BRUINIER-YANG'S MAIN CONJECTURE

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This note was written when I prepared my seminar talk, any mistakes are due to myself. If you find any typos or have any suggestions, feel free to contact me.

Contents

In this talk, I'll give a closer look at Bruinier-Yang's conjecture. The goal of today's talk is to convince you that the thing we will do in the April is really interesting. I will sketch the theory of regularized theta lift without specifying details. This will be one of the topic we want to take in details during the April.

1. Borcherds product

The main ingredient of everything is the Borcherds lift and its generalization. Roughly speaking, regularized theta lift is trying to lift a "modular forms" of some specific weight relate to the dimension of orthogonal group, to a "modular forms" on $\mathrm{SO}(n,2)$. Note that this is opposite to the direction we did previously: now we want to do theta lifts from SL_2 to $\mathrm{SO}(V)$!

Let's specify notation here: $\mathbb{H}$ is the Poincare upper half plane, $\mathbb{D}$ is the symmetric space of orthogonal group of signature $(n,2)$, is the space of negative definite oriented 2-subspaces. K' is a level structure of metaplectic group, K is a level structure of orthogonal group. X_K is the Shimura variety with respect to the orthogonal group.

1.1. Theta lifting on $\mathrm{SO}(n,2)$.

Recall that the group $\mathrm{SL}_2(\mathbb{R})$ has a double cover $\mathrm{Mp}_{2,\mathbb{R}}$ whose elements can be written in the form

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi(\tau)\right),$$

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where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $\phi(\tau) = \pm\sqrt{c\tau + d}$ is considered as a holomorphic function of τ in $\mathcal{H}$ whose square is $c\tau + d$. This will related to the maximal compact subgroup $\mathrm{SO}(2) \subset \mathrm{SL}_2(\mathbb{R})$, as Simon mentioned. The matrix multiplication is defined by

$$(A, f(\cdot))(B, g(\cdot)) = (AB, f(B(\cdot))g(\cdot)),$$

where $A, B \in \mathrm{SL}_2(\mathbb{R})$ and f and g are suitable functions on $\mathbb{H}$. The group $\mathrm{Mp}_2(\mathbb{Z})$ is the discrete subgroup of $\mathrm{Mp}_2(\mathbb{R})$ of elements of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. We can still restrict ourselves into SL_2 without losing anything crutial. But in this talk, most things I talked will be conceptional, it does not harm to stick into metaplectic group, and somehow this is good for my personal mental health.

Let (V, Q) be a quadratic space over $\mathbb{Q}$. We will be interested the Weil representation with respect to the dual pair $(\mathrm{SL}_2, \mathrm{SO}(V))$. Given a Weil representation ω on $\mathrm{Sp}(W \otimes V)$, by restricting, we get a Weil representation on $\mathrm{Mp}_{2,\mathbb{A}} \times \mathrm{SO}(\mathbb{A})$. Such representation is automorphic in the sense that is $\mathrm{SL}(\mathbb{Q}) \times \mathrm{SO}(\mathbb{Q})$-invariant, and stabilized by some compact open subgroups. The theta function will be a function

$$\theta : \mathrm{Mp}_{2,\mathbb{A}} \times \mathrm{SO}(\mathbb{A}) \times \mathcal{S}(V(\mathbb{A})) \to \mathbb{C}.$$

defined by

$$\theta(g, h; \varphi) = \sum_{x \in V(\mathbb{Q})} \omega(g)\varphi(h^{-1}x).$$

Recall that when (V, Q) has signature $(n, 0)$, given an even lattice L, denote $\widehat{L} = L \otimes \mathbb{Z}_p$, we choose and compute a specific Schwartz function $\varphi = \varphi_\infty \otimes \varphi_p$ such that

$$\varphi_\infty = e^{-\pi Q[x]}, \quad \varphi_p = \mathrm{char}(\widehat{L}).$$

From now on we will be interested in the orthogonal group of signature $(n, 2)$. We will start with an integral lattice $L \subset L^\sharp$. Recall that it is unimodular when $L = L^\sharp$. Feel free to stick on this case for the rest part of the talk. The stabilizer of such lattice will give a maximal open compact subgroup in the finite adele.

