. *A prime $p \neq 2$ is the sum of two squares if and only if $p \equiv 1 \mod 4$.*

Using the observation

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad - bc)^2.$$

We further have

Theorem 1.1.2. *A positive integer n is of the form $n = x^2 + y^2$ if and only if each prime factor $p \equiv 3 \mod 4$ of n appears to an even power.*

To answer this question, we naturally define the *representation number*

$$r(n) := \#\{(x, y) \in \mathbb{Z}^2 : n = x^2 + y^2\}.$$

We will use baby version of the Siegel-Weil formula to prove

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Theorem 1.1.3 (Jacobi)**.**

$$r(n) = 4 \left(\sum_{\substack{d|n \\ d\equiv 1 \mod 4}} 1 - \sum_{\substack{d|n \\ d\equiv 3 \mod 4}} 1 \right).$$

As a byproductm Jacobi's formula shows that

$$p \equiv 1 \mod 4 \Rightarrow r(p) = 4(2-0) = 8,$$
$$p \equiv 3 \mod 4 \Rightarrow r(p) = 4(1-1) = 0.$$

1.2. **Theta function.** Consider the Jacobi's theta series

$$\theta(q) := \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots .$$

Then the representation numbers $r(n)$ naturally appear as the nth coefficients of the square of Jacobi's theta series

$$\theta^2 = (\sum_{n\in\mathbb{Z}} q^{n^2})^2 = \sum_{n\geq 0} r(n)q^n = 1 + 4q + 4q^2 + 4q^4 + 8q^5 + 4q^8 + \cdots .$$

Let $\tau = u + iv$, it's clear that we have $\theta(\tau+1) = \theta(\tau)$. Using the Possion summation formula, we have

$$\theta(-\frac{1}{4\tau}) = (-2i\tau)^{1/2}\theta(\tau).$$

This shows that $\theta \in M_{1/2}(\Gamma_1(4))$ is a weight half modular form. A precise meaning would be it's an automorphic form of the Metaplectic group.

Remark 1.2.1. If we apply the Mellon transformation, then theta function becomes Riemann zeta function, and the second equality implies the functional equation.

For our interests, we have $\theta^2(\tau) \in M_1(\Gamma(4))$ is a weight 1 modular form of level $\Gamma_1(4)$.

Proposition 1.2.2.

$$\dim M_1(\Gamma_1(4)) = 1.$$

Therefore, it is the only modular form of such weight and level. Now we have another way to construct the modular form, basically the Eisenstein series.

1.3. **Eisenstein series.** Let $\chi : (\mathbb{Z}/4\mathbb{Z})^\times \xrightarrow{\sim} \{\pm 1\}$ be the unique nontrivial character. We define an Eisenstein series

$$G_k^\chi(\tau) = \sum_{(0,0)\neq(c,d)\in\mathbb{Z}^2, 4|c} \frac{\chi(d)}{(c\tau+d)^k}.$$

We can normalize it into

$$E_k^\chi(\tau) = \sum_{\Gamma^\infty\backslash\Gamma_1(4)} \frac{\chi(d)}{(c\tau+d)^k} = 1 + c_k^\chi \cdot \sum_{n\geq 1} \left(\sum_{d|n} \chi(d) d^{k-1} \right) q^n.$$

Where c_k^χ is a constant related to the special L-values. The last equality is an standard exercise, the main ingredient is applying twisted Possion summation formula to the function $\frac{1}{(\tau-u)^k}$. See Bump for more details.

Disregarding some convergent issue, which is not really a issue using some analysis, we have $E_1^\chi(\tau) \in M_1(\Gamma_1(4))$. Moreover, by comparing the constant term, we get

$$\theta^2(\tau) = E_1^\chi(\tau).$$

Comparing the coefficient before q, we obtain $c_1^\chi = 4$, and we conclude the Jacobi's theorem.

The equality is exactly the Siegel-Weil formula. General pattern would be, theta function have more arithmetic meaning inside, while Eisenstein series are more easy to compute by hand. Siegel-Weil formula builds the bridge between the arithmetic and analytic.

2. Weil representation

2.1. **Setup.** Let V be a quadratic space over $\mathbb{Q}$ of dimension m with a symmetric bilinear form $(\ ,\ )$. It associated to a quadratic form $Q[x] = \frac{1}{2}(x,x)$. We recover the symmetric bilinear form by

$$(x,y) = Q[x+y] - Q[x] - Q[y].$$

For simplicity we assume that m is *even*. We have

$$M(\mathbb{Q}) = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{Q} \right\},$$
$$N(\mathbb{Q}) = \left\{ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Q} \right\}.$$

$s_0 = \dim V/2 - 1 = \frac{m}{2} - 1$.

2.1.1. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$. We fix the standard additive character $\psi : \mathbb{A} \to \mathbb{C}^\times$ whose archimedean component is given by $\psi_\infty : \mathbb{R} \to \mathbb{C}^\times, x \mapsto e^{2\pi i x}$.

