Shimura varieties and their special cycles in the case of $O(0,2)$, $O(1,2)$, and $O(2,2)$.[1]

Outline of the talk:

(1) Primer on Shimura Varieties.
(2) Introduction to Shimura Varieties of Orthogonal Type.
(3) Definition of Natural, Shifted, Weighted, and Special Cycle.
(4) Examples of each of the cycles we consider.

References for the talk are:

(1) Kudla 1997 - Algebraic Cycles on Shimura Varieties of Orthogonal Type
(2) Kudla - Special Cycles and Derivatives of Eisenstein Series
(3) Bruinier & Yang 2008 - Faltings Heights of CM Cycles and Derivatives of L-functions
(4) Gross & Kohnen & Zagier - Heegner Points and Derivatives of L-Series.
(5) Kai-Wen Lan - An Example-Based Introduction to Shimura Varieties

Motivation. *I'm not an expert on the history myself.* **I might return to add an explanation of the history.** The key work is **Heegner points** i.e. special cycles should generalize Heegner points. We show this in our talk. Furthermore, special cycles should also behave well w.r.t. pull backs and much can be said about their cohomology. I think Kudla's paper on special cycles proved a formula for the intersection of special cycles. In another direction, the generating series of special cycles is of interest and Bruinier-Yang & the work of Bruinier-Rapoport-Yang develop the theory in the case of Simura curves.

0.1 Review of Shimura Varieties. There are two ways to try and think about Shimura varieties, as far as I know. One is more direct i.e. thinking about symmetric domains and various quotients. The second is the more abstract approach of Delgine. The crucial point is that they coincide.

Definition 0.1 (Low-Brow Definition of a Shimura Variety)**.** A Shimura variety is is roughly as follows. Let G be a reductive group over $\mathbb{Q}$. Let $\mathbb{D}$ be a symmetric domain on which $G(\mathbb{Q})$ acts. Then for a sufficiently small open compact subgroup $K \subseteq G(\mathbb{A}_f)$, we get

$$X_K(\mathbb{C}) := G(\mathbb{Q})\backslash \mathbb{D} \times G(\mathbb{A}_f)/K$$

where the action on the left is $g[x,h] \to [gx,h]$ and on the right $[x,h]g = [x,hg]$.

Throughout this talk, I write $X_K(\mathbb{C})$ if I am suggestively thinking about the complex points.

Definition 0.2 (Definition of Shimura Variety)**.** A **Shimura datum** is a pair (G,X) consisting of

(1) G a reductive group[2] over $\mathbb{Q}$,
(2) X a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ from the Deligne torus,

subjected to the conditions

SV1 for all $h \in X$, the Hodge structure on $Lie(G_{\mathbb{R}})$ defined by $Ad \circ h$ is of type $(-1,1),(0,0),(1,-1)$,
SV2 for any $h \in X$, $ad(h(i))$ is a Cartan involution of $G^{ad}_{\mathbb{R}}$,
SV3 G^{ad} has no $\mathbb{Q}$-factor such that the projection h is trivial.

Recall G^{ad} is the quotient G/Z.

From this Shimura datum, one obtains a Shimura variety via the same double coset construction.

Remark 0.3. Let $h \in X$. The action of $Ad \circ h : \mathbb{S} \to G_{\mathbb{R}} \to End(LieG_{\mathbb{R}})$ gives a Hodge structure on $Lie(G_{\mathbb{R}})$ depending on how $(z_1,z_2) \in \mathbb{S}_{\mathbb{C}}$ act. There is a sign difference here $h_{\mathbb{C}}(z_1,z_2) = z_1^{-p} z_2^{-q} v$ has $v \in V^{p,q}$ a weight (p,q) vector.

Let θ be an involution of $G^{ad}_{\mathbb{R}}$. It is a **Cartan involution** if $\{g \in G^{ad}_{\mathbb{R}}(\mathbb{C}) : g = \theta(\overline{g})\}$ is compact.

Example 0.4 (Casimir Kothari's Notes)**.** Let $G := \mathrm{GL}_2$. Let X be the $\mathrm{GL}_2(\mathbb{R})$-conjgates of the homomorphism $h_0 : \mathbb{S} \to \mathrm{GL}_{2,\mathbb{R}}$ given by $a + bi \to \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Then (G,X) is a Shimura datum. As a check,

(1) The Lie algebra is just 2×2 real matices. The action of $\mathrm{Ad} \circ h$ is just studying the adjoint action on the Lie algebra. But because of h_0's form, these all of the declared type. Essentially, $h(z)$ acts on the tangent space by $\frac{z}{\bar{z}}$.

[1]If there are any mistakes or typos *please* email them to me ktdao@wisc.edu.

[2]For example, think of GL_n, SL_n, Sp_{2g}, U_n, SO_n, or any of the other classical groups.

(2) $ad(h(i))$ being a Cartan involution just means we check $\{g \in G_{\mathbb{R}}^{ad}(\mathbb{C}) : g = \theta(\bar{g})\}$ is compact. One can show we just get $SU_2(\mathbb{R})$ for so clearly compact.
(3) G^{ad} has no $\mathbb{Q}$-factor s.t. the projection on h is trivial is easy since it is just SL_2 for the adjoint.

The bijection $X \leftrightarrow \mathbb{C} \setminus \mathbb{R}$ is given by identifying h_0 above with $i \in \mathbb{C} \setminus \mathbb{R}$.

Proposition 0.5. For any $K \subseteq G(\mathbb{A}_f)$ an open compact subgroup, we can form the associated Shimura variety

$$X_K(\mathbb{C}) := G(\mathbb{Q}) \backslash (\mathbb{D} \times G(\mathbb{A}_f)/K) \cong \coprod_j \Gamma_j \backslash \mathbb{D}^+. \tag{1}$$

where $\Gamma_j = G(\mathbb{Q})_+ \cap g_j K g_j^{-1}$. where $g_1, \ldots, g_r$ are coset representatives of the **finite** double quotient[3]

$$G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K. \tag{2}$$

Proof of Double Coset Decomposition. First off, the double coset $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$ is finite.

Secondly, we show that the map

$$\varphi : \Gamma_g \backslash \mathbb{D}^+ \to G(\mathbb{Q})_+ \backslash \mathbb{D}^+ \times G(\mathbb{A}_f)/K$$

is an injective map given by $[x] \mapsto [x, g]$. Suppose $[x_1], [x_2] \in \Gamma_g \backslash \mathbb{D}^+$ map to the same point of $X_K(\mathbb{C})$ i.e. $[x_1, g] = [x_2, g]$. So there exists an $h \in G(\mathbb{Q})_+$ and a $k \in K$ such that

$$[x_1, g] \overset{\text{literal}}{=} [hx_2, hgk].$$

The equality $hgk = g$ implies $gkg^{-1} = h^{-1} \in gKg^{-1}$. But then $h \in G(\mathbb{Q}) \cap gKg^{-1}$. This proves injectivity.

