HARMONIC ANALYSIS AND DISCONTINUOUS GROUPS IN WEAKLY SYMMETRIC RIEMANNIAN SPACES WITH APPLICATIONS TO DIRICHLET SERIES

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IN THE following lectures we shall give a brief sketch of some representative parts of certain investigations that have been undertaken during the last five years. The center of these investigations is a general relation which can be considered as a generalization of the so-called Poisson summation formula (in one or more dimensions). This relation we here refer to as the "trace-formula."

1. Let S be a Riemannian space, whose points we denote by x and the (local) coordinates by x^1, $x^2, \ldots,$ x^n, with a positive definite metric

$$ds^2 = \sum g_{ij}\, dx^i\, dx^j.$$

We shall assume the g_{ij} to be analytic in the coordinates. Further we assume that we have a locally compact group G of isometries of S (not necessarily the full group of isometries), whose elements we denote by m, and that G acts transitively on S so that given x and y in S, there exists an $m \in G$ such that $x = my$. We shall be concerned with the linear operators on functions $f(x)$ defined on S, which have the property that the operators are invariant under G, or otherwise expressed, linear operators that commute with the isometries m in G. We restrict ourselves here to the class of linear operators that are differential operators of finite order, integral operators of the form $\int\limits_S k(x, y)\, f(y)\, dy$ (where dy denotes the invariant element of volume derived from the metric), or any

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finite combination (by addition or multiplication) of such. This class evidently forms a ring.

Turning first to the integral operators, one observes that in order that the operator

$$\int_S k(x, y) f(y)\, dy$$

should be invariant, it is necessary and sufficient that the kernel satisfy the relation

$$k(mx, my) = k(x, y), \tag{1.1}$$

for all x, y in S and all m in G. We shall refer to such a kernel as a "point-pair invariant". If we consider such a "point-pair invariant" $k(x, y)$ as a function of one of the arguments, say x, keeping the other point y fixed, we see that $k(x, y)$ is invariant under the subgroup of G that leaves y fixed. This subgroup we denote by Ry and call it the rotation group of y. We express this property of k by saying that it has as a function of x rotational symmetry around the point y. Let x_0 be a chosen fixed point in S and R^0 with elements r^0 the rotation group of x_0. R^0 is isomorphic to a compact (or possibly finite) subgroup of the orthogonal group of n elements. Norming the bi-invariant Haar measure on R^0 so that $\int_{R^0} dr^0 = 1$, we can define for a function $f(x)$ a symmetrized function

$$f(x\,;x_0) = \int_{R^0} f(r^0 x)\, dr^0\,; \tag{1.2}$$

$f(x\,;x_0)$ clearly has rotational symmetry around the point x_0. Furthermore, if we have a function $f(x\,;x_0)$ with rotational symmetry around x_0, we can define a point-pair invariant $k(x, y)$ by the relation

$$k(x, y) = f(mx\,;x_0), \quad \text{where} \quad my = x_0,$$

this definition is seen not to depend on the particular choice of m if there is more than one m satisfying the relation $my = x_0$. Therefore the study of point pair invariants is equivalent to the study of functions with rotational symmetry around some point x_0.

We observe also the following facts, before turning to the consideration of differential operators. Because G acts transitively on S, an invariant operator, say L, of our class is completely characterized by its action at one point, say x_0. By this we mean that, introducing the notation $[Lf(x)]_{x=x_0}$ to denote the value of the function $Lf(x)$ at the point $x = x_0$, we can for an arbitrary point x_1 express $[Lf(x)]_{x=x_1}$ by means of the relation

$$[L f(x)]_{x=x_1} = [L f(mx)]_{x=x_0},$$

where m is a solution of $mx_0 = x_1$. Conversely if we have an operator $\mathscr{L}$ (not necessarily invariant), we can from its action at x_0, construct an invariant operator L by the relation