Recall the Schrödinger model of the Weil representation $\omega = \omega_{V,\psi}$ is the representation of $G(\mathbb{A}) \times H(\mathbb{A})$ on the space of Schwartz functions $\mathcal{S}(V(\mathbb{A}))$ such that for any $\varphi \in \mathcal{S}(V(\mathbb{A}))$ and $x \in V(\mathbb{A})$,

$$\begin{aligned}
\omega(m(a))\varphi(x) &= \chi_V(a)|a|^{m/2}\varphi(x \cdot a), && m(a) \in M(\mathbb{A}), \\
\omega(n(b))\varphi(x) &= \psi(bQ(v))\varphi(x), && n(b) \in N(\mathbb{A}), \\
\omega(w)\varphi(x) &= \hat{\varphi}(x), && w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
w(h)\varphi(x) &= \varphi(h^{-1}x), && h \in H(\mathbb{A}).
\end{aligned}$$

Here

- $\chi_V : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ is the quadratic character that corresponds to the quadratic extension $\mathbb{Q}(\sqrt{\mathrm{disc}(V)})/\mathbb{Q}$, where

$$\mathrm{disc}(V) := (-1)^{\binom{m}{2}} \det((x_i, x_j)) \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2.$$

for any $\mathbb{Q}$-basis $\{x_1, \cdots, x_m\}$ of V.

- $\hat{\varphi}$ is the Fourier transform of φ using the additive character ψ:

$$\hat{\varphi}(x) := \int_{V(\mathbb{A})} \varphi(y)\psi((x, y))dy.$$

Associated to $\varphi \in \mathcal{S}(V(\mathbb{A}))$, we define the *theta function*

$$\theta(g, h, \varphi) := \sum_{x \in V(\mathbb{Q})} \omega(g)\varphi(h^{-1}x), \quad g \in G(\mathbb{A}), h \in H(\mathbb{A}).$$

Then $\theta(g, h, \varphi)$ is invariant under $G(\mathbb{Q}) \times H(\mathbb{Q})$. We will be interested in the theta integral

$$\int_{H(\mathbb{Q})\backslash h(\mathbb{A})} \theta(g, h, \varphi)dh.$$

In other words, the theta lift of the constant function on $H(\mathbb{A})$ to $G(\mathbb{A})$.

2.2. **Example of the theta integral.** Assume that V is positive definite, with a lattice $\Lambda \subset V$ over $\mathbb{Z}$. We consider the Schwartz function $(\otimes_p \varphi_p) \otimes \varphi_\infty \in \mathcal{S}(V(\mathbb{A})^n)$ such that

- φ_p is the characteristic function of $\Lambda \otimes \mathbb{Z}_p$,
- φ_∞ is the Gaussian function $\varphi_\infty(x) = e^{-\pi(x,x)} = e^{-2\pi Q[x]}$.

Let $K \subset H(\mathbb{A})$ be the stabilizere of Λ, then the double quotient

$$H(\mathbb{Q})\backslash H(\mathbb{A})/K \xrightarrow{\sim} \mathrm{Gen}(\Lambda), \quad h \longmapsto h(\Lambda \otimes \widehat{\mathbb{Z}}) \cap V.$$

Where Gen is the isomorphic class of lattices with the same genus as Λ. Recall that two lattices have the same genus if they can be transformed by $H(\mathbb{A})$. Let $\{h_i\}$ be a complete set of representatives of $H(\mathbb{Q})\backslash H(\mathbb{A})/K$ and let $\{\Lambda_i\}$ be the corresponding representatives. Then

$$\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \theta(g, h, \varphi)dh = \sum_i \int_{H(\mathbb{Q})\backslash H(\mathbb{Q})h_i K} \theta(g, h, \varphi)dh.$$

Using $H(\mathbb{Q}) \cap h_i K h_i^{-1} = \mathrm{Aut}(\Lambda_i)$, each summand evaluates to

$$\int_{H(\mathbb{Q})\backslash H(\mathbb{Q})h_k K h_i^{-1}} \theta(g, hh_i, \varphi)dh = \frac{1}{\#\,\mathrm{Aut}(\Lambda_i)} \int_{h_i K h_i^{-1}} \theta(g, hh_i, \varphi)dh = \frac{1}{\#\,\mathrm{Aut}(\Lambda_i)} \int_K \theta(g, h_i h, \varphi)dh.$$

Assume $g = g_\tau$ for $\tau = u + iv \in \mathbb{H}$, a.k.a., $g_\tau = n(u)m(v^{1/2})$ and identity for the nonarchimedean places. Thanks to the strong approximation, this will be enough to determine the theta function. Then we have

$$\int_K \theta(g_\tau, h_i h, \varphi)dh = \int_K \sum_{x \in V(\mathbb{Q})} \omega_\infty(g_\tau)\varphi(h^{-1}h_i^{-1}x)dh.$$

By our choice of $\{\varphi_p\}$, the summation and integration is 1 if and only if

$$h^{-1}h_i^{-1}x \in \Lambda \Leftrightarrow x \in hh_i\Lambda = \Lambda_i.$$

Therefore the summand evaluates to

$$\mathrm{vol}(K) \sum_{x \in \Lambda_i} \omega_\infty(g_\tau)\varphi_\infty(x).$$