A standard fact in the theory is that the natural map

$$G(\mathbb{Q})_+ \backslash \mathbb{D}^+ \times G(\mathbb{A}_f)/K \leftrightarrow G(\mathbb{Q}) \backslash \mathbb{D} \times G(\mathbb{A}_f)$$

is a bijection. So one gets $\varphi : \Gamma_g \backslash \mathbb{D}^+ \to X_K$ an injective map. But this implies X_K is a disjoint union of these images for various g. But the various g must vary over coset representatives of $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$. ☐

We can recover the curves $Y(\Gamma(N))$ as follows. Let $K := \ker(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}))$. This is an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$. Then,

$$X_K(\mathbb{C}) = \coprod_{(\mathbb{Z}/N\mathbb{Z})^\times} Y(\Gamma(N))$$

because

$$\mathrm{GL}_2(\mathbb{Q})_+ \backslash \mathrm{GL}_2(\mathbb{A}_f)/K \cong \mathrm{GL}_2(\mathbb{Q})_+ \backslash \mathrm{GL}_2(\mathbb{Q})\, \mathrm{GL}_2(\widehat{\mathbb{Z}})/K \tag{3}$$
$$\cong \mathrm{GL}_2(\mathbb{Q})_+ \backslash \mathrm{GL}_2(\mathbb{Q})\, \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \tag{4}$$
$$\overset{\det}{\cong} (\mathbb{Z}/N\mathbb{Z})^\times. \tag{5}$$

Our choice of K makes K normal so all the Γ_r are the same. One can check that X_K is defined over $\mathbb{Q}$ by considering the associated moduli problem:

(1) $Y(\Gamma(N))$ paramterizes elliptic curves with extra data $(E, E[N] \cong (\mathbb{Z}/N\mathbb{Z})^2)$ and up to isomorphism and preserving the symplectic form coming from a choice of polarization.
(2) There is always the Weil pairing which pairs the N-torsion points and lands in μ_N the Nth roots of unity.
(3) Now we want to be able to solve this moduli problem over $\mathbb{Q}$. The issue is we need to idenfiy $E[N] \cong \mu_N^2$ by picking an Nth root of unity. We can also require that E itself is defined over $\mathbb{Q}$. So the issue of getting a $\mathbb{Q}$-point corresponds to

$$(E, \mu_N^2),\ (E, (\sigma\mu_N)^2,\ , (E, (\sigma^2\mu_N)^2), \ldots, \ldots,\ (E, (\sigma^{\varphi(N)-1}\mu_N)^2).$$

In this case, we obtain a Shimura variety defined over $\mathbb{Q}$.

[3]The group $G(\mathbb{Q})_+$ is defined as the group of elements whose image under $G \to G^{ad}$ land in the identity component. For example, $\mathrm{GL}_2(\mathbb{Q})_+$ is the group of positive determinant matrices.

Example 0.6 (Recovering $Y_0(N)$ and $Y_1(N)$)**.** Since $Y_0(N)$ and $Y_1(N)$ are obtained by quotienting the upperhalfplane by $\Gamma_0(N)$ and $\Gamma_1(N)$, we make the obvious modification to K as a kernel. Indeed, let K be the preimage of the subgroup

$$\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) : c \equiv 0 \bmod N\} \subseteq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$$

under the map $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Then K is a normal subgroup. Furthermore,

$$\mathrm{GL}_2(\mathbb{Q})_+ \backslash \mathrm{GL}_2(\mathbb{A}_f)/K \cong \mathrm{GL}_2(\mathbb{Q})_+ \backslash \mathrm{GL}_2(\mathbb{Q}) \cdot \left(\mathrm{GL}_2(\widehat{\mathbb{Z}})/K\right) \tag{6}$$

$$\cong \mathrm{GL}_2(\mathbb{Q})_+ \backslash \mathrm{GL}_2(\mathbb{Q})(\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\Gamma_0(N)) \tag{7}$$

$$\cong 1 \tag{8}$$

via the determinant map.

We now state one important theorem which is crucial to the arithmetic theory of Shimura varieties.

Theorem 0.7 (Existence of a Reflex field and Canonical Model)**.** Let K be an open compact subgroup of $G(\mathbb{A}_f)$ where $(G, \mathbb{D})$ is a **Shimura datum**. Then the Shimura varieties

$$X_K = G(\mathbb{Q})\backslash \mathbb{D} \times G(\mathbb{A}_f)/K$$

have the structure of an algebraic (quasi-projective) variety with multiple connected components all of which are defined over an algebraic number field called the **reflex field** and in particular, the number field is independent of the choice of K. Furthermore, each X_K has a **Satake-Baily-Borel compactifiication** $\overline{X_K}$ which is a projective variety defined over the reflex field.

Let K be an open compact subgroup of $G(\mathbb{A}_f)$ for $G := GSpin(V)$ where (V, Q) has signature $(n, 2)$. Then the Shimura varieties

$$X_K = G(\mathbb{Q})\backslash \mathbb{D} \times G(\mathbb{A}_f)/K$$

have the structure of an algebraic (quasi-projective) variety with multiple connected components which is defined over the **reflex field** which is an algebraic number field.

Furthermore, each X_K has a **Satake-Baily-Borel compactification** aka **minimal compactification** $\overline{X_K}$ which has the structure of a projective variety defined over the reflex field.

Remark 0.8 (Motivations for existence of reflex fields)**.** Shimura varieties X_K are allowed to have mutliple components. This gives them a higher chance of having a smaller field of definition. Here's an of example demonstrating this principle.

Let us consider for the curve $V(X - i) \in \mathbb{A}^2_{\mathbb{C}}$. This is a curve whose first coordinate is always i. It is clear that there is no chance for this curve to be defined over a smaller field such as $\mathbb{R}$ since its defining equation does not have coefficients in $\mathbb{R}$. However, we can consider instead

$$V((X - i)(X + i)) = V(X^2 + 1) \subseteq \mathbb{A}^2_{\mathbb{C}}$$

which *does* have coefficients in $\mathbb{R}$. This also adds an additional component, but at least the algebra set is preserved under the Galois action $Gal(\mathbb{C}/\mathbb{R})$ and is therefore defined over $\mathbb{R}$ (e.g. an Exercise in Hartshorne II.4).

Example 0.9. The modular curves $Y_0(N), Y_1(N)$ are by themselves defined over $\mathbb{Q}$ and are quasiprojective. The number of cusps one has to add to get a projective variety can be computed in the case of modular curves.

On the other hand, each of the $Y(\Gamma(N))$ are only defined over $\mathbb{Q}(\zeta_N)$. But putting them together $\coprod_{(\mathbb{Z}/N\mathbb{Z})^\times} Y(\Gamma(N))$ produces a quasiprojective variety defined over $\mathbb{Q}$.

Later, I shall give more examples of Shimura curves, but I think we have enough to see the flavor of how they might be constructed. To even begin talking about "sub"-Shimura varieties, we need a notion of maps of Shimura varieties. In particular, we give a criterion for a map of Shimura varieties to be a closed embedding.

Definition 0.10. Fix $(G, \mathbb{D})$ and $(G', \mathbb{D}')$. A **map of Shimura varieties** will always arise from a map of algebraic groups $G \to G'$ sending X to X'. For fixed open compact subgroups $K \subseteq G, K' \subseteq G'$, a map

$X_K(\mathbb{C}) \to X_{K'}(\mathbb{C})$ need not always be well-defined. In this talk, they always will be and are explicitly given by a map of double cosets

$$G(\mathbb{Q})\backslash\mathbb{D} \times G(\mathbb{A}_f)/K \to G'(\mathbb{Q})\backslash\mathbb{D}' \times G'(\mathbb{A}_f)/K'.$$

Theorem 0.11 (Criterion for Closed Embeddings)**.** A map of Shimura datum $(G, \mathbb{D}) \to (G', \mathbb{D}')$ defines a map inverse systems of Shimura varieties $Sh(G, \mathbb{D}) \to Sh(G', \mathbb{D}')$. The inverse system of Shimura variety associated to $(G, \mathbb{D})$ is defined as

$$Sh(G, \mathbb{D}) = \{X_K\}_{K \text{ small open compact subgroups } \subseteq G(\mathbb{A}_f)}.$$

If $G \hookrightarrow G'$ is an injective homomorphism, then the map on total Shimura varieties is a **closed immersion** i.e. for any open compact subgroup $K \subseteq G(\mathbb{A}_f)$ there is an open compact subgroup $K' \subseteq G'(\mathbb{A}_f)$ such that

$$Sh(G, \mathbb{D})_K \to Sh(G', \mathbb{D}')_{K'}$$

is a closed immersion of algebraic varieties.