The maximal compact subgroup at archimedean place is $S(O(2) \times O(n)) \subset \mathrm{SO}(V)$. In particular, you see that the group $\mathrm{SO}(n, 2)$ is not compact, since our quadratic form is not positive definitive. We don't want to think about functions with too deep level structure, let's stick on this level structure, which we will denote by $K = K_\infty K_f$. On the contrary, we will denote an open compact of $\mathrm{Mp}_{2,\mathbb{A}_f}$ by K'_f.

We first consider a specific Schwartz function in the archimedean place. Note that we require Schwartz function to be exponentially decay, and $\varphi_\infty = e^{-\pi Q[x]}$ does not satisfy this.

Recall that the hermitian symmetric space $\mathbb{D}$ of $H(\mathbb{R})$ parameterize oriented negative definite 2-dimensional subspaces of $V(\mathbb{R})$. For any $z \in \mathbb{D}$, we can construct a positive defnite quadratic form

$$(x,x)_z := (x_{z^\perp}, x_{z^\perp}) - (x_z, x_z).$$

Then we define the Gaussian

$$\varphi_\infty(x,z) = \exp(-\pi(x,x)_z) \in S(V(\mathbb{R})).$$

Moreover, for any $h \in H(\mathbb{R})$, we have $\varphi_\infty(hx,hz) = \varphi_\infty(x,z)$. Now fix a point $z_0 \in \mathbb{D}$, and $\varphi_f \in \mathcal{S}(V(\mathbb{A}_f))$, we consider the theta function

$$\theta(g,h;\varphi_f) := \theta(g,h;\varphi(\cdot,z_0)\otimes\varphi_f(\cdot)).$$

This theta function has weight $n/2-1$ at the archimedean place of the Weil representation. In general, given a quadratic space with signature (b^+,b^-), the archimedean Schartz functions will contribute a automorphic representation of weight $\frac{b^+b^-}{2}$ (But actually I don't know how to compute the Weil representation here).

Thanks to the strong approximation theorem and multiplying with some automorphic factors to separating the action, we only need to consider the theta function restrict into the symmetric space with some action from Weil representation:

$$\theta : \mathbb{H}\times\mathbb{D}\times H(\mathbb{A}_f)\times\mathcal{S}(V(\mathbb{A}_f)) \to \mathbb{C}.$$

The precise formula can be found in [BY09, Section 2], and we won't do it here. We will specify the level structure soon. The point is that Weil representation in adeles are rather complicated since it is hard to describe the Weil representation precisely in non-archimedean places like in the archimedean place. But thanks to the fact that Weil representation is global with nice compatibility, and we have strong approximation theorem for metaplectic group, we can save our lift by restricting into archimedean place.

Another advantage of this form is that we can see how $\mathbb{D}$ involve into the story more transparently. Now the Weil representation becomes

$$\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \phi(\tau)\right)\cdot\theta(\tau,z,h_f;\varphi) = \phi(\tau)^{n-2}\theta(\tau,z,h_f;\varphi'_f).$$

Here φ'_f is changed via the Weil representation.

What we want to do next is to focus on a small class of Schwartz functions with respect to the Weil representations. We are interested in is the following representation: we will start with a vector space $\mathbb{C}[L^\sharp/L]$ spanned by the basis e_γ, $\gamma\in L^\sharp/L$. This admits a unitary representation of $\mathrm{Mp}_2(\mathbb{Z})$ defined by

$$\rho_L(T)(e_\gamma) = e^{2\pi i Q[\gamma]}e_\gamma,$$

$$\rho_L(S)(e_\gamma) = \frac{\sqrt{i}}{\sqrt{|L^\sharp/L|}}\sum_{\delta\in L^\sharp/L} e^{-2\pi i(\gamma,\delta)}e_\delta,$$

where $T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right)$ and $S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right)$ are the standard generator of $\mathrm{Mp}_2(\mathbb{Z})$. This is in fact arise from the natural Weil representation, but I don't know what's the right way to think about this. The action will naturally factor through some quotient of the arithmetic subgroup.