Using the definition, we have

$$\begin{aligned}\omega_\infty(g_\tau)\varphi_\infty(x) &= \omega(n(u)m(v^{1/2}))\varphi_\infty(x), \\ &= \chi_V(v^{1/2})|v|^{m/4}\psi(uQ[x])\varphi_\infty(v^{1/2}x), \\ &= |v|^{m/4}e^{2\pi i u Q[x]}e^{-2\pi v Q[x]}, \\ &= |v|^{m/4}e^{2\pi i Q(x)\tau}.\end{aligned}$$

Combine all the computations, we deduce that the theta integral equals

$$\mathrm{vol}(K)|v|^{m/4} \sum_i \frac{1}{\#\,\mathrm{Aut}(\Lambda_i)} \cdot \sum_{x \in \Lambda_i} e^{2\pi i Q(x)\tau}.$$

Let's see what we will get in special example. Let $V = \mathbb{Q}^2$ with $Q[(x,y)] = x^2 + y^2$. Let $\Lambda = \mathbb{Z}^2$, there is only one lattice in the equivalence of the genus class. Then the summation becomes

$$\sum_{(a,b) \in \mathbb{Z}^2} q^{(a^2+b^2)} = \sum_{n \in \mathbb{N}} \tau(n) q^n.$$

This recovers our sum of square function.

2.3. **Metaplectic group and Weil representation over $\mathbb{R}$.** In practice using the strong approximation, we will mostly focus on the archimedean place. In this case, Weil representation is not that scary.

2.3.1. We define $\mathrm{Mp}_2(\mathbb{R})$ to be the double cover of $\mathrm{SL}_2(\mathbb{R})$ whose elements can be written in the form

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau + d}\right),$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $\sqrt{c\tau+d}$ is considered as a holomorphic function of τ in the upper half-plane whose square is $c\tau + d$. Misleadingly, we will write it as $G'(\mathbb{R})$.

We will see later in the computation of the Eisenstein series that $c\tau + d$ plays a role in the angle of γ, so it's not hard to convince that this produce a double cover in the Iwasawa decomposition.

Remark 2.3.2. Maybe Simon can give a better explanation?

Taking $K'_\infty = \widetilde{\mathrm{SO}(2)}$ is the double cover of the $\mathrm{SO}(2)$, this is the maximal compact of the metaplectic group. Therefore, the Iwasawa decomposition becomes

$$G'(\mathbb{R}) = NAK'.$$

We identify the character group of K'_∞ with $\frac{1}{2}\mathbb{Z}$ and note that the value of the character ν_ℓ on $-1 \in \mu_2 \cap K'_\infty$ is $(-1)^{2\ell}$. For $\ell \in \mathbb{Z}$, the character ν_ℓ factors through $K = \mathrm{SO}(2)$ and we write $\nu_\ell(k_\infty) = e^{i\ell\theta}$.

Globally, we have a metaplectic double cover $G'(\mathbb{A}) \to G(\mathbb{A})$. The compact open subgroup $K_0(4)$ has a split section.

Remark 2.3.3. $K_0(4) = K_0(4)_2 \times \prod_{p\neq 2} K_p$, it has things to do with a independent choice of square root, but I don't konw how to explain this. Maybe Simon can. See also `https://math.stackexchange.com/questions/2802562/why-are-half-integral-weight-modular-forms-defined`

Recall that $\mathrm{SL}_2(\mathbb{Q})$ has a natural splitting. We denote the image as $G'(\mathbb{Q})$. For any $K' \subset K_0(4)$, we have strong approximation

$$G'(\mathbb{A}) = G'(\mathbb{Q})G'(\mathbb{R})K'.$$

Therefore, we may focus on the automorphic representation of the metaplectic group over the archimedean place.

We extend the quadratic character χ_V to the double cover by defining

$$\chi_V^\psi(x,z) = \chi_V(x) \cdot \begin{cases} z \cdot \gamma(x,\psi)^{-1} & \text{if m is odd,} \\ 1 & \text{if m is even.} \end{cases}$$

Where γ is the Weil index. In our case, $\psi(x) = e^{2\pi i x}$, we have

$$\gamma_\mathbb{R}(a,\psi) = \begin{cases} 1 & \text{if } a > 0, \\ -i & \text{if } a < 0. \end{cases}$$

The Weil representation now gives by (cf. [Kudla94, (3.1)-(3.3)], [Kudla96, Proposition 4.3], and [KRY06, Lemma 8.5.6])

$$\begin{aligned}
\omega(m(a))\varphi(x) &= \chi_V^\psi(a)|a|^{m/2}\varphi(x\cdot a), & m(a) &\in M(\mathbb{A}), \\
\omega(n(b))\varphi(v) &= \psi(bQ(x))\varphi(x), & n(b) &\in N(\mathbb{A}), \\
\omega(w)\varphi(x) &= \gamma(V)\hat{\varphi}(x), & w &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
w(h)\varphi(v) &= \varphi(h^{-1}v), & h &\in H(\mathbb{A}).
\end{aligned}$$

Here assume V has signature (p,q), then

$$\gamma(V) = \chi_V(-1)e^{2\pi i(q-p+1)}.$$

Furthermore. when m is odd, the central $\mathbb{C}^1$ (μ_2) in G' acts by scalar, while when m is even, the central acts trivially.