In practice, it is often possible to make the choice of K', which is dependent on K, very explicit.

0.2 Set-up for Shimura Varieties of Orthogonal Type. Throughout, we use the following set-up.

(1) V is an inner product space over $\mathbb{Q}$ of signature $(n, 2)$. We write $Q(x) := \frac{1}{2}(x, x)$ for the associated quadratic form.
(2) $SO(V) = SO(n, 2)$ is the special orthgonal group of signature $(n, 2)$
(3) $G := GSpin(n, 2)$.
The reason why people prefer to work with $GSpin(n, 2)$ is because it leads to Shimura varieties of Hodge type. This is an advantage because it leads to a moduli interpretation. One positive consequence is it then lets one figure out the field of definition / reflex field more easily via the moduli perspective.
(4) For $K \subseteq G(\mathbb{A}_f)$ open compact we can form

$$X_K := G(\mathbb{Q})\backslash(D \times G(\mathbb{A}_f)/K)$$

where

$$\mathbb{D} := \{w \in V(\mathbb{R}) : (w, w) = 0, (w, \bar{w}) < 0\}/\mathbb{C}^\times$$

is the Grassmannian of oriented negative definite 2-planes. One has

$$\mathbb{D} = \mathbb{D}^+ \coprod \mathbb{D}^- \text{ where the two are interchanged by complex conjugation.}$$

(5) Decompose for any K open compact

$$X_K = \coprod_{1 \le j \le r} \Gamma_j \backslash \mathbb{D}^+ \text{ for } \Gamma_j = g_j K g_j^{-1} \cap G(\mathbb{Q})_+$$

and where the number of connected components correspond to the number of coset representatives of

$$G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K.$$

(6) $Sh(G, \mathbb{D}) := \varprojlim_{K \subseteq G(\mathbb{A}_f)} X_K$.

We discuss some generalities of $GSpin(V)$. First off, there is always a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & GSpin(V) & \xrightarrow{\psi} & SO_V & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \Vert & & \\ 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & Spin(V) & \longrightarrow & SO_V & & \end{array}$$

where $\psi(g)(v) = gvg^{-1}$.

Definition 0.12. Let V be as before. Let $C(V)$ denote the **Clifford algebra** where $C(V) := \frac{T(V)}{v^2 - Q(v)}$. Then one can decompose $C(V)$ into its even part $C^{even}(V)$ and its odd part $C^{odd}(V)$. One gets a linear inclusion

$$V \hookrightarrow C^{odd}(V).$$

The Clifford algebra comes with what is known as the spinor norm $\nu(x)$ for $x \in C(V)$. Then

$$GSpin(V) := \{g \in C^{even}(V)^\times : gVg^{-1} = V\}.$$

Example 0.13. Since I assume people have not seen much of the Clifford algebra, let me compute one example of $GSpin(V)$.

Let $V := \mathbb{R}^2$ and $Q(x) = -x_1^2 - x_2^2$. Let e_1, e_2 be the standard basis. Then

$$-2 = (e_1 + e_2) \otimes (e_1 + e_2) = -2 + (e_1 \otimes e_2 + e_2 \otimes e_1).$$

This implies $e_1 \otimes e_2 = -e_2 \otimes e_1$. Then

$$C(V) \text{ has a basis given by } 1, e_1, e_2, e_1 \otimes e_2.$$

As $\mathbb{R}$-algebras, we get $C(V) \cong \mathbb{H}$ as $\mathbb{R}$-algebras

$$e_1 \to i, \qquad e_2 \to j, \qquad e_1 \otimes e_2 \ \to\ k.$$

Then, for $GSpin(V)$ we get

$$GSpin(V)(\mathbb{R}) \cong \{g \in C^{even}(V)^\times : gVg^{-1} = V\} = \{a + b(e_1 \otimes e_2) : (a, b) \neq 0\} \cong \mathbb{C}^\times.$$

Definition 0.14. The spin group $Spin(V)$ is defined as follows. It is the algebraic subgroup $Spin(V) \subseteq GSpin(V)$ consisting of $x \in GSpin(V)$ with **Clifford norm** $N(x) = x^t x = 1$. Here, x^t just means for any pure tensor $x_1 \otimes \cdots \otimes x_n$ one reverses the order of the entries $x^t = x_n \otimes \cdots \otimes x_1$.

Theorem 0.15 (General Facts about Shimura Varieties of Orthogonal Type)**.** Assume the situation above.

(1) X_K is a quasiprojective variety over $\mathbb{Q}$. It is projective if and only if V is an anisotropic quadratic space. This would force if $n \leq 2$ by Meyer's Theorem.
(2) $\dim X_K = n$,

Due to accidental isomorphisms in low dimensions, the Shimura varieties of orthogonal type can be classified and put into a table;

$\dim X$	$G := GSpin(n, 2)$	X
0	$k^\times$ rank 2 tori	Points
1	GL_2 or $B^\times$ for B an indefinite quaternion algebra	Modular Curves / Shimura Curves
2	$\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ and $\mathrm{GL}_{2,F}^{\det \in \mathbb{Q}^\times}$	Product of Modular Curves / Hilbert Modular Surfaces.

For the same of completeness, we compute some examples corresponding to the dimension 0 case.

Example 0.16. The easiest is $n = 0$ in which (V, Q) is to be a 2-dimension $\mathbb{Q}$-vector space with $(0, 2)$-quadratic form. Let v_1, v_2 be an orthogonal basis of V and assume $Q(v_i) = q_i \in \mathbb{Q}^\times$. The Clifford algebra is

$$C(V) \cong \mathbb{Q} \oplus \mathbb{Q}v_1 \oplus \mathbb{Q}v_2 \oplus \mathbb{Q}(v_1 \otimes v_2). \tag{9}$$

The even part of the Clifford algebra is $C^{even}(V) = \mathbb{Q} \oplus \mathbb{Q}(v_1 \otimes v_2)$ and as a $\mathbb{Q}$-algebra

$$C^{even}(V) \cong \frac{\mathbb{Q}[X]}{(X^2 + q_1 q_2)} \tag{10}$$

because orthogonality implies $0 = (v_1, v_2) = \frac{1}{2}[Q(v_1 + v_2) - Q(v_1) - Q(v_2)]$ which means

$$0 = \frac{1}{2}[(v_1 + v_2)(v_1 + v_2) - v_1^2 - v_2^2] = \frac{1}{2}[v_2 v_1 + v_2 v_1] \ \Rightarrow\ \frac{1}{2}v_2 v_1 = -\frac{1}{2}v_1 v_2. \tag{11}$$

in the Clifford algebra. Since $char(\mathbb{Q}) \neq 2$, the relation we get is $X^2 = (v_1 \otimes v_2)^2 = v_1 v_2 v_1 v_2 = -v_1(v_1 v_2)v_2 = -q_1 q_2$. It follows that

$$C^{even}(V)^\times \cong \mathbb{Q}(\sqrt{-q_1 q_2})^\times$$

and since these elements always preserve $V \hookrightarrow C^{odd}(V)$, we know $GSpin(V) \cong \mathbb{Q}(\sqrt{-q_1 q_2})^\times$.