$$[L f(x)]_{x=x_0} = [\mathscr{L} f(x)]_{x=x_0},$$

provided

$$[\mathscr{L} f(r^0 x)]_{x=x_0} = [\mathscr{L} f(x)]_{x=x_0},$$

for every element r^0 in the rotation group R^0 of x_0. Finally if $\mathscr{L}$ does not have this property we may define

$$[L f(x)]_{x=x_0} = [\mathscr{L} f(x\,;x_0)]_{x=x_0},$$

where $f(x\,;x_0)$ is the symmetrized function of f around x_0 defined by (1.2), because $f(r^0x)$ and $f(x)$ have the same symmetrized function around x_0. If $\mathscr{L}$ is invariant then $L = \mathscr{L}$.

Furthermore, one observes that an invariant operator applied to a function with rotational symmetry around a point, gives a function which again is rotationally symmetric around the same point. Also an invariant operator applied to a point-pair invariant as a function of say the first point, gives as result again a point-pair invariant.

Consider now the class of invariant differential operators of finite order, and let for simplicity the local co-ordinates around x_0 be chosen such that the matrix (g_{ij}) at $x = x_0$ is the identity matrix E_n. Let D be an invariant differential operator, its action at the point $x = x_0$ is identical to that of a differential operator $D^{(0)}$ with constant coefficients,

$$D^{(0)} = \sum a_{i_1, i_2, \dots, i_n} \left(\frac{\partial}{\partial x^1}\right)^{i_1} \left(\frac{\partial}{\partial x^2}\right)^{i_2} \cdots \left(\frac{\partial}{\partial x^n}\right)^{i_n}.$$

By the highest homogeneous part of $D^{(0)}$ we mean the aggregate of terms in the above sum, where $a_{i_1 \dots i_n} \neq 0$ and $i_1 + i_2 + \dots + i_n$ attains its maximal value; we denote this by $\overline{D}^{(0)}$, and write

$$\overline{D}^{(0)} = p_D \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n}\right),$$

where p_D is a homogeneous polynomial. The rotation group R^0 induces on the tangent space of S at x_0 a subgroup R of the orthogonal group $\mathcal{O}_n$, and the polynomial $p_D(u_1, u_2, \dots, u_n)$ is seen to be invariant under this group R of orthogonal transformations. Conversely, if we have a homogeneous polynomial $p(u_1 \dots u_n)$ which is invariant under the group R, we may define an invariant differential operator D_p by the relation

$$[D_p f(x)]_{x=x_0} = \left[p\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n}\right) f(x; x_0) \right]_{x=x_0}.$$

It should be observed that whereas $p_{(D_p)} = p$, all one can say about $D_{(p_D)} - D$ is that it is an invariant operator of lower order than D. One also easily shows that if p_1 and p_2 are two such homogeneous polynomials invariant under R, we have that $D_{p_1 p_2} - D_{p_1} D_{p_2}$ is an operator of lower order than $D_{p_1 p_2}$. Using these facts, and a well-known result by Hilbert which says that the polynomials p have a finite basis of homogeneous polynomials $p_1, p_2, \dots, p_l$, $1 \leqslant l$, such that every homogeneous polynomial p can be written as a polynomial (not necessarily in a unique way) of $p_1, p_2, \dots, p_l$, with constant coefficients, one obtains the result that $D_{p_1}, D_{p_2}, \dots, D_{p_l}$ generate the ring of the invariant differential operators in the sense that any invariant differential operator D can be written as a finite expression

$$D = \sum A_{\nu_1, \nu_2, \dots, \nu l} \, D_{p_1}^{\nu_1} D_{p_2}^{\nu_2} \dots D_{pl}^{\nu l}, \tag{1.3}$$

where the A's are constants. Writing $D_{p_i} = D_i$ for $i = 1, 2, \dots, l$, we shall call $D_1, D_2, \dots, D_l$ a set of fundamental operators and we may assume that it is so chosen that l is minimal.