The theta integral is defined as usual, we set

$$\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \theta(g', h; \varphi) dh, \quad g' \in G'(\mathbb{A}), \varphi \in \mathcal{S}(V).$$

Example 2.3.4. Let $V = \mathbb{Q}$ with lattice $\mathbb{Z}$ and quadratic form $Q[x] = x^2$. Using the same computation as 2.2, we get

$$\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \theta(g'_\tau, h, \varphi) dh = \mathrm{vol}(K) \sum_{x \in \mathbb{Z}} \omega_\infty(g'_\tau)\varphi_\infty(x).$$

Now apply the Weil representation, we have

$$\begin{aligned}
\omega_\infty(g'_\tau)\varphi_\infty(x) &= \omega(n(u)m(v^{1/2}))\varphi_\infty(x), \\
&= \psi(uQ[x])\chi_V^\psi(v^{1/2})|v|^{1/4}\varphi_\infty(x \cdot v^{1/2}), \\
&= |y|^{1/4} e^{2\pi u Q[x]} \cdot e^{-2\pi v Q[x]}, \\
&= |y|^{1/4} e^{2\pi i \tau Q[x]}.
\end{aligned}$$

Therefore, we recover the Jacobi theta function:

$$\theta(\tau, \varphi) := |y|^{-1/4} \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \theta(g'_\tau, h, \varphi) dh = \sum_{x \in \mathbb{Z}} e^{2\pi i \tau x^2}.$$

2.4. **Some examples of quadratic space.**

3. Siegel-Weil formula

3.1. **Siegel Eisenstein series.**

Definition 3.1.1. We denote by $I(s, \chi_V)$ the principal series representation of $G(\mathbb{A})$ induced by $\chi_V|\cdot|^s$; a.k.a, $I(s_0, \chi_V) := \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})} \chi_V|\cdot|^s$.

To be more precise, it consists of all smooth functions $\Phi(g, s)$ on $G(\mathbb{A})$ such that

$$\Phi(n(b)m(a)g, s) = \chi_V(a)|a|^{s+1}\Phi(g, s).$$

Example 3.1.2. The example we are interested in comes from the Weil representation. There is a $G(\mathbb{A})$-intertwining map

$$\lambda : S(V(\mathbb{A})) \to I(s_0, \chi_V), \quad \lambda(\varphi)(g) = (\omega(g)\varphi)(0).$$

This follows from the fact that

$$(\omega(m(a))\varphi)(0) = \chi_V(a)|a|^{m/2}\varphi(0) = \chi_V(a)|a|^{s_0+1}\varphi(0) = \chi_V(a)|a|^{s_0+1}\lambda(\varphi).$$

A section $\Phi(s) \in I(s, \chi_V)$ is called *standard*, it its restriction to $K_\infty K$ is independent of s. One can show that any $\lambda(\varphi) \in I(s_0, \chi_V)$ can be extended into a standard section.

For any standard section $\Phi(s)$, we define the Eisenstein series

$$E(g,s;\Phi) := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Phi(\gamma g, s).$$

It converges for $\mathrm{Re}(s) > 1$, and defines an automorphic form on $G(\mathbb{A})$. When $\Phi = \lambda(\varphi)$, it's called the Siegel Eisenstein series.

3.2. **Relation to the classical Eisenstein series.** Here we compute an easy case to recover the classical Eisenstein series. Let ℓ be an even integer, and $\Phi^{\ell}_{\infty}(S)$ be the normalized eigenfunction of weight ℓ in $I_{\infty}(s,\chi)$, i.e.,

$$\Phi^{\ell}_{\infty}(g'k', s) = \nu_{\ell}(k')\Phi^{\ell}_{\infty}(gq, s), \quad \Phi^{\ell}_{\infty}(1,s) = 1$$

for $k' \in K'_{\infty}$, where ν_{ℓ} is the character of K'_{∞} of weight ℓ. For our purpose, we can take $\ell = 1$. Let $\tau = u + iv$, with the correspond $g_{\tau} = n(u)m(v^{1/2})$.

Lemma 3.2.1. *Suppose $\gamma g_{\tau} = g_{\gamma\tau} k_{\theta}$, then $e^{i\theta} = \frac{c\bar{\tau}+d}{|c\tau+d|}$.*

Proof. We have

$$\gamma\tau = \frac{a\tau+b}{c\tau+d} = \frac{1}{|c\tau+d|^2}\left(ac|\tau|^2 + bd + (ad+bc)\mathrm{Re}(\tau) + i\,\mathrm{Im}(\tau)\right).$$

Therefore,

$$g_{\gamma\tau} = \begin{pmatrix} 1 & \frac{ac|\tau|^2+bd+(ad+bc)\mathrm{Re}(\tau)}{|c\tau+d|^2} \\ & 1 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{Im}(\tau)^{1/2}}{|c\tau+d|} & \\ & \left(\frac{\mathrm{Im}(\tau)^{1/2}}{|c\tau+d|}\right)^{-1} \end{pmatrix}.$$