Proof of claim. Let $g \in C^{even}(V)^\times$. Focus on pure tensors so suppose $g = v_1 v_2$. Then $g^{-1} = v_2 v_1 \frac{1}{q_2 q_1}$. It follows that

$$gv_1 g^{-1} = \frac{1}{q_2q_1} v_1v_2v_1v_2v_1 = \frac{-q_2q_1}{q_2q_1} v_1 = -v_1 \tag{12}$$

$$gv_2 g^{-1} \frac{1}{q_2q_1} v_1v_2v_2v_2v_1 = \frac{1}{q_1} v_1v_2v_1 = -\frac{q_1}{q_1} v_2 = -v_2. \tag{13}$$

Example 0.17 (Orthogonal Shimura Variety associated to $O(0,2)$)**.** Let (V,Q) be a quadratic space over $\mathbb{Q}$ of signature $(0,2)$ as above. Set $k := \mathbb{Q}(\sqrt{d})$ with $d < 0$ as the even part of the Clifford algebra. Then

$$G := GSpin(k) \cong Res_{k/\mathbb{Q}}(\mathbb{G}_{m,k}). \tag{14}$$

It follows $G(\mathbb{A}_f) \cong \mathbb{A}_{k,f}^\times$, the symmetric domain $\mathbb{D}$ consists of two points corresponding to the two orientations of $V \otimes \mathbb{R}$, and if we pick $K := \widehat{O_k}^\times$, then

$$X_K = G(\mathbb{Q})\backslash\{z_0^\pm\} \times G(\mathbb{A}_f)/K \cong \coprod_{k^\times\backslash\mathbb{A}_{k,f}^\times/\widehat{O_k}^\times} \{z_0^\pm\} \cong \{z_0^\pm\} \times (k^\times\backslash\mathbb{A}_{k,f}^\times/\widehat{O_k}^\times) = \{z_0^+\} \times Cl(k). \tag{15}$$

The last isomorphism is given by

$$(k^\times\backslash\mathbb{A}_{k,f}^\times/\widehat{O_k}^\times) \xrightarrow{\cong} Cl(k) \qquad (a_v)_v \mapsto \prod_v v^{ord_v(a_v)} \tag{16}$$

and we remark that $k^\times \cong G(\mathbb{Q})_+$ since the adjoint $G^{ad}(\mathbb{Q})$ is trivial.
Warning. There is a potential for confusion here since we wrote $X_K = \coprod_{G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K} \Gamma_{g_j}\backslash\mathbb{D}^+$ above. However, $\mathbb{D}^+$ does not mean we get $\{z_0^+\}$.

One can see why there must be two points if one writes out what happens to the action of $G(\mathbb{Q})$ on $z_0^\pm$. By definitoin, $G(\mathbb{Q}) \cong k^\times$ consists of units and so $G(\mathbb{Q})_+ := \{x \in k^\times : N(x) > 0\}$. Therefore, $G(\mathbb{Q})_+$ does not change the orientation of z_0^+ to z_0^-.

Remark 0.18. The case of $O(0,2)$ is somewhat exceptional. Notice how I did not work directly with V except to identify $C^{even}(V)$. In low dimensions, it is possible to do this and I summarize this. Obviously this list is not complete since I do not write down what happens for $O(3,2)$.

$\dim V = n+2, V$ over $\mathbb{Q}$	$C^{even}(V)$	$GSpin(n,2)$
$\dim V = 2$	Imaginary Quadratic Field k	$Res_{k/\mathbb{Q}}\mathbb{G}_{k,m}$
$\dim V = 3$	Quaternion Algebra over $\mathbb{Q}$	GL_2 or $B^\times$ (B an indefinite quaternion algebra)
$\dim V = 4$	Quaternion algebra on $Z(C^{even}(V))$	$\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$, $\mathrm{GL}_{2,F}^{\det\in\mathbb{Q}^\times}$, $B^\times \times_{\mathbb{G}_m} B^\times$

Remark 0.19. Ryan Tamura asked during the seminar talk: *Why do we focus solely on $O(n,2)$?* The answer is that these are the only orthogonal groups which give rise to a Shimura variety. The 2 here is to ensure that we can get a complex structure on $\mathbb{D}$ and then $O(n,2)$ produces a Shimura variety. As a fun fact, $O(4,4)$ *does* produce a complex structure, but there is no associated Shimura variety.

Example 0.20 (Example in the case of $O(1,2)$ i.e. $X_0(N)$)**.** *Following §7 of Bruinier-Yang.* In this situation, one can recover two of the three[4] classes of Shimura curves one might have.

The goal is to pick a set-up which gives $X_K = Y_0(N)$. This follows the presentation of Bruinier-Yang. Let N be a positive integer. Let

$$V := M_2(\mathbb{Q})^{Tr=0} \text{ with quadratic form } Q(x) := N\det(x) \tag{17}$$

and $(x,y) = -N \cdot Tr(xy^\iota)$ for $x, y \in V$ and where y^ι denotes the adjoint of a (2×2)-matrix.

Then $GSpin(V) \cong \mathrm{GL}_2$ by letting

$$\mathrm{GL}_2(\mathbb{Q}) \curvearrowright V \text{ by conjugation } \gamma.x = \gamma x\gamma^{-1} \text{ for } \gamma \in \mathrm{GL}_2(\mathbb{Q}), x \in V. \tag{18}$$

[4]The classical modular curves, the orthogonal Shimura curves, and the unitary Shimura curves.

Furthermore, we identify $\mathbb{D}$ with $\mathbb{H}^{\pm}$ by the bilomorphic map

$$z := x + iy \mapsto \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \bmod \mathbb{C}^\times \in \mathbb{D} \tag{19}$$

and under this identification, the action of $\mathrm{GL}_2(\mathbb{R})$ is the usual linear fractional action.

For our compact open subgroup, let $K_p \subseteq G(\mathbb{A}_f)$ be the subgroup

$$K_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \mid c \in N\mathbb{Z}_p \right\} \tag{20}$$

and take $K := \prod K_p \subseteq G(\mathbb{A}_f)$. This matches our previous choice of K.

One can check directly $G(\mathbb{A}_f) = G(\mathbb{Q})K$ because of $G(\mathbb{A}_f) \cong G(\mathbb{Q}) \cdot G(\widehat{\mathbb{Z}})$. So there is a single component. Therefore,

$$Y_0(N) = \Gamma_0(N) \xrightarrow{\sim} \mathbb{H} \to X_K(\mathbb{C}) = G(\mathbb{Q})\backslash \mathbb{D} \times G(\mathbb{A}_f)/K, \qquad \Gamma_0(N)z \mapsto G(\mathbb{Q})(z,1)K \tag{21}$$

is an isomorphism.

0.3 Definition and Properties of Special Cycles. We now describe the method to get sub-Shimura varieties of X_K.

1. Given (V, Q), pick an $x \in V(\mathbb{Q})$ a vector with $Q(x) > 0$.
2. Let $V_x := x^\perp$ be the orthogonal complement. The quadratic form Q descends to V_x but now has signature $(n-1, 2)$.
3. Let $G_x := GSpin(V_x)$ denote the stabilizer of V_x in $G := GSpin(V)$.
4. Let $\mathbb{D}_x := \{z \in \mathbb{D} : z \perp x\}$. This is a divisor on $\mathbb{D}$ and gives rise a natural map

$$G_x(\mathbb{Q})\backslash \mathbb{D}_x \times G_x(\mathbb{A}_f)/(G_x(\mathbb{A}_f \cap K) \to X_K \qquad (z, g) \to (z, g). \tag{22}$$

The image defines a divisor $Z(x)$ of X_K which is rational over $\mathbb{Q}$.