The fundamental operators in general do not commute, and as commutativity is essential for our later considerations, we shall make an additional assumption about G and S, which will imply commutativity (as we do not know, however, whether this assumption is necessary for commutativity, we should note that it is only the commutativity that is really necessary for the following developments).

We assume that there is a fixed isometry μ of S (possibly not in G), such that $\mu G \mu^{-1} = G$, $\mu^2 \in G$, and that for any pair of points x and y in S, there exists an m in G for which $mx = \mu y$ and $my = \mu x$. We may call a space for which there is some group of isometries G with these properties (if that is the case then the full group of all isometries will have these properties too) a " weakly symmetric " Riemannian space. This concept is more general than E. Cartan's concept of a symmetric space, as symmetric implies weakly symmetric, whereas it can be shown by examples that weakly symmetric does not imply symmetric.

Under this assumption we can prove that all the invariant operators commute. We first show that they commute when applied to point-pair invariants $k(x, y)$ considered as functions of the first point x. We first notice that if L is an invariant operator then so is also $\widetilde{L}$ defined by

$$\widetilde{L} f(x) = [L f(\mu^{-1} x)\,]_{x \to \mu x}.$$

Also from our assumption about G follows that for any point-pair invariant $k(x, y)$ we have

$$k(\mu y, \mu x) = k(mx, my) = k(x, y).$$

Denoting by a subscript the argument (x or y) that the operator is to act on, we have

$$L_x k(x, y) = k'(x, y),$$

where $k'(x, y)$ again is a point-pair invariant. Now we have

$$\begin{aligned}\widetilde{L}_y k(x, y) &= \widetilde{L}_y k(\mu y, \mu x) \\ &= [L_y k(y, \mu x)]_{y \to \mu y} = [k'(y, \mu x)]_{y \to \mu y} \\ &= k'(\mu y, \mu x) = k'(x, y).\end{aligned}$$

Thus

$$L_x k(x, y) = \widetilde{L}_y\, k(x, y),$$

so that we may shift the operator from the first to the second argument by replacing it with $\widetilde{L}$. If we now have two operators $L^{(1)}$ and $L^{(2)}$ we may write

$$\begin{aligned}L_x^{(1)} L_x^{(2)} k(x, y) &= L_x^{(1)} \widetilde{L}_y^{(2)}\, k(x, y) \\ &= \widetilde{L}_y^{(2)} L_x^{(1)} k(x, y) = L_x^{(2)} L_x^{(1)}\, k(x, y),\end{aligned}$$

(since the operators clearly may be interchanged when they act on different arguments). Thus we have commutativity when our operators are applied to point-pair invariants. Therefore we have also commutativity if our operators are applied to a function with rotational symmetry around a point, say x_0. For a function without rotational symmetry we notice that

$$[L^{(1)} L^{(2)} f(x)]_{x=x_0} = [L^{(1)} L^{(2)} f(x; x_0)]_{x=x_0},$$

where $f(x ; x_0)$ is the function with rotational symmetry defined by (1.2). From this follows

$$[L^{(1)} L^{(2)} f(x)]_{x=x_0} = [L^{(2)} L^{(1)} f(x)]_{x=x_0},$$

or what is the same

$$L^{(1)} L^{(2)} f(x) = L^{(2)} L^{(1)} f(x),$$

that is, the operators commute.

It can be shown that the operator $\bar{\tilde{L}}$ where the bar denotes conjugation, is the formal adjoint of the operator L.

Returning to (1.3) we may now write

$$D = P\ (D_1, D_2, \ldots, D_l), \tag{1.4}$$

where P is a polynomial with constant coefficients. It should be noted that though our fundamental operators were chosen so that l was minimal, there may sometimes still be algebraic relations between them, so that the representation (1.4) may not necessarily be unique. Further it can be shown that one can always choose a set of fundamental operators with minimal l, such that each of them is self-adjoint.