Then

$$\begin{aligned} k_{\theta} &= \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \begin{pmatrix} \frac{|c\tau+d|}{v^{1/2}} & \\ & \frac{v^{1/2}}{|c\tau+d|} \end{pmatrix} \begin{pmatrix} av^{1/2} & auv^{-1/2}+bv^{-1/2} \\ cv^{1/2} & cuv^{-1/2}+dv^{-1/2} \end{pmatrix}, \\ &= \begin{pmatrix} \frac{|c\tau+d|}{v^{1/2}} & * \\ & \frac{v^{1/2}}{|c\tau+d|} \end{pmatrix} \begin{pmatrix} av^{1/2} & (au+b)v^{-1/2} \\ cv^{1/2} & (cu+d)v^{-1/2} \end{pmatrix}, \\ &= \begin{pmatrix} * & * \\ \frac{cv}{|c\tau+d|} & \frac{cu+d}{|c\tau+d|} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \end{aligned}$$

Therefore, we have

$$e^{i\theta} = \cos\theta - \sin\theta = \frac{c(u-iv)+d}{|c\tau+d|} = \frac{c\bar{\tau}+d}{|c\tau+d|}.$$

□

From the proof, we also obtain

Corollary 3.2.2.

$$\mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}.$$

Lemma 3.2.3.

$$\Phi_\infty^\ell(g_\tau, s) = |v|^{\frac{1}{2}(s+1)}.$$

Proof.

$$\begin{aligned}
\Phi_\infty^\ell(g_\tau, s) &= \Phi_\infty^\ell\left(\begin{pmatrix}1 & u\\ & 1\end{pmatrix}\begin{pmatrix}v^{1/2} & \\ & v^{-1/2}\end{pmatrix}, s\right),\\
&= \chi(v^{1/2})|v|^{\frac{1}{2}(s+1)} = |v|^{\frac{1}{2}(s+1)}.
\end{aligned}$$

□

Corollary 3.2.4.

$$\Phi_\infty^\ell(\gamma g_\tau, s) = \Phi_\infty^\ell(g_{\gamma\tau}k_\theta, s) = \left(\frac{c\bar{\tau}+d}{|c\tau+d|}\right)^\ell \cdot \mathrm{Im}(\gamma\tau)^{\frac{1}{2}(s+1)}.$$

We are ready to compute the Eisenstein series. We define

$$\begin{aligned}
E(\tau, s, \Phi_\infty^\ell \otimes \Phi_f) &:= \mathrm{Im}(\tau)^{-\frac{\ell}{2}} E(g_\tau, s, \Phi_\infty^\ell \otimes \Phi_f),\\
&= \mathrm{Im}(\tau)^{-\frac{\ell}{2}} \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} \Phi_\infty(\gamma g_\tau, s),\\
&= \sum_\gamma \left(|c\tau+d|^2\,\mathrm{Im}(\gamma\tau)\right)^{-\frac{\ell}{2}} \left(\frac{c\bar{\tau}+d}{|c\tau+d|}\right)^\ell \cdot \mathrm{Im}(\gamma\tau)^{\frac{1}{2}(s+1)},\\
&= \sum_\gamma \left(\frac{c\bar{\tau}+d}{|c\tau+d|^2}\right)^\ell \cdot \mathrm{Im}(\gamma\tau)^{\frac{1}{2}(s+1-\ell)},\\
&= \sum_\gamma \frac{1}{(c\tau+d)^\ell}\,\mathrm{Im}(\gamma\tau)^{\frac{1}{2}(s+1-\ell)}.
\end{aligned}$$

This recovers our classical Eisenstein series.

Remark 3.2.5. We can further impose character into the principal series to get twisted Eisenstein series. Nothing change in previous computation, except now non-archimedean places will contribute a character:

$$\begin{aligned}
E(\tau, s, \Phi_\infty^\ell \otimes \Phi_f) &:= \mathrm{Im}(\tau)^{-\frac{\ell}{2}} E(g_\tau, s, \Phi_\infty^\ell \otimes \Phi_f),\\
&= \mathrm{Im}(\tau)^{-\frac{\ell}{2}} \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} \Phi_\infty^\ell(\gamma g_\tau, s)\Phi_f(\gamma, s),\\
&= \sum_\gamma \frac{\chi(d)}{(c\tau+d)^\ell}\,\mathrm{Im}(\gamma\tau)^{\frac{1}{2}(s+1-\ell)}.
\end{aligned}$$

When $\ell = 1$, we get

$$E(\tau, s, \Phi) = \sum_\gamma \frac{\chi(d)}{(c\tau+d)} \cdot \frac{v^{s/2}}{|c\tau+d|^s}.$$

This is the Eisenstein Maass form.

We can further impose character into the principal series to get twisted Eisenstein series.

3.3. **Siegel-Weil formula.** Assume V is anisotropic or $m-r>3$ (Weil's convergence condition), then the Eisenstein series $E(g,s;\lambda(\varphi))$ is holomorphic at s_0, and we have the Siegel-Weil formula

$$\frac{\alpha}{2}\int_{O(\mathbb{Q})\backslash O(\mathbb{A})}\theta(g,h;\varphi)dh = E(g,s_0;\lambda(\varphi)).$$

Where α is the Tamagawa number, which we will discuss later.

4. Proof of the Siegel-Weil formula

4.1. **Update after seminar.** During the seminar, Simon give a wonderful way to think about the Siegel-Weil formula, which is better than give a dry proof below. I have forgot the precise statements so I will write down something vague but hopefully is the moral of his claim.