Example 1.1.1. For unimodular lattice L, this becomes

$$\rho_L(T)(x) = e^{2\pi i}x, \quad \rho_L(S)(x) = \sqrt{i}e^{-2\pi i}x.$$

Now the answer is clear: we want to consider the subspace S_L of Schwartz functions in $S(V(\mathbb{A}_f))$ which is the linear combination of the characteristic functions

$$\phi_\mu = \mathrm{char}(\mu + \widehat{L}), \quad \mu \in L^\sharp/L.$$

Restricting into this, the theta function becomes a function

$$\theta_L : \mathbb{H} \times \mathbb{D} \times H(\mathbb{A}_f) \times S_L \to \mathbb{C}.$$

Equivalently, this defines a function

$$\theta : \mathbb{H} \times \mathbb{D} \times H(\mathbb{A}_f) \to S_L^\vee.$$

Suppose we have an automorphic function of the form

$$f : \mathbb{H} \to S_L.$$

Then we can define a theta lifting

$$\Phi(z, h, f) := \int_{\Gamma\backslash\mathbb{H}} ((f(\tau), \theta(\tau, z, h_f)))d\mu(\tau).$$

Here $((v, w))$ is the following natural pairing:

$$((-,-)) : S_L \times S_L^\vee \to \mathbb{C}.$$

1.2. **Regularized theta lift.** Therefore, it's natural now to introduce so called vector valued holomorphic modular form. These are S_L-valued functions on $\mathbb{H}$ such that each entries are holomorphic modular forms. Such function admits a Fourier expansion

$$f(\tau) = \sum_{m\geq 0} c(m)q^m, \quad c(m) \in S_L.$$

And each coefficient can be expanded uniquely as a linear combination

$$c(m) = \sum_{\mu \in L^\sharp/L} c(m, \mu) \cdot \varphi_\mu.$$

This sounds to be a perfect story, except that we didn't get anything interesting for such theta lifts. The problem is that since our theta function has weight $n/2 - 1$, we need to lift a modular form of weight $1 - n/2$ since we took a dual representation. To look closer, we are expect our inner product invariant under the action of Γ, which means the product has weight 0. We will have to face with similar situation even when we use adele formalism. Trouble: we don't have such modular form even in very small level structure.

Remark 1.2.1. If we consider $(2, n)$-signature, then the theta lift exists, and it's the theta correspondence.

Therefore, we introduce weakly holomorphic modular forms, which allows functions to have poles at the cusp. In other words, the Fourier expansion now becomes of the form

$$f(\tau) = \sum_{m>>-\infty} c(m)q^m.$$

The arithmetic subgroup acting on them by

$$(\gamma, \phi(\tau)) \cdot f(\tau) := \phi^{2k}\rho_L \cdot f(\tau).$$

Now it comes to another trouble: we want to integral along functions with singularities, but such integration might diverges. This comes to the story of regularized theta lift, or singular theta lift. This will be one of the topic we want to explore in the April. Once this is settled, we get some meromorphic modular forms on $\mathrm{SO}(n, 2)$. The Borcherds lift declare the following statement:

Theorem 1.2.2. *The function $\Phi(z, h, f)$ is smooth on $X_K \backslash Z(f)$, where*

$$Z(f) = \sum_{\mu \in L^\sharp/L} \sum_{m>0} c(-m, \mu) Z(m, \mu).$$

It has a logarithmic singularity along the divisor $-2Z(f)$. The $(1,1)$-form $dd^c\Phi(z, h, f)$ can be continued to a smooth form on all of $X_K = H(\mathbb{A}) \cap K \backslash \mathbb{D}$.

One can further define the Borcerds lift for more general S_L-valued functions call *harmonic weak Maass forms*, can get similar results. There is a exact sequence among these two class of functions:

$$0 \to M^!_{k,\rho_L} \to H_{k,\rho_L} \xrightarrow{\xi,\bar{\rho}_L} S_{2-k} \to 0.$$

The same theory of regularized theta lift exists, except now the coefficients will corresponds to the holomorphic part $c^+(-m, \mu)$ of the function. We will call such lift an *automorphic Green function*, the reason will be explained later.