Now let $f(x)$ be a function which is an eigenfunction of all our fundamental operators D_i so that

$$D_i\, f(x) = \lambda_i\, f(x), \quad i = 1, 2, \ldots, l, \tag{1.5}$$

where the λ_i are constants; because of (1.4) it will then be an eigenfunction of all the invariant differential operators, and in particular of the Laplace operator derived from the metric, therefore $f(x)$ will be analytic in the coordinates. If we take a point x_0 such that $f(x_0) \neq 0$, and form $f(x\,; x_0)$ defined by (1.2), this will again satisfy the equations (1.5) and will not vanish identically in x since $f(x_0\,; x_0) = f(x_0) \neq 0$. We now write

$$f(x\,; x_0) = f(x_0)\ \omega_\lambda(x, x_0), \tag{1.6}$$

where the subscript λ is an abbreviation for the l-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ so that $\omega_\lambda(x_0, x_0) = 1$. We call this the " normed " eigenfunction with rotational symmetry around x_0, and shall show that it is unique, that is to say a function with rotational symmetry around x_0 which takes the value 1 at the point x_0 and which satisfies the equations (1.5) is identical with $\omega_\lambda(x, x_0)$. To prove this we observe that for such a function $g(x)$, we have, because $g(x) = g(x\,; x_0)$,

$$\left[\left(\frac{\partial}{\partial x^1}\right)^{\nu_1}\left(\frac{\partial}{\partial x^2}\right)^{\nu_2}\cdots\left(\frac{\partial}{\partial x^n}\right)^{\nu_n} g(x)\right]_{x=x_0}$$

$$=\left[\left(\frac{\partial}{\partial x^1}\right)^{\nu_1}\left(\frac{\partial}{\partial x^2}\right)^{\nu_2}\cdots\left(\frac{\partial}{\partial x^n}\right)^{\nu_n} g(x\,;\,x_0)\right]_{x=x_0} = [D\, g(x)]_{x=x_0},$$

where D is an invariant differential operator depending only on $(\nu_1, \nu_2, \ldots, \nu_n)$. Because of (1.4) and since $g(x)$ satisfies the equations (1.5) we thus get on using $g(x_0) = 1$,

$$\left[\left(\frac{\partial}{\partial x^1}\right)^{\nu_1}\left(\frac{\partial}{\partial x^2}\right)^{\nu_2}\cdots\left(\frac{\partial}{\partial x^n}\right)^{\nu_n} g(x)\right]_{x=x_0} = P(\lambda_1, \lambda_2, \ldots, \lambda_l),$$

where P is a polynomial depending only on $(\nu_1, \nu_2, \ldots, \nu_n)$. This shows that all the partial derivatives of $g(x)$ at the point x_0 are uniquely determined by the l-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ and so since $g(x)$ is analytic in the coordinates, $g(x)$ is unique, that is, it coincides with $\omega_\lambda(x, x_0)$. We may from $\omega_\lambda\,(x, x_0)$ construct the point-pair invariant $\omega_\lambda(x, y)$ which will, because of the relation

$$D_x\, \omega_\lambda(x, y) = \widetilde{D}_y\, \omega_\lambda\,(x, y),$$

be a normed eigenfunction also in y with rotational symmetry around the point x. Therefore we must have

$$\omega_\lambda(x, y) = \omega_{\widetilde{\lambda}}\,(y, x), \tag{1.7}$$

where $\widetilde{\lambda}$ denotes an l-tuple $(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_l)$ not necessarily identical to the original one. $\omega_\lambda(x, y)$ is now easily seen to be an eigenfunction (considered as a function of x) of our whole class of invariant operators for the reason that

$$L_x\, \omega_\lambda(x, y)$$

because of the commutativity of L and the D_i, $i = 1, 2, \ldots, l$, again satisfies the equations (1.5), and furthermore it is again a function with rotational symmetry around y, and differs therefore only by a factor independent of x (and hence since the factor is a point pair invariant it is independent of y also) from $\omega_\lambda(x, y)$, that is to say

$$L_x\, \omega_\lambda(x, y) = \Lambda\, \omega_\lambda(x, y),$$

where Λ is a constant depending on L and the l-tuple λ only.