This process produced cycles of codimension 1 aka divisors. If we want to obtain higher codimension cycles as well, we could try and generalize this construction as follows. Let $x \in V(\mathbb{Q})^r$ be an r-tuple of points. Let $V_x, G_x, \mathbb{D}_x$ be defined as above. So $\dim \mathrm{span}_{1 \le i \le r}(x_i) = r(x)$ and $Q(x) := \frac{1}{2}(x_i, x_j)$ be a positive semidefinite matrix of rank $r(x)$ whose restriction to V_x has signature $(n - r(x), 2)$ and then we get a cycle

$$Z(x; K) : Sh(G_x, \mathbb{D}_x)_K \to Sh(G, \mathbb{D})_K = X_K$$

of codimension $r(x) = rank(Q(x)) \le r$.

Proposition 0.21. If $rank(Q(x)) < r(x)$ or $Q(x)$ is not positive semidefinite, then $Z(x) = \emptyset$.

Example 0.22. Consider the case where V has type $(0, 2)$. Then the X_K are disjoint unions of points. There are no natural cycles of codimension > 0 in this case since there are no vectors $x \in V(\mathbb{Q})$ such that $Q(x) > 0$.

Definition 0.23 (Shifted Cycles $Z(x, g; K)$)**.** Let $g \in G(\mathbb{A}_f)$. Then a **shifted cycle** $Z(x, g; K)$ is the image of the map

$$Z(x, g; K) : G_x(\mathbb{Q})\backslash(\mathbb{D}_x \times G_x(\mathbb{A}_f)/(G_x(\mathbb{A}_f) \cap gKg^{-1}) \to X_K(\mathbb{C}) \qquad (z, h) \mapsto (z, hg).$$

The cycle $Z(x, g; K)$ is of course defined over $\mathbb{Q}$ once more. One good reason for this & for many classical special cycles is that we should have a moduli interpretation of what points on these cycles are.

Proposition 0.24 (Properties of Shifted Cycles $Z(x, g; K)$)**.** The shifted cycles we have just defined satisfy the following properties:

1. For $k \in K$, $Z(x, g; K) = Z(x, gk; K)$.
2. For $h \in G_x(\mathbb{A}_f)$, $Z(x, hg; K) = Z(x, g; K)$.
3. For $f \in G(\mathbb{Q})$ one has $Z(fx, fg; K) = Z(x, g; K)$.

Proof. The proof of (1)-(3) are rather straightforward:

1. The image is defined up to a right action of K.
2. The points of $Z(x, g; K)$ are given by the images of $Sh(G_x, \mathbb{D}_x)$ sitting inside of $X_K(\mathbb{C})$. So the shift by h is just an automorphism of the cycle and so the shifted cycle is preserved.

(3) Given a point $(z, hg) \in Z(x, g; K)$ we get that $(fz, fhg) = (z, hg)$ since $G(\mathbb{Q})$ acts on $\mathbb{D} \times G(\mathbb{A}_f)$ diagonally.

□

Remark 0.25. There is an issue with shifted cycles which makes life a little bit harder. The shifted cycles are not invariant under pullbacks $Sh(G, \mathbb{D})_K \to Sh(G, \mathbb{D})_{K'}$. The next best thing we can do is to try and put together shifted cycles so that we get something that is invariant.

Definition 0.26 (Weighted Cycles)**.** Let $\varphi \in S(V(\mathbb{A}_f)^r)^K$ be a K-invariant Schwarz function on $V(\mathbb{A}_f)^r$. Let $T \in Sym_{r \geq 0}(\mathbb{Q})$. Define

$$\Omega_T := \{x \in V^r : Q(x) = T\}.$$

If $\Omega_T \neq \emptyset$, we may fix an $x \in \Omega_T(\mathbb{Q})$ and by K-invariance we get

$$\mathrm{Supp}(\varphi) \cap \Omega_T(\mathbb{A}_f) := \coprod_j K g_j^{-1} x \qquad \text{for } g_j \in G(\mathbb{A}_f).$$

(1) The decomposition varies over a finite indexing set. The LHS is compact since $\mathrm{Supp}(\varphi)$ is compact and $\Omega_T(\mathbb{A}_f)$ is closed. The decomposition follows from just taking K-orbits.
(2) Define the **weighted cycle** as

$$Z(T, \varphi; K) := \sum_j \varphi(g_j^{-1} x) Z(x, g_j; K) = \sum_{K \backslash \mathrm{Supp}(\varphi) \cap \Omega_T(\mathbb{A}_f) \ni y} \varphi(y) Z(g_j y, g_j; K).$$

which have codimension $rank(T)$. The g_j in the second sum is fixed. In modern language, people call these **special cycles**. Additionally,

$$Z(T, \varphi; K) \in CH^{rank(T)}(X_K)_{\mathbb{Q}(\text{Coefficients of } Z(T,\varphi;K))}. \tag{23}$$

Since our Schwarz functions will typically be valued in $\mathbb{Z}$ or $\mathbb{Q}$, we get integral/rational classes in the Chow groups.

Proposition 0.27. Let $pr : X_{K'} \to X_K$ be projection coming from $K \subseteq K'$. Then,

$$pr^* Z(T, \varphi; K) \cong Z(T, \varphi; K').$$

Remark 0.28. For the rest of the talk, we will specialize to the case $r = 1$ for clarity.

Lemma 0.29 (Bruinier-Yang Lemma 4.1)**.** Let $r = 1$. Suppose $G(\mathbb{A}_f) = G(\mathbb{Q})_+ K$ and $\Gamma_K := G(\mathbb{Q})_+ \cap K$. Then

$$Z(m, \varphi; K) = \sum_{x \in \Gamma_K \backslash \Omega_m(\mathbb{Q})} \varphi(x) pr(\mathbb{D}_x, 1) \tag{24}$$

where $pr : \mathbb{D} \times G(\mathbb{A}_f) \to X_K$ is the natural projection.

Remark 0.30. This is a variant of a result in Kudla's *Algebraic Cycles on Shimura Varieties of Orthogonal Types* Proposition 5.4. In Kudla's statement, he accounts for multiple connected components while in Bruinier-Yang Lemma 4.1, they deal only with a single connected component.

For the sake of completeness, here we also present's Kudla's results.

Definition 0.31. Let $K \subseteq G(\mathbb{A}_f)$ be an open compact subgroup. For any $g \in G(\mathbb{A}_f)$, one can define

$$\Gamma_g := G(\mathbb{Q})_+ \cap gKg^{-1} \qquad \& \qquad \Gamma_{g,x} := G_x(\mathbb{Q}) \cap \Gamma_g$$

where $x \in V$ is a positive norm vector. Then, we let

$$pr(\mathbb{D}_x, g) := \text{ image of } \mathbb{D}_x \times \{g\} \to X_K(\mathbb{C}).$$

In our formulas, g will be a coset reprensentative of

$$G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$$

parameterizing the connected components of $X_K(\mathbb{C})$. In this way, $pr(\mathbb{D}_x, g)$ is always a **connected cycle** and so Bruinier-Yang Lemma 4.1 is decomposing the special cycle into its connected cycles. This interpretation makes clear of the formula

$$pr(\mathbb{D}_x, g) = pr(\mathbb{D}_x, gk) \ \forall k \in K.$$

Proposition 0.32 (Kudla Proposition 5.4)**.** Let $G(\mathbb{A}_f) = \coprod_j G(\mathbb{Q})_+ g_j K$ and let $\Gamma_{g_j} := G(\mathbb{Q})_+ \cap g_j K g_j^{-1}$. Then the special cycles can be written as

$$Z(m, \varphi; K) = \sum_j \sum_{x \in \Omega_m(\mathbb{Q}) \bmod \Gamma_{g_j}} \varphi(g_j^{-1}x) pr(\mathbb{D}_x, g_j).$$