It is a good time to recall the special cycles and CM cycles we defined previous. Recall that given a Schwartz function $\varphi \in S(V(\mathbb{A}))^K$, we define a special cycle on X_K

$$Z(m, \varphi; K) := \sum_j \varphi(g_j^{-1}x) Z(x, g_j; K).$$

Here $m \in \mathbb{N}$, index j and elements g_j are defined using

$$\mathrm{Supp}(\varphi) \cap \Omega_m(\mathbb{A}_f) := \bigsqcup_j K g_j^{-1} x, \quad \Omega_m(\mathbb{A}_f) := \{x \in V : Q[x] = m\}.$$

At this point we can also mention the advantage of the harmonic weak Maass forms: we can always fund a weak harmonic Maass form f such that

$$Z(m,f) = Z(m,\phi_\mu).$$

While this is impossible for weakly holomorphic modular forms. This is the reason we would like to focus on harmonic weak Maass forms. This will be the another topic we want to study during the April.

1.3. **Bruinier-Yang result.** Next, we briefly discuss the main work of Schofer and Bruinier-Yang.

We define CM cycles on X_K as follows. Let $U \subset V$ be a negative definite 2-dimensinoal rational subspace of V. It determines a two point subset $\{z_U^\pm\} \subset \mathbb{D}$ given by $U(\mathbb{R})$ with the two possible choices of orientation. Let $V_+ \subset V$ be the orthogonal complement of U over $\mathbb{Q}$. Then V_+ is a positive definite subspace of dimension n, and we have the rational splitting

$$V = V_+ \oplus U.$$

Let $T = \mathrm{GSpin}(U)$, which we view as a subgroup of H acting trivially on V_+, and put $K_T = K \cap T(\mathbb{A}_f)$. We obtain the CM cycle

$$Z(U) = T(\mathbb{Q})\backslash(\{z_U^\pm\} \times T(\mathbb{A}_f)/K_T) \to X_K.$$

Here each point in the cycle is counted with multiplicity $\frac{2}{w_{K,T}}$, where $w_{K,T} = \#(T(\mathbb{Q}) \cap K_T)$.

The splitting induces definite lattices

$$N = L \cap U, \quad P = L \cap V_+.$$

Then $N \oplus P \subset L$ is a sublattice of finite index. The main result of Bruinier-Yang, which is another topic we are going to pursue, is the following:

Theorem 1.3.1. *The value of the automorphic Green function $\Phi(z,h,f)$ at the CM cycle $Z(U)$ is given by*

$$\Phi(Z(U),f) = \deg(Z(U)) \cdot \big(\text{constant term}(\langle f^+(\tau), \theta_P(\tau) \otimes \mathcal{E}_N(\tau)\rangle) + L'(\xi(f),U,0).\big)$$

Here θ_P is the theta series with respect to the lattice P. The function $\mathcal{E}_N(\tau)$ is some specific function constructed with respect to N. To be more fancy, it is the holomorphic part of the harmonic weak Maass form of the derivative of the Siegel Eisenstein series: $E_N'(\tau,0;1)$.

Remark 1.3.2. Nowadays, $L'(\xi(f),U,0)$ is also written as $L'(\xi(f),\theta_P,\text{center})$, viewed as a Rankin-Selberg intergral.

When f is weakly holomorphic, the last term $L'(\xi(f),U,0) = 0$, this is essentially the result of Schofer. Schofer compute the CM value of the Green function in order to reprove Gross-Zagier's result about singular moduli. At that moment, he didn't realize its relation to the arithmetic intersection theory.

This brilliant observation was done by Bruinier-Yang, after they find an additional term related to the derivative of the L-function. Let's take a example to see what we are supposed to get for the derivative of the L-function: Consider the Riemann zeta function:

$$\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{n^s}.$$

Suppose s_0 is a zero of zeta function with real part lies in $\frac{1}{2}$. Then $\zeta'(s_0)$ equals

$$\sum_{n>0} \log n \frac{1}{n^{s_0}} = \sum_{p} (\mathrm{ord}_p(n) n^{s_0}) \log p.$$

Now it becomes a rational numbers linear combination of the linear independent terms $\log p$. A phenomenon observed by Gross-Zagier, Gross-Keating, Kudla, Kudla-Rapoport, among others, is that the derivative of L-functions are related to the arithmetic intersection theory.

2. Arithmetic Chow group

I'm not intended to give a full introduction to the arithmetic intersection theory. According to Tonghai, the fundation is full of brilliant ideas. I'll sketch the main spirit of the theory which will be enough for the applications.