The starting point is the theta correspondence. Recall that globally we define the theta integral which assign an automorphic forms $f\in\mathcal{A}(H)$ to

$$f\longmapsto\int_{H(\mathbb{Q})\backslash H(\mathbb{A})}\overline{f(h)}\theta(g,h;\varphi)dh.$$

This will give a automorphic form in $\mathcal{A}(H)$. From representation theory perspective, this is roughly speaking a homomorphism

$$\mathrm{Hom}(\mathcal{A}(H)^\vee\times\mathcal{S}(V(\mathbb{A})),\mathcal{A}(G)).$$

Equivalently, it gives a homomorphism

$$\mathrm{Hom}(\mathcal{S}(V(\mathbb{A})),\mathcal{A}(H)\times\mathcal{A}(G)).$$

The image will be of the form

$$\oplus\pi_H\otimes\pi_G.$$

The theory of theta correspondence says this gives a correspondence between the representations of H and G. This defines a function

$$\Theta:\mathrm{Irr}(H)\to\mathrm{Irr}.$$

If you write one of the automorphic representation into local components:

$$\pi_H=\otimes_v\pi_v,$$

then the global theta also compatible with the local theta.

Now if we start with a trivial representation in $\mathcal{A}(H)$, then the theta integral becomes a homomorphism:

$$\mathrm{Hom}(\mathcal{S}(V(\mathbb{A})),\mathcal{A}(G)).$$

However, we know that there is another such homomorphism, which sends the Schwartz functions into the Siegel Eisenstein series. If you assuming the uniqueness (which need to be more careful in the archimedean place), then two homomorphisms should agree, which is the Siegel-Weil formula.

4.2. **A fake reduction.** First observe that the action on both sides of the Siegel-Weil formula are compatible. In fact, the action of $G(\mathbb{A})$ are inherit from the Weil representation.

To be more precise, for $g' \in G(\mathbb{A})$, and $\varphi \in \mathcal{S}(V(\mathbb{A}))$, define

$$\varphi_{g'}(x) := (\omega(g')\varphi)(x).$$

Then

$$\begin{aligned}\theta(gg', h; \varphi) &= \sum_{x \in V(\mathbb{Q})} (\omega(gg', h)\varphi)(x); \\ &= \sum_{x \in V(\mathbb{Q})} \omega(g, h)(\omega(g')\varphi)(x); \\ &= \sum_{x \in V(\mathbb{Q})} \omega(g, h)\varphi_{g'}(x); \\ &= \theta(g, h; \varphi_{g'}).\end{aligned}$$

On the other hand,

$$\begin{aligned}E(gg', s_0; \lambda(\varphi)) &= \sum_{\gamma \in P(\mathbb{Q})/G(\mathbb{Q})} \lambda(\varphi)(\gamma gg', s_0); \\ &= \sum_{\gamma \in P(\mathbb{Q})/G(\mathbb{Q})} (\omega(\gamma gg')\varphi)(0); \\ &= \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} (\omega(\lambda g)\varphi_{g'})(0); \\ &= E(g, s_0; \lambda(\varphi_{g'})).\end{aligned}$$

It's not clear to me whether we can do this in the archimedean place. For instance, if we thinking of $(\mathfrak{g}, K)$-module, then this is not what we are looking for.

But still, to save our life, we reduce to show the special case of the Siegel-Weil formula to the case when $g = 1$, in this case, we are expected to get an equality between two numbers. In fact, in the later computation, if we replace φ by $\omega(g')\varphi$, the same computation works and we get Siegel-Weil formula for general g'. The main reason is that G and H actions are commute.

4.3. **Tamagama measure on orthogonal groups.** One of the main ingredient of the proof of the Siegel-Weil formula is the well-compatibility of different measures, which we will justify now. Let A be an $\mathbb{Q}$-algebra with measure, in our case we can take A to be $\mathbb{A}$, $\mathbb{R}$, and $\mathbb{Q}_p$.

The philosophy is that, given any linear algebraic group G or its homogeneous space G/H, we have natural measure defined on $G(A)$ and $G/H(A)$, which will be left or right invariant under the group action.

Consider the vector space $V(A)$, where A could be $\mathbb{A}, \mathbb{R}, \mathbb{Q}_p, \mathbb{Q}$, or any $\mathbb{Q}$-algebra. For any $\alpha \in \mathbb{G}_m(A)$, denote $V_A[\alpha] = \{x \in V(A) \mid \frac{1}{2}(x, x) = Q[x] = \alpha\}$.

Proposition 4.3.1. *Take $0 \neq x \in V(A)$ such that $Q[x] = \alpha$, we have identification*

$$V_A[\alpha] - \{0\} \simeq O_x(A)\backslash O(A),$$

where $O_x(A)$ is the stabilizer of x. We will later also consider the case when $\lambda = 0$.