We can now show that any function which satisfies the equations (1.5) will be an eigenfunction of our class of invariant operators, namely we have

$$\begin{aligned}[L\, f(x)]_{x=x_0} &= [L\, f(x\,;\, x_0)]_{x=x_0} \\ &= [L\, f(x_0)\, \omega_\lambda\, (x, x_0)]_{x=x_0} \\ &= \Lambda\, f(x_0)\, \omega_\lambda\, (x_0, x_0) = \Lambda\, f(x_0).\end{aligned}$$

Since this holds for any point x_0, we have

$$L\, f(x) = \Lambda\, f(x),$$

and we see that the eigenvalue Λ does depend only on L and the l-tuple λ, but not on the particular function $f(x)$.

Thus for an integral operator we may write

$$\int_S k(x, y)\, f(y)\, dy = h(\lambda)\, f(x), \tag{1.8}$$

where $h(\lambda) = h(\lambda_1, \lambda_2, \ldots, \lambda_l)$ depends on k and λ only. In order to get an expression for $h(\lambda)$ it is therefore enough to produce a "representative" set of eigenfunctions, that is, one that exhausts all the possibilities for the l-tuple $(\lambda_1, \ldots, \lambda_l)$, that is, l-tuples for which there really do exist functions satisfying the equations (1.5).

In a number of cases that are of particular interest for applications, such a set can be obtained from the following lemma:

Let T with elements t be a subgroup of G which is simply transitive on S, that is, such that the equation $x = tx_0$, where x is any point in S and x_0 a chosen fixed point, always has one and only one group element t as a solution. Further suppose that we have a continuous non-vanishing function $\phi(t)$ on T that satisfies the relation

$$\phi(t_1 t_2) = \phi(t_1)\,\phi(t_2),$$

for all t_1 and t_2 in T. If we now define $f(x) = f(tx_0) = \phi(t)$, where $tx_0 = x$, then $f(x)$ is an eigenfunction of our operators, because

$$L\,f(x) = [L\,f(tx)]_{x=x_0} = \phi(t)\,[L\,f(x)]_{x=x_0} = [L\,f(x)]_{x=x_0}\,f(x).$$

If we have several such multiplicatively independent functions $\phi_1(t), \phi_2(t), \ldots, \phi_\kappa(t)$, then

$$\phi_1^{s_1}(t)\,\phi_2^{s_2}(t)\,\ldots\,\phi_\kappa^{s_\kappa}(t)$$

will also be one (where, if T or what is the same S, is not simply connected the exponents $s_1, s_2, \ldots, s_\kappa$ have to be chosen such that the resulting function is single-valued). It is of course not always so that different choices of the κ-tuple $s_1, \ldots, s_\kappa$ necessarily lead to different l-tuples $\lambda_1, \ldots, \lambda_l$. In many cases one gets all possibilities for which eigenfunctions exist covered by this construction.

The nature of the set of possible λ's may differ from the completely discrete set that would occur if S is compact,† to the situation for many non-compact spaces where the set of all l-tuples of complex numbers $\lambda_1, \ldots, \lambda_l$, which satisfy the possible algebraic relations between the D_i, $i = 1, 2, \ldots, l$, does occur. Intermediary situations can of course also occur. In the case when the set of all l-tuples λ of complex numbers satisfying the algebraic relations between the D_i's does occur, it is easily shown that $\omega_\lambda(x, y)$ as a function of λ is an analytic function on the algebraic variety defined by these relations, which is regular whenever all λ_i's are finite.