The model in mind is the following: assume we start with a comapct algebraic curve and a rational function f, then the divisor $\mathrm{div}(f)$ would have degree zero.

Now when we look at $\mathrm{Spec}\,\mathbb{Z}$ and $r \in \mathbb{Q}$ a rational function, we no longer have such nice property. But we know that we have the following equality if we set up the height properly:

$$\sum_{p} p^{v_p(r)} \times |r|_\infty = 1.$$

Equivalently,

$$\sum_{v} v_p(r) \log p + \log |r|_\infty = 0.$$

The arithmetic intersection theory start with this observation, they first "compactify" the $\mathrm{Spec}\,\mathbb{Z}$ by adding the archimedean contribution. Then they define the intersection number with some weight by logarithm. The amazing thing is that, this really defines a good theory, in the sense that it satisfies many properties we know for the classical intersection theory.

2.1. **Arithmetic Chow group.**

We will start with a arithmetic scheme $\mathcal{M} \to \mathrm{Spec}\,\mathbb{Z}$.

Definition 2.1.1. Suppose $\mathcal{Z}$ is a Cartier divisor on $\mathcal{M}$. A *Green function* for $\mathcal{Z}$ is a smooth real-valued function Φ on the $\mathcal{M}(\mathbb{C}) \setminus \mathcal{Z}(\mathbb{C})$ satisfying

(i) If f is a meromorphic funtion on a holomorphic chart $U \to \mathcal{M}(\mathbb{C})$ satisfying

$$\mathrm{div}(f) = \mathcal{Z}(\mathbb{C})|_U,$$

then $\Phi|_U + 2\log|f|$, initially defined on $U \setminus \mathcal{Z}(\mathbb{C})|_U$, extends to a smooth function on U;

(ii) Pullback by complex conjugation on $\mathcal{M}(\mathbb{C})$ fixes Φ.

Definition 2.1.2. An *arithmetic divisor* on $\mathcal{M}$ is a pair $\widehat{\mathcal{Z}} = (\mathcal{Z}, \Phi)$ consisting of a Cartier divisor $\mathcal{Z}$ and a Green function Φ for $\mathcal{Z}$. An arithmetic divisor as above is principal if it has the form

$$\widehat{\mathcal{Z}} = (\mathrm{div}(f), -2\log|f|)$$

for some rational function f on $\mathcal{M}$.

The codimension one arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{M})$ is the quotient of the group of all arithmetic divisors on $\mathcal{M}$ by the subgroup of principal arithmetic divisors.

Note that the Cartier divisor could be really mysterious in the arithmetic setting. For instance, consider $\mathcal{M} = \mathbb{Z}[x, y]$ and a Cartier divisor given by $xy - N$. Over the generic fiber and those special fibers $p \nmid N$, this gives a hyperbola. When $p \mid N$, this separate into two lines. On the other hand, p is also a Cartier divisor, which is the whole special fiber at p. These two different classes are called vertical and horizontal divisors.

Now if we look back to the Borcherds product, we see that $\Phi(z, h, f)$ is exactly a Green function. Therefore, we can re-interpret Borcherds product in the following way:

Assuming there is a good theory of integral model of X_K, and a good definition for $\mathcal{Z}(f)$ in the integral model, then

$$\widehat{\mathcal{Z}}(f) = (\mathcal{Z}(f), \Phi(f)) \in \widehat{\mathrm{CH}}^1(\mathcal{X}_K).$$

Such theory does exists (when the level structure is big enough)! It's about the integral model of GSpin Shimura varieties. We won't do that here. It's another totally different and interesting worlds, and worth another semester's reading seminar.

I leave a comment about the reason why we want to compare the singularity with $-2\log|f|$, it's related to the Poincaré-Lelong formula: One of the key ingredient here is the following theorem

Theorem 2.1.3 (Poincaré-Lelong)**.** *Let $\widehat{L}$ be an hermitian line bundle on X and s a meromorphic section of L. Then we have the following formula in the* $(1,1)$*-distributed form*

$$dd^c(-\log||s||^2) + \delta_{\mathrm{div}(s)} = c_1(\widehat{L}_{\mathbb{C}}).$$

The last term also suggests that given a Green function, we can relate them with some line bundle, this idea will be realized in the next section.