Proof. The proof idea is as follows. First it's clear that that $O(A)$ acts on the space $V_A[\alpha] - \{0\}$. Next $O(A)$ acts transitively on $V_A[\alpha] - \{0\}$: it was by Witt's theorem which says that any isometry between two subspaces of the non-degenerate quadratic space will extend to an isometry of the whole space. In particular, we get a transitively action by extending the map $x \mapsto y$ for any $x, y \in V_A[\alpha] - \{0\}$. □

Next, we given $\alpha \neq \beta \in A$ when A is p-adic, we can find a linear transformation $\gamma \in \mathrm{GL}(V, A)$ such that $(\gamma x, \gamma y) = \alpha^{-1}\beta(x, y)$ for any $x, y \in V_A$. In this way, it constructs an isomorphism

$$V[\alpha] \simeq O_x(A)\backslash O(A) \simeq O_{\gamma x}(A)\backslash O(A) \simeq V[\beta].$$

Therefore, with some artificial choice, we can write $V_A - V_A[0] \simeq V[\alpha] \times \mathbb{G}_m$. By some linear transformation, one can further identify $V_A[\alpha] \simeq V_A[0]$ Hence $V_A - \{0\} = V_A[\alpha] \times \mathbb{G}_a(A)$. We will have a natural volume form $\omega = dx_\alpha \wedge d\alpha$ for $\alpha \in \mathbb{G}_a(A)$, and dx_α is a volume form on $O_x\backslash O$. The magic fact is the following equality:

$$\int_{V(A)} f(x)dx = \int_A \int_{V_A[\alpha]} f(x)dx_\alpha d\alpha.$$

The same procedure can apply to $\mathbb{R}$ and $\mathbb{A}$ as well, those the discussion might differs.

Example 4.3.2. For $V = \mathbb{R}^2$ with $Q[(x, y)] = x^2 + y^2$. We have

$$dx \wedge dy = \frac{1}{r} d\theta \wedge dr$$

for $r > 0$. When $r \leq 0$, we assign with 0 measure.

4.4. **Fourier expansion of the theta integral.** We compute the Fourier expansion of the theta series

$$\begin{aligned}
\int_{O(\mathbb{Q})\backslash O(\mathbb{A})} \theta(1, h; \varphi)dh &= \sum_{x \in V(\mathbb{Q})} \int_{O(\mathbb{Q})\backslash O(\mathbb{A})} (\omega(h)\varphi)(x)dh; \\
&= \sum_{x \in V(\mathbb{Q})} \int_{O(\mathbb{Q})\backslash O(\mathbb{A})} \varphi(h^{-1}x)dh; \\
&= \sum_{r \in \mathbb{Q}} \sum_{\substack{x \in V(\mathbb{Q}) \\ Q[x] = r}} \int_{O(\mathbb{Q})\backslash O(\mathbb{A})} \varphi(h^{-1}x)dh.
\end{aligned}$$

Therefore for $r \neq 0$, we have

$$\begin{aligned}
\sum_{\substack{x\in V(\mathbb{Q})\\ Q[x]=r}} \int_{O(\mathbb{Q})\backslash O(\mathbb{A})} \varphi(h^{-1}v)dh &= \sum_{\gamma\in O_r(\mathbb{Q})\backslash O(\mathbb{Q})} \int_{O(\mathbb{Q})\backslash O(\mathbb{A})} \varphi(h^{-1}\gamma x)dh; \\
&= \int_{O_r(\mathbb{Q})/O(\mathbb{A})} \varphi(h^{-1}x)dh; \\
&= \int_{O_r(\mathbb{Q})\backslash O_r(\mathbb{A})} \int_{O_r(\mathbb{A})\backslash O(\mathbb{A})} \varphi(h^{-1}x)dh; \\
&= \int_{O_r(\mathbb{Q})\backslash O_r(\mathbb{A})} \int_{V_{\mathbb{A}}[r]} \varphi(x)dx_r \\
&= \tau(O_r) \int_{V_{\mathbb{A}}[r]} \varphi(x)dx_r.
\end{aligned}$$

The last equality holds because, $\varphi(x)$ is $O_r(\mathbb{A})$-invariant.

For $r = 0$, we have

$$\begin{aligned}
\sum_{\substack{x\in V(\mathbb{Q})\\ Q[x]=0}} \int_{O(\mathbb{Q})\backslash O(\mathbb{A})} \varphi(h^{-1}x)dh &= \int_{O(\mathbb{Q})\backslash O(\mathbb{A})} \varphi(0)dh + \sum_{\gamma\in O_r(\mathbb{Q})\backslash O(\mathbb{Q})} \int_{O(\mathbb{Q})\backslash O(\mathbb{A})} \varphi(h^{-1}x)dh; \\
&= \tau(O)\varphi(0) + \tau(O_0) \int_{V_{\mathbb{A}}[0]-\{0\}} \varphi(x)dx.
\end{aligned}$$

Now the area $\tau(O)$ has been computed by Weil:

Proposition 4.4.1 (Weil)**.** $\tau(O) = \tau(O_r) = 2$ *is a constant when* $n \geq 3$.