As an illustration we may for instance consider the space of n by n positive definite symmetric matrices $Y = (y_{ij})$ with the metric

$$ds^2 = \sigma\,(Y^{-1}\,dY\,Y^{-1}\,dY),‡$$

† Because we require our functions to be regular globally, if one admits " local " eigenfunctions (that cannot be continued everywhere in S, or that by such continuation would not be single-valued) the situation is different as shown by the examples of the surface of a sphere or the periphery of a circle.

‡ σ here and in the following denotes the trace.

where $dY = (dy_{ij})$, and the group G may be taken as the group of all non-singular real n by n matrices A, the isometries being

$$Y \to A\,Y\,A'$$

(A' is the transposed of A); finally the isometry μ may be taken as

$$Y \to Y^{-1}.$$

It is then easily established that all our requirements are satisfied. The point-pair invariants are easily seen to be of the form that $k(Y_1, Y_2)$ is a symmetric function of the n eigenvalues of the matrix $Y_2\,Y_1^{-1}$, or if one prefers it, $k(Y_1, Y_2)$ is a function of the n arguments $\sigma(\,(Y_2\;Y_1^{-1})^\nu)$, $\nu = 1, 2, \ldots, n$. Conversely any such function is a point-pair invariant.

A set of fundamental operators can be obtained as follows: let $\dfrac{\partial}{\partial Y}$ denote the matrix $\left(\dfrac{1+\delta_{i,j}}{2}\,\dfrac{\partial}{\partial\,y_{ij}}\right)$, where $\delta_{i,j}$ is the Kronecker symbol; then the operators

$$D_i = \sigma\left(\left(Y\frac{\partial}{\partial Y}\right)^i\right), \quad i = 1, 2, \ldots, n \tag{1.9}$$

are a set of fundamental operators, and they are algebraically independent.

To obtain a representative set of eigenfunctions, consider the subgroup of G formed by the "triangular" matrices $T = (t_{ij})$ with $t_{ij} = 0$ for $i < j$ and $t_{ii} > 0$ for $i = 1, 2, \ldots, n$. This group acts simply transitive on our space, and for any complex n-tuple $s = (s_1, s_2, \ldots, s_n)$ the function

$$\phi_s(T) = \prod_{i=1}^{n} t_{ii}{}^{2s_i + i - (n+1)/2} \tag{1.10}$$

is single-valued and continuous on this group and has the property

$$\phi_s(T_1)\,\phi_s(T_2) = \phi_s\,(T_1\,T_2).$$

Thus defining for $Y = TT'$

$$f_s(Y) = \phi_s(T), \tag{1.11}$$

this is an eigenfunction. One can show that

$$D_i\, f_s(Y) = \lambda_i\,(s)\, f_s(y),$$

where $\lambda_i(s) = \lambda_i(s_1, s_2, \ldots, s_n)$ is a polynomial in the s_j of degree i which is symmetric in the s_j and of the form

$$\lambda_i(s_1, s_2, \ldots, s_n) = s_1{}^i + s_2{}^i + \ldots + s_n{}^i + \text{terms of lower degree.}$$

From this one sees that λ_i are a basis for the symmetric polynomials of the s_j, so that the s_j are determined as roots of an algebraic equation of nth degree whose coefficients are rational in the λ_i, so they are determined up to a permutation of the s_j. From this it follows that we may by suitable choice of the s_j make the λ_i any n-tuple of finite complex numbers. One also can show that

$$\widetilde{\lambda}_i(s_1, s_2, \ldots, s_n) = \lambda_i(-s_1, -s_2, \ldots, -s_n).$$