Rough Idea of Proof. The cycles $Z(y, 1; K)$ can be rewritten as a sum of connected cycles $pr(\mathbb{D}_x, g_j)$. In the case of Bruinier-Yang Lemma 4.1, the $Z(y, 1; K)$ break up as a sum of copies of $pr(\mathbb{D}_x, 1)$. Since $G(\mathbb{A}_f) = G(\mathbb{Q})_+ K$, the decomposition

$$\operatorname{Supp}(\varphi) \cap \Omega_m(\mathbb{A}_f) = \coprod_{j, g_j \in G(\mathbb{A}_f)} K g_j^{-1} x = \coprod_{j, g_j \in G(\mathbb{Q})_+} K g_j^{-1} x$$

which implies that $K \backslash \operatorname{Supp}(\varphi) \cap \Omega_m(\mathbb{A}_f) = K \backslash \operatorname{Supp}(\varphi) \cap \Omega_m(\mathbb{Q})$ and we can rewrite

$$Z(m, \varphi; K) = \sum_{g_j \in K \backslash \operatorname{Supp}(\varphi) \cap \Omega_m(\mathbb{A}_f)} \varphi(g_j^{-1}x) Z(x, g_j; K) \tag{25}$$

$$= \sum_{g_j \in K \backslash \operatorname{Supp}(\varphi) \cap \Omega_m(\mathbb{Q})} \varphi(g_j^{-1}x) Z(x, g_j; K) \tag{26}$$

$$= \sum_{g_j \in K \backslash \operatorname{Supp}(\varphi) \cap \Omega_m(\mathbb{Q})} \varphi(g_j^{-1}x) \sum_{h_i \in G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K} pr(\mathbb{D}_x, h_i) \tag{27}$$

$$= \sum_{g_k \in K \backslash \operatorname{Supp}(\varphi) \cap \Omega_m(\mathbb{Q})} \varphi(g_j^{-1}x) pr(\mathbb{D}_x, 1) \qquad \text{there's only a single connected component of } X_K \tag{28}$$

$$= \sum_{g_j \in (K \cap G(\mathbb{Q})_+) \backslash \Omega_m(\mathbb{Q})} \varphi(g_j^{-1}x) pr(\mathbb{D}_x, 1) \tag{29}$$

and this last term is exactly what we wanted. □

Remark 0.33. To handle the general case, Kudla spent a lot of time working with double cosets and rewriting the sums that appear in the formulas for special cycles.

0.4 Examples of Special Cycles for Orthogonal Shimura Varieties. Now we present some examples of special cycles on orthogonal Shimura varieties. The focus will be on recovering many classical cycles / divisors that have been showing up before Kudla's 1997 work.

Example 0.34 (Special Cycles for $O(0, 2)$)**.** For $O(0, 2)$, there are no special cycles to consider for the reason I mentioned before.

Remark 0.35 (Sketch of how to construct special cycles using lattices in general)**.** The point of this remark is to write down an outline on how to construct special cycles in general.

1. Assume the set from before i.e. $G := GSpin(V)$ and (V, Q) signature $(n, 2)$.
2. Let $L \subseteq V$ be an integral even lattice.
3. Define a Schwarz function $\varphi := \sum_{\mu \in L^\vee/L} char(\mu + \widehat{L})$ where $L \otimes \widehat{\mathbb{Z}} = \widehat{L}$.
4. Pick $K \subseteq G(\mathbb{A}_f)$ as to preserve $L^\vee/L$. Sometimes, we go the other way and pick K first and make an informed decision on L.
5. Form the special cycles $Z(m, \varphi; K) = \sum_{g_j \in \coprod_j K g_j^{-1} x} \varphi(g_j^{-1}x) Z(x, g_j; K)$ where $x \in \Omega_m(\mathbb{Q})$. In the case where X_K is a single connected component, Bruinier-Yang Lemma 4.1 makes life significantly more easier.

Example 0.36 (The Modular curve $X_0(N)$ following Bruinier-Yang)**.** First, we define a lattice $L \subseteq M_2(\mathbb{Q})^{Tr=0}$;

$$L := \left\{ \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \qquad \& \qquad L^\vee = \left\{ \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}. \tag{30}$$

The discriminant group $L^\vee/L$ is identified with $\mathbb{Z}/2N\mathbb{Z}$ by sending $r \in \mathbb{Z}/2N\mathbb{Z}$ to $\mu_r := \operatorname{diag}(r/2N, -r/2N)$.

From our definition of K, we see K acts on L'/L trivially and that K maps L to itself. In other words, $\phi_\mu := char(\mu + \widehat{L})$ are K-invariant Schwarz function.

Describing the set $\mathrm{Supp}(\phi_\mu) \cap \Omega_m(\mathbb{Q})$**.** For $m \in \mathbb{Z}_{>0}$ and $\mu \in L^\vee/L$, we would like to study

$$\mathrm{Supp}(\phi_\mu) \cap \Omega_m(\mathbb{Q}) = \{x \in \mu + L \mid Q(x) = m\}. \tag{31}$$

Let $D := -4Nm \in \mathbb{Z}$ be a negative discriminant. If $r \in \mathbb{Z}$ and $\mu = \mu_r \bmod L$, then

$$D = -4Nm = -4NQ(\mu_r) = -4N\frac{-r^2}{4N} = r^2 \equiv r^2 \bmod 4N$$

and so

$$x = \begin{pmatrix} \frac{r}{2N} & \frac{1}{N} \\ \frac{D-r^2}{4N} & \frac{-r}{2N} \end{pmatrix} \in \mathrm{Supp}(\phi_\mu) \cap \Omega_m(\mathbb{Q}). \tag{32}$$

OTOH, we know that for $D < 0$ and $D \equiv r^2 \bmod 4N$, if $m := \frac{-D}{4N}$ and $\mu := \mu_r$ then $m \in Q(m) + \mathbb{Z}$ is positive. This gets us most of the way to describing the choices of x we can make.

Furthermore, we get from **Bruinier-Yang Lemma 4.1** that

$$Z(m, \phi_\mu) = P_{D,r} + P_{D,-r}$$

are the Heegner divisors defined in Gross-Kohnen-Zagier i.e.

$$P_{D,r} := \{z \in \mathbb{H}/\Gamma_0(N) : az^2 + bz + c = 0, b^2 - 4ac = D, a > 0, a \equiv 0 \bmod N, b \equiv r \bmod 2N\}$$

counted with multiplicity.

Exercise: Check that $Z(m, \phi_\mu) = P_{D,r} + P_{D,-r}$**.** For sake of exposition, we leave the complete check that this equality holds to the reader. The main difficulty is describing $\mathrm{Supp}(\varphi) \cap \Omega_m(\mathbb{Q})$ which is an exercise in linear algebra.