Therefore, we conclude that when $n \geq 4$, we have

$$\frac{1}{2} \int_{O(\mathbb{Q})\backslash O(\mathbb{A})} \theta(1, h; \varphi) = \varphi(0) + \sum_{r\in\mathbb{Q}} \int_{V_{\mathbb{A}}[r]-\{0\}} \varphi(x)dx_r. \tag{4.4.1}$$

4.4.2. If we work harder, we can see it is in fact the Fourier coefficients of the theta integral.

Let $n(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$. We compute

(4.4.2)

$$\begin{aligned}
\int_{O(\mathbb{Q})\backslash O(\mathbb{A})} \theta(n(\alpha), h; \varphi) dh &= \sum_{x \in V(\mathbb{Q})} \int_{O(\mathbb{Q})\backslash O(\mathbb{A})} (\omega(n(\alpha), h)\varphi)(x) dh; \\
&= \sum_{x \in V(\mathbb{Q})} \int_{O(\mathbb{Q})\backslash O(\mathbb{A})} \varphi(h^{-1}x)\psi(\alpha Q[x]) dh; \\
&= \sum_{r \in \mathbb{Q}} \left(\sum_{\substack{x \in V(\mathbb{Q}) \\ Q[x]=r}} \int_{O(\mathbb{Q})\backslash O(\mathbb{A})} \varphi(h^{-1}x) dh \right) \psi(r\alpha); \\
&= 2\varphi(0) + 2\int_{V_{\mathbb{A}}[0]-\{0\}} \varphi(x) dx_0 + 2\sum_{r \in \mathbb{Q}^*} \left(\int_{V_{\mathbb{A}}[r]-\{0\}} \varphi(x) dx_r \right) \psi(\alpha r).
\end{aligned}$$

4.5. Fourier expansion of the Siegel-Eisenstein series.

4.5.1. *Bruhat decomposition.* Our first goal is finding representative elements in the quotient $B\backslash G$. Recall the Bruhat decomposition

$$G = B \sqcup BJB,$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Further, we can decompose

$$B\backslash BJB \simeq B \cap J^{-1}BJ\backslash B \quad BJb \longmapsto b.$$

Therefore we can choose representative elements:

$$B(\mathbb{Q})\backslash G(\mathbb{Q}) = BI_2 \sqcup \bigsqcup_{r \in \mathbb{Q}} BJn(r).$$

Remark 4.5.2. For $c \neq 0$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix}.$$

4.6. Fourier expansion.

First let's compute

$$\begin{aligned}
(\omega(Jn(b))\varphi)(0) &= \int_{V(\mathbb{A})} (n(b)\varphi)(x)\psi((0,x)) dx; \\
&= \int_{V(\mathbb{A})} (n(b)\varphi)(x) dx; \\
&= \int_{V(\mathbb{A})} \psi(\frac{1}{2}b(x,x))\varphi(x) dx; \\
&= \int_{V(\mathbb{A})} \psi(bQ[x])\varphi(x) dx.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
E(n(\alpha), s_0; \lambda(\varphi)) &= \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \lambda(\varphi)(\gamma n(\alpha), s_0); \\
&= \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} (\omega(\gamma n(\alpha))\varphi)(0); \\
&= (\omega(n(\alpha))\varphi)(0) + \sum_{r\in\mathbb{Q}} (\omega(n(r+\alpha))\varphi)(0); \\
&= \varphi(0) + \sum_{r\in\mathbb{Q}} \int_{V(\mathbb{A})} \psi((r+\alpha)Q[x])\varphi(x)dx.
\end{aligned}$$

Check the partial summation part, the summation equals

$$\begin{aligned}
&\sum_{r\in\mathbb{Q}} \int_{V(\mathbb{A})} \psi((r+\alpha)Q[x])\varphi(x)dx; \\
=& \sum_{r\in\mathbb{Q}} \int_{\mathbb{A}} \left(\int_{V(\mathbb{A})[\rho]} \psi((r+\alpha)\rho)\varphi(x)dx_\rho \right) d\rho; \\
=& \sum_{r\in\mathbb{Q}} \int_{\mathbb{A}} \left(\int_{V(\mathbb{A})[\rho]} \varphi(x)dx_\rho \right) \psi((r+\alpha)\rho)d\rho.
\end{aligned}$$

If we set $\alpha = 0$, the summation becomes

$$\sum_{r\in\mathbb{Q}} \int_{\mathbb{A}} \left(\int_{V(\mathbb{A})[\rho]} \varphi(x)dx_\rho \right) \psi(r\rho)d\rho.$$

Denote $f(\rho) := \int_{V(\mathbb{A})[\rho]} \varphi(x)dx_\rho$, then we have

$$\sum_{r\in\mathbb{Q}} \hat{f}(r) = \sum_{r\in\mathbb{Q}} f(r) = \sum_{r\in\mathbb{Q}} \int_{V(\mathbb{A})[r]} \varphi(x)dx_r.$$

Conpare with (4.4.1), we finishes the proof of the Siegel-Weil formula.

To get the Fourier coefficient, recall how we proof the Possion summation formula. Given $\rho \in \mathbb{A}$, the summation $\sum_{r\in\mathbb{Q}} f(r+\rho)$ is $\mathbb{Q}$-invariant. Using Fourier expansion, we have

$$\sum_{r\in\mathbb{Q}} f(r+\rho) = \sum_{r\in\mathbb{Q}} \hat{f}(r)\psi(r\rho).$$

In particular, if we apply $\rho = 0$, we get the Possion summation formula. From the proof, we see that we have a more general formula

$$\sum_{r\in\mathbb{Q}} \hat{f}(r+\rho) = \sum_{r\in\mathbb{Q}} f(r)\psi(r\rho).$$

This gives the Fourier expansion of the Eisenstein series. Compare with (4.4.2), we get identification between their Fourier coefficients.

4.7. **General Siegel-Weil formula for** Sp**.** singular term

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