Example 2.1.4. We provide here an example to illustrate why we want a Green function has the same singularity as $-\log|f|^2$.

Given a differential $(1,1)$-form ω. we define the distribution δ_Z:

$$\delta_Z(\omega) := \int_Z \omega.$$

If you are familiar with Poincaré duality, you can recognize this is exactly how we define the intersection pair in algebriac topology.

On the other hand, we consider the distribution

$$\log|f|^2(\omega) := \int_X \log|f|^2 \omega.$$

Now we considering the distribution $dd^c \log|s|^2$. For simplicity, let's consider $s = z$ and $\omega = z$. Then

$$\begin{aligned} dd^c \log|z|^2(z) &= \int_{\mathbb{C}} \log|z|^2 \cdot dd^c z, \\ &= \int_{\mathbb{C}} \log(z\bar{z}) \cdot dd^c z, \\ &= \int_{\gamma} d^c \log(z\bar{z}) dz, \\ &= \int_{\gamma} \frac{1}{\bar{z}} dz, \\ &= -2\pi. \end{aligned}$$

For general Cartier divisor f, we can do this computation locally, it turns out that using the integration by part, we can always identify these two distributions. For general rational sections in line bundles, we do similar things, except that when we to the trivializztion, we will get additional terms in the chern class.

2.2. **Automorphic vector bundle.** There is an equivalent way to view the codimension one arithmetic Chow group.

Definition 2.2.1. A *metrized line bundle* on $\mathcal{M}$ is a pair

$$\widehat{\mathcal{L}} = (\mathcal{L}, ||\cdot||)$$

consisting of a line bundle on $\mathcal{M}$, and a smoothly varying family of Hermitian metrics on its complex fiber. We further require the metrics to be invariant under pullback by complex conjugation on $\mathcal{M}(\mathbb{C})$.

If we denote by $\widehat{\mathrm{Pic}}(\mathcal{M})$ the group of metrized line bundles (under tensor product), there is a canonical isomorphism

$$\widehat{\mathrm{Pic}}(\mathcal{M}) \simeq \widehat{\mathcal{H}}^1(\mathcal{M})$$

sending a metrized line bundle $\widehat{\mathcal{L}}$ to the arithmetic divisor

$$\widehat{\mathrm{div}}(s) = (\mathrm{div}(s), -2\log||s||)$$

for any nonzero rational section s of $\mathcal{L}$. This is like a arithmetic version of relations between Weil divisors and line bundles.

A natural question arise now: we have been talking about metrized line bundle and sections on $\mathcal{M}$, but why we want this form. The key observation is that automorphic functions are exactly the rational sections on the Shimura varieties. Let's do the complex version here.

Let G be any connected semisimple Lie group, $K \subset G(\mathbb{R})$ a maximal compact subgroup, (ρ, W) a finite dimensional complex representation. $\Gamma \subset G(\mathbb{R})$ a torsion-free discrete arithmetic subgroup. Recall that the symmetric space is defined by $\mathbb{D} = G/K$. We define the homogeneous vector bundle

$$\mathcal{W} := \Gamma \backslash (G \times W)/K \to X_\Gamma = \Gamma \backslash X.$$

Here Γ acts trivially on W but acts on $G \times W$ on the right by $(g, w)k = (gk, \tau(k)^{-1}w)$.

Let's look at the global section. From the construction, the global section of $\mathcal{W}$ will correspond to functions $f : G(\mathbb{R}) \to W$ such that

$$f(\gamma g k) = \tau(k)^{-1} f(g).$$

This defines an automorphic form on $G(\mathbb{R})$. Similarly, given $K = K_\infty K_f$, and a representation $\rho : K_\infty \to \mathrm{GL}(W)$, we can define

$$\mathcal{W} = G(\mathbb{Q}) \backslash G(\mathbb{A}) \times W/(ZK \times K_f) \to \mathrm{Sh}_K(G, X).$$

And the rational sections are automorphic functions with level structure K. Such construction is compatible with the Borel compactification in a complicated sense, hence is non-trivially an algebraic vector bundle. Such construction can be generlize to the case when V admits a hermitian structure, and you won't be surprise if I told you this give you a hermitian vector bundles.

Example 2.2.2. Let's do the example in the modular curve.