OTOH, we do check that the x above gives a quadratic whose solutions are in $P_{D,r}$. The inner product gives

$$(w(z), x) = -N \cdot Trace \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \begin{pmatrix} \frac{-r}{2N} & -\frac{1}{N} \\ -\frac{D-r^2}{4N} & \frac{r}{2N} \end{pmatrix} = (-N)(-\frac{r}{2N} + \frac{D-r^2}{4N}z^2 - \frac{1}{N} - \frac{r}{2N}z) \tag{33}$$

$$= \frac{r^2 - D}{4}z^2 + rz + 1. \tag{34}$$

Then the discriminant is

$$r^2 - 4(\frac{r^2 - D}{4}) = D.$$

The congruence relations for a and b clearly hold;

$$a = \frac{D - r^2}{4} \equiv 0 \bmod N \text{ because } D \equiv r^2 \bmod 4N \;\Rightarrow\; 4N \mid D - r^2 \;\Rightarrow\; N \mid \frac{D - r^2}{4} \tag{35}$$

$$b \equiv r \bmod 2N. \tag{36}$$

Meanwhile, the quadratic form $Q(x)$ gives

$$N \det \begin{pmatrix} \frac{r}{2N} & \frac{1}{N} \\ \frac{D-r^2}{4N} & \frac{-r}{2N} \end{pmatrix} = N \cdot \left(-\frac{r^2}{4N^2} - \frac{D - r^2}{4N^2}\right) = N \cdot \frac{-D}{4N^2} = -\frac{D}{4N} = m.$$

Remark 0.37. If one considers, instead of $M_2(\mathbb{Q})$, an indefinite quaternion algebra $B/\mathbb{Q}$ and defined V similarly, then one obtains Shimura curves which are not classical modular curves. One can then define analogues of the Heegner divisors in these cases.

Example 0.38 (Heenger Divisors for Shimura Curves)**.** First, we define the Shimura curve as follows.

(1) Fix B an indefinite quaternion algebra[5] over $\mathbb{Q}$.
(2) Let $V := B^{Tr=0}$ and let $Q(x) := -x^2$ and so $(x, y) = Tr(xy^\iota)$ where ι denotes the adjgate transpose.
(3) The action by conjugation implies that $B^\times \cong GSpin(V)$.
(4) Identify the Grassmannian of negative definite orientated 2-planes

$$\mathbb{H}^\pm \to \mathbb{D} \qquad z \mapsto w(z) = \begin{pmatrix} z & -z^2 \\ 1 & z \end{pmatrix} \bmod \mathbb{C}^\times.$$

[5] i.e. $B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$

(5) Because we work with an indefinite quaternion algebra, we need to be a little bit careful with how we pick our open compact subgroup $K \subseteq B^\times(\mathbb{A}_f)$.

Let S be the set of primes p where $B_p = \mathbb{Q}_p \otimes B$ a division algebra. Let $D(B) := \prod_{p\in S} p$ and there are only finintely many such primes by definition. Let $O_B \subseteq B$ be a maximal order and so

$$B(\mathbb{A}_f) \xrightarrow{\sim} \prod_{p\in S} B_p \times M_2(\mathbb{A}_f^S) \tag{37}$$

$$O_B \otimes \widehat{\mathbb{Z}} \xrightarrow{\sim} \prod_{p\in S} O_{B,p} \times M_2(\widehat{\mathbb{Z}}^S) \tag{38}$$

Let R denote the Eichler order of discriminant $N \cdot D(B)$. In this way,

$$K := R \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \cong \prod_{p\in B} O_{B,p} \times \{x \in M_2(\widehat{\mathbb{Z}}^S) : c \equiv 0 \bmod N\}.$$

(6) Now K is an open compact subgroup of $G(\mathbb{A}_f)$. Then we define the Shimura curve

$$X_0^B(N) := X_K(\mathbb{C}) = B^\times(\mathbb{Q})\backslash\mathbb{D} \times B^\times(\mathbb{A}_f)/K = \Gamma\backslash\mathbb{D}^+$$

where $\Gamma \cong G(\mathbb{Q})^+ \cap K \cong R^\times$.

Now we construct the Heegner divisors on Shimura curves. We are shall be very brief for the sake of time.

(1) Identify $V(\mathbb{Q})$ with traceless matrices of $M_2(\mathbb{R})$. Then $Q(x) = -(b^2 - 4ac)$ whenever $x = \begin{pmatrix} b & 2c \\ -2a & -b \end{pmatrix}$.

In this way,

$$\mathbb{D}_x := \{z \in \mathbb{H}^\pm \mid x \perp w(z) \text{ or just } -2(az^2 + bz + c) = 0\}.$$

(2) $\Omega_d(\mathbb{Q}) := \{x \in V(\mathbb{Q}) : Q(x) = d\}$ and we pick φ, a K-invariant Schwarz-function, as follows

$$\varphi := \sum_{\mu \in L^\vee/L} \mathrm{char}(\mu + L^\vee)$$

and where $L := R \cap V(\mathbb{Q})$ which is a lattice. Then by **Bruinier-Yang Lemma 4.1** we get

$$Z(d, \varphi; K) = \sum_{x\in\mu\in O\subseteq L^\vee/L \bmod \Gamma, x\in\Omega_d(\mathbb{Q})} pr(\mathbb{D}_x, 1)$$

where μ determines an orbit of the K-action on $L^\vee/L$. This generalizes the classical Heegner points to the case of *any* indefinite quaternion algebra.

Example 0.39 (Recovering the Modular Correspondence)**.** Recall the definition of the modular correspondence on $Y(1) \times Y(1)$ where $Y(1) := \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$. One definition[6] is that the modular correspondences are the cycles

$$T_m := \{(z, w) \in Y(1) \times Y(1) \mid w = Az \text{ for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}), \det(A) = m\}.$$

Using our definition of a special cycle, we can recover this up to a change of variable. Another definition is to use modular polynomials as in the paper of Gross-Keating and a third is via a moduli interpretation.

First, we can realize $Y(1) \times Y(1)$ as an orthogonal Shimura variety. [7] Let $V := M_2(\mathbb{Q})$ and define the form as $Q(x) = \det(x)$ where $x \in V$. This is a signature $(2, 2)$ quadratic form since the associated symmetric square matrix is $\begin{pmatrix} & & & 1/2 \\ & & -1/2 & \\ & -1/2 & & \\ 1/2 & & & \end{pmatrix}$ where $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$ corresponds to the vector $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ is a symmetric matrix with 2 negative eigenvalues and two positive eigenvalues. The associated inner product is $(x, y) = Tr(xy^\iota)$.

---

[6] i.e. as in up to a change of variables

[7] The signature is $(2, 2)$ since the matrix giving the form is $\begin{pmatrix} & & & 1/2 \\ & & -1/2 & \\ & -1/2 & & \\ 1/2 & & & \end{pmatrix}$. There are 2 minus signs and 2 plus signs.

One can check that $GSpin(V) \cong \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ and that pairs (g,h) with $\det(g) = \det(h)$ act on V by $v \mapsto gvh^{-1}$. This clearly preserves the quadratic form,. Furthermore, if $K := \prod_p K_p$ where $K_p = \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2(\mathbb{Z}_p)$, then the associated Shimura variety is

$$X_K \cong Y(1) \times Y(1)$$

which is clearly already defined over $\mathbb{Q}$. There is only a single connected component because

$$G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K \cong G(\mathbb{Q})_+ \backslash G(\mathbb{Q})/\{\pm 1\} \cong 1.$$

Since the $Y(1)$'s are not compact, we must compactify to get the usual modular curve $X(1)$. We will not worry about this.

Modular Correspondence. We identify $\mathbb{D}$ with $\mathbb{H}^{\pm} \times \mathbb{H}^{\pm}$ by

$$\gamma : (z,w) \mapsto \begin{pmatrix} zw & w \\ z & 1 \end{pmatrix} \bmod \mathbb{C}^{\times}.$$

Now let us pick an easy lattice to consider $L := M_2(\mathbb{Z}) \subseteq V$. The dual lattice is clearly itself.

Let $\phi := \sum_{\mu \in L^\vee/L} char(\mu + \widehat{L})$ be a sum of characteristic functions. The first two things to check are that ϕ is K-invariant and that $L^\vee/L$ is preserved. First off, we can compute the dual lattice

$$L^\vee = \{x \in V : (x, L) \in \mathbb{Z}\} = M_2(\mathbb{Z}).$$

The discriminant group is then trivial and clearly preserved by K. Furthermore, K itself must preserve the lattice. Let $m \in \mathbb{Q}_{>0}$. Then, we know for an $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L$, with a fixed determinant $\det(x) = m$, we get

$$\mathbb{D}_x = \{(z,w) \in \mathbb{H}^{\pm} \times \mathbb{H}^{\pm} \mid (\gamma(z,w), x) = -azw + bw + cz - d = 0\}.\}$$

since one can check

$$(\gamma(z,w), x) = Tr \begin{pmatrix} -zw & z \\ w & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = Tr \begin{pmatrix} -azw + cz & * \\ ** & bw - d \end{pmatrix} = -azw + bw + cz - d.$$

But observe that **Bruinier-Yang Lemma 4.1** gives

$$Z(m, \phi) = T_m$$

where the T_m are Hirzebruch-Zagier cycles / modular correspondences on a product of modular curves (c.f. this paper).

Indeed, one could interpret the divisors T_m as

$$\{(z,w) \in Y(1) \times Y(1) \mid (z,w) = (z, Az) \text{ for } A = \begin{pmatrix} -c & d \\ -a & b \end{pmatrix} \in M_2(\mathbb{Z}), \det(A) = m\}.$$

Indeed, $Az = \frac{-cz+d}{-az+b}$ and so the equation in $\mathbb{D}_x$'s definition gives

$$-azw + bw + cz - d = 0 \;\Rightarrow\; w = \frac{-cz+d}{-az+b}.$$

Remark 0.40. This is not a formal proof. For any K-invariant Schwarz function φ of the form $\sum_{\mu \in L^\vee/L} char(\mu + \widehat{L})$ where L is an even integral lattice of V, and (V, Q) with open compact K the data determining the Shimura variety $Y(1) \times Y(1)$, the special cycles are all, up to isomorphism, described by the modular correspondences.

Example 0.41 (Hilbert Modular Surfaces and the Hirzbruch Zagier Surfaces)**.** Let $F := \mathbb{Q}(\sqrt{p})$ be a real quadratic field where $p \equiv 1 \bmod 4$. Let O_F be the ring of integers. Let $\partial_F := \sqrt{p}O_F$ be the different. Define a quadratic space

$$V := \{x \in M_2(F) \mid x' = x^t\} = \{x = \begin{pmatrix} a & \lambda \\ \lambda' & b \end{pmatrix} \mid a, b \in \mathbb{Q}, \lambda \in F\}$$

with quadratic form $Q(x) = \det(x)$ which has signature $(2,2)$. Then the bilinear form is $(x,y) = Tr(xy^\iota)$. Define a lattice

$$L := V \cap M_2(O_F) \text{ and so its dual lattice is } L' = \left\{ \begin{pmatrix} a & \lambda \\ b & \lambda' \end{pmatrix} \mid a, b \in \mathbb{Z}, \lambda \in \sqrt{p}O_F \right\}.$$

In this way, we can identify

$$G := GSpin(V) \cong \{g \in \mathrm{GL}_2(F) \mid \det(g) \in \mathbb{Q}^\times\} \cong Res_{F/\mathbb{Q}} \mathrm{GL}_2$$

which acts on V by $g \cdot x = \frac{1}{\det g} g x {}^t g'$. Let

$$K := G(\widehat{\mathbb{Z}}) = \{g \in \mathrm{GL}_2(O_F \otimes \widehat{\mathbb{Z}}) \mid \det(g) \in \widehat{\mathbb{Z}}^\times\}.$$

Now identify the Grassmannian of negative definite oriented 2-planes by

$$(\mathbb{H}^\pm)^2 \xrightarrow{w} \mathbb{D}, \qquad w((z_1, z_2)) = \begin{pmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{pmatrix}.$$

Furthermore $G(\mathbb{A}_f) = G(\mathbb{Q})_+ K$, because we can write

$$G(\mathbb{A}_f) \cong G(\mathbb{Q}) G(\widehat{\mathbb{Z}}) \cong G(\mathbb{Q}) K \cong G(\mathbb{Q})_+ K$$

and the last isomorphism follows since $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in K$ lets us swap the sign of the determinant. This isomorphism implies in particular that $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$ is trivial and so there is only a single conected component that appears in the double coset

$$X_K(\mathbb{C}) = G(\mathbb{Q}) \backslash \mathbb{D} \times G(\mathbb{A}_f)/K \cong \mathrm{SL}_2(O_F) \backslash \mathbb{H}^2.$$

So far, we used/shall use some of the following facts. The object on the RHS is precisely the Hilbert modular surface as defined in Hirzebruch and Zagier's paper[8].

(1) K is a normal algebraic subgroup of $G(\mathbb{A}_f)$,
(2) $\Gamma = G(\mathbb{Q})_+ \cap K$ is isomorphic to $\mathrm{SL}_2(O_F)$,
(3) the double quotient $G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/K$ is trivial,
(4) one has that

$$Trace \begin{pmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{pmatrix} \begin{pmatrix} -b & \lambda \\ \lambda' & -a \end{pmatrix} = -b z_1 z_2 + \lambda' z_1 + \lambda z_2 - a.$$

(5) K acts transitively on $L^\vee/L \cong \mathbb{Z}/p\mathbb{Z}$ and we know $\Gamma_K \backslash \Omega_{\frac{m}{p}}(\mathbb{Q}) \cong L_{\frac{m}{p}}/\{\pm 1\}$.

The RHS is the Hilbert modular surface defined e.g. in Gross-Hirzebruch. After picking an x and using the identification above, we find that

$$\mathbb{D}_x \cong \{(z_1, z_2) \in (\mathbb{H}^\pm)^2 \mid -b z_1 z_2 + \gamma' z_1 + \gamma z_2 - a = 0\}.$$

Now if we take the Schwarz function $\varphi = \sum_{\mu \in L^\vee/L} char(\mu + \widehat{L})$, we obtain the special cycles

$$Z(\frac{m}{p}, \varphi; K) \overset{(*)}{=} \sum_{x \in (G(\mathbb{Q})_+ \cap K) \backslash \Omega_{\frac{m}{p}}(\mathbb{Q})} \varphi(x) pr(\mathbb{D}_x, 1) \tag{39}$$

$$= \sum_{x \in \Gamma_K \backslash \Omega_{\frac{m}{p}}(\mathbb{Q})} \sum_{\mu \in L^\vee/L} char(\mu + \widehat{L})(x) pr(\mathbb{D}_x, 1) \tag{40}$$

$$= \sum_{x \in L_{\frac{m}{p}}/\{\pm 1\}} \{(z_1, z_2) \in \mathbb{H}^2 \mid b z_1 z_2 - \gamma' z_1 - \gamma z_2 + a = 0\}. \tag{41}$$

The last equality follows from the fact that Γ_K acts on $L^\vee/L$ transitively and modulo this action gives me $L_{\frac{m}{p}}/\{\pm 1\}$ where

$$L_{\frac{m}{p}} := \{x = \begin{pmatrix} a & \lambda \\ \lambda' & b \end{pmatrix} \in V(\mathbb{Q}) \mid Q(x) = \frac{m}{p}\}.$$

The step made at (*) uses Kudla's Proposition 5.4 and/or Bruinier-Yang's Lemma 4.1.

The special cycle $Z(\frac{m}{p}, \varphi; K)$ is precisely the Hirzebruch-Zagier cycle up to a change of variables or sign on the Hilbert modular surface.

[8] In [HZ], the quotient is via the action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \curvearrowright \mathbb{H}^2$ sending $(z, w) \mapsto (\frac{az+b}{cz+d}, \frac{a'z+b'}{c'z+d'})$