Now let's consider the representation of $K = \mathrm{SO}_2 = S^1$:

$$\rho : S^1 \to \mathbb{C}^\times, \quad z \longmapsto z^k.$$

Then $\mathrm{SL}_2(\mathbb{R}) \times \mathbb{C}/K$ consists of collections (g, z) such that $(gk_\theta, z) \sim (g, e^{ik\theta} z)$. A section of $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) \times \mathbb{C}/\mathrm{SO}(2) \to \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ is a function

$$F : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R}) \times \mathbb{C}, \quad g \longmapsto (g, f(g)),$$

satisfy the compatibility you expect. To be more precise, first we have

$$(gk_\theta, F(gk_\theta)) = (g, e^{ik\theta} F(gk_\theta)) = (g, F(g)),$$

hence we have

$$F(g) = e^{ik\theta} F(gk_\theta).$$

And $\gamma \cdot (g, F(g)) = (\gamma g, F(g)) = (\gamma g, F(\gamma g))$, hence $F(\gamma g) = F(g)$. Therefore, our section F is equivalent to the function $f : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ such that

$$F(\gamma g) = f(g), f(gk_\theta) = e^{ik\theta} F(g).$$

This is exactly the automorphic forms with weight k. Recall that we recover our modular form if we multiple back the automorphic factor and mix the action of arithmetic subgroup and maximal compact subgroup:

$$f(\tau) := F(g_\tau) j(g_\tau, i)^k.$$

This also justify the name automorphic vector bundle rather than modular vector bundle.

Based on the computation here, it might be interesting to ask whether we can translate the Maass forms into automorphic vector bundles. I guess so except that the representation is no longer algebra and we can only define them on the Riemann surface and cannot descent it into Shimura varieties.

2.3. **Borcherds lift.** When f is harmonic weak Maass form, there exists so called the Borcherds lift that related to the regularized theta lift. Now rephrase into the following way: assuming there is a good theory of integral model $\mathcal{X}_K$ and special cycles $\mathcal{Z}_K$, then Borcherds lift give an element in the arithmetic Chow group:

$$\widehat{\mathcal{Z}}(f) = (\mathcal{Z}(f), \Phi(f)) = \widehat{\mathrm{div}}(\Psi),$$

where Ψ is the Borcherds lift, which is a rational section on hermitian line bundle $\omega^{\otimes \frac{1}{2}c(0,0)}$, where ω is the canoncial bundle. In particular, when $c(0,0) = 0$, which means the weakly holomorphic modular form does not have constant term, them $\widehat{\mathcal{Z}}(f)$ is a principal divisor in the arithmetic chow group of the integral model of the orthogonal Shimura varieties.

3. Main conjecture and consequences

3.1. **Faltings height pair.** Now we are ready to reformulate Bruinier-Yang's computation and estiblish the main conjecture.

Given X_K the orthogonal Shimura variety, with a nice integral model $\mathcal{X}_K$. When K is the stabilizer of a self-dual lattice, the level structure is hyperspecial and gives a good reduction. In general, the integral model does not admit a good reduction, but in those cases, the integral model is still normal and Cohen-Macaulay. There also have regular integral model with semi-stable reduction but is no longer canonical.

Such integral model has relative dimension n, hence absolute dimension $n+1$. We consider now $Z^n(\mathcal{X})$ the cycles of codimension n. We can define a height pairing

$$\widehat{\mathrm{CH}}^1(\mathcal{X}) \times Z^n(\mathcal{X}) \to \mathbb{R}.$$

When $\widehat{x} = (x, g_x) \in \widehat{\mathrm{CH}}^1(\mathcal{X})$ and $y \in Z^n(\mathcal{X})$ such that x and y intersect properly, then in particular it has dimension 0, then either they lies in the finite place or the generic fiber. The intersection multiplicity is defined by

$$\langle \widehat{x}, y \rangle_{\mathrm{Fal}} = \langle x, y \rangle_{\mathrm{fin}} + \langle \widehat{x}, y \rangle_\infty,$$

where

$$\langle \widehat{x}, y \rangle_\infty = \frac{1}{2} g_x(y(\mathbb{C})),$$

and $\langle x, y\rangle_{\text{fin}}$ denotes the intersection pairing at the finite places. It's not the naive intersection pairing, but we will count them with multiplicities. To be more precise, it can be written of the form

$$\sum_p \sum_{t \in \mathcal{Z}(\mathbb{F}_p^{\text{alg}})} \text{mult}(t) \cdot \log(p).$$

In general, there is a arithmetic version of rational equivalence to help computing the intersection pair.

3.2. **Main conjecture.** Note that in our case, we have

$$\langle \widehat{\mathcal{Z}}(f), \mathcal{Z}(U)\rangle_\infty = \frac{1}{2}\Phi(Z(U), f).$$

Therefore, we can view Bruinier-Yang's computation as a computation on the archimedean place of the Faltings height.

Now assuming f is a weakly holomorphic modular form, so that $\xi(f) = 0$, and furthermore assume it has constant term $c(0,0) = 0$. Then is corresponds to a rational sections on the trivial line bundle on $\mathcal{X}_K$. In particular, $\widehat{\mathcal{Z}}(f) = \widehat{\text{div}}(\Phi(f))$ is rational equivalent to 0. Therefore, we expect the Faltings intersection height equals 0:

$$0 = \langle \widehat{\mathcal{Z}}(f), \mathcal{Z}(U)\rangle_{\text{Fal}} = \langle \mathcal{Z}(f), \mathcal{Z}(U)\rangle_{\text{fin}} + \frac{1}{2}\Phi(Z(U), f).$$

Therefore, we get

$$\langle \mathcal{Z}(f), \mathcal{Z}(U)\rangle_{\text{fin}} = -\frac{\deg(\mathcal{Z}(U))}{2}\text{constant}(\langle f^+(\tau), \theta_P(\tau) \otimes \mathcal{E}_N(\tau)\rangle).$$

Now LHS eauals

$$\sum_{\mu \in L^\sharp/L} \sum_{m>0} c(-m, \mu)\langle \mathcal{Z}(m, \mu), \mathcal{Z}(U)\rangle_{\text{fin}}.$$

with $c(-m, \mu)$ the singular coefficients of the weakly holomorphic modular form f. Next we look at the analytic side. Note that θ_P and $\mathcal{E}_N$, note that these two functions are defined by the lattice L, hence are two constant functions when f and c changes! We simply denote

$$\theta_P(\tau) \otimes \mathcal{E}_N(\tau) := \sum_{\mu \in L^\sharp/L} \sum_{m>>0} \nu(m, \mu) q^n \phi_\mu.$$

If we denote

$$f^+(\tau) = \sum_{\mu \in L^\sharp/L} \sum_n c^+(n, \mu) q^n \phi_\mu.$$

Then the right hand side equals

$$\sum_{\mu \in L^\sharp/L} \sum_{m>0} c^+(-m, \mu)\nu(m, \mu).$$

Now we can formulate Bruinier-Yang's conjecture:

Conjecture 3.2.1. $\langle \mathcal{Z}(m, \mu), \mathcal{Z}(U)\rangle_{\mathit{fin}} = -\frac{\deg(Z(U))}{2}\nu(m, \mu)$.

Using this, we can impose the global conjuecture on the Faltings height:

Conjecture 3.2.2. *For any $f \in H_{1-n/2}, \bar{\rho_L}$, one has*

$$\langle \widehat{\mathcal{Z}}(f), \mathcal{Z}(U) \rangle_{Fal} = \frac{\deg(Z(U))}{2}(c^+(0,0)\kappa(0,0) + L'(\xi(f), U, 0)).$$

Where $\kappa(0,0)$ is the constant term of the $\mathcal{E}_N$.

Since we can choose f such that $\mathcal{Z}(f) = \mathcal{Z}(m, \mu)$ for a specific m and μ, these two conjectures are in fact equivalent.

The amazing thing is that, we didn't do anything on the finite place, we didn't even define the possible integral model, but we can predict the intersection number from the archimedean place!

Also note that when we taking $V = M_2^{\mathrm{tr}=0}$ with the det as our quadratic form, then the orthogonal Shimura variety becomes modular curve. In this case, special cycle and CM cycles are both Heegner divisors. The inresection pair in the conjecure is exactly the arithmetic intersection of the Heegner points. This is how this conjecture relates to the Gross-Zagier formula.

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