To find an expression for the $h(\lambda)$ defined in (1.8),

$$\int_S k(Y_1, Y)\, f_s(Y)\, dY = h(\lambda)\, f_s(Y_1),$$

where dY is the invariant element of volume

$$dY = \frac{2^{\frac{1}{2}n(n-1)}}{|Y|^{\frac{1}{2}(n+1)}} \prod_{i \leqslant j} dy_{ij},$$

we may write

$$k(Y_1, Y) = k(\sigma\,(YY_1{}^{-1}), \sigma(\,(YY_1{}^{-1})^2), \ldots, \sigma(\,(YY_1{}^{-1})^n\,)\,),$$

and take $Y_1 = E_n$ the identity matrix so that $f_s(Y_1) = 1$; further we may introduce the t_{ij}, $i \geqslant j$, in $Y = TT'$ as new coordinates in our space, the element of volume then becomes $\dfrac{2^{n(n+3)/4}}{t_{11}\, t_{22}^2 \cdots t_{nn}^n} \prod_{i \geqslant j} dt_{ij}$ and the relation becomes

$$2^{\frac{1}{2}n(n+3)} \int k(\, \sigma(TT'), \sigma(\, (\, TT')^2), \ldots, \sigma(\, (TT')^n)\,)\, \times$$

$$\times \prod_{i=1}^{n} t_{ii}^{\,2s_i - \frac{1}{2}(n+1)} \prod_{i \geqslant j} dt_{ij} = h(\lambda),$$

where the integration is carried from 0 to ∞ over the t_{ii} and from $-\infty$ to ∞ over the t_{ij} with $i > j$. For special forms of k these integrations can be carried out explicitly, for instance if

$$k(Y_1, Y) = |\, YY_1^{-1}\,|^{\alpha}\, e^{-\beta\sigma(YY_1^{-1})},$$

where the real part of β is positive, and the real part of $2s_i + 2\alpha > \frac{1}{2}(n-1)$ for $i = 1, 2, \ldots, n$; the integral then becomes

$$2^{\frac{1}{2}n(n+3)} \int \left\{ \exp\left(-\beta \sum_{i \geqslant j} t_{ij}^{\,2} \right) \right\} \prod_{i=1}^{n} t_{ii}^{\,2s_i + 2\alpha - (n+1)/2} \prod_{i \geqslant j} dt_{ij},$$

and splits into a product of $\dfrac{n(n+1)}{2}$ simple integrals, each of which is expressible in terms of Gamma functions.[§]

2. Let now Γ be a discrete subgroup of G which acts properly discontinuous on the space S, and let there be given a representation of Γ by unitary ν by ν matrices $\chi(M)$, where we denote the elements of Γ by M. Consider function vectors $F(x)$, that are column vectors, whose ν components are scalar functions of the point x, and which furthermore satisfy the relation

$$F(M\,x) = \chi(M)\, F(x), \tag{2.1}$$

for all x in S and M in Γ. Such a function $F(x)$ is then of course fully determined by its values on a fundamental domain $\mathscr{D}$ of Γ in S. Applying one of the invariant integral operators to such a function $F(x)$ one sees that

$$\int_S k(x, y)\, F(y)\, dy = \int_{\mathscr{D}} K(x, y; \chi)\; F(y)\, dy,$$

§ For this special choice of k, the resulting form of formula (1.8) has in the meantime been derived by different means by H. Maass, *Journal of the Indian Math. Soc.* 19 (1955), 1-24.

where the kernel K is a matrix given by

$$K(x, y; \chi) = \sum_{M \in \Gamma} \chi(M)\, k(x, My). \tag{2.2}$$

Considering now the Hilbert space defined by the inner product

$$(F_1, F_2) = \int_{\mathscr{D}} \bar{F}_1'(x)\, F_2(x)\, dx,$$

where $\bar{F}_1'$ is the conjugate transposed of F_1, one sees easily that the operator

$$\int_{\mathscr{D}} K(x, y; \chi)\, F(y)\, dy \tag{2.3}$$

is normal since the adjoint operator has a kernel that is derived from the right-hand side of (2.2) by replacing $k(x, y)$ by $\bar{\tilde{k}}(x, y) = \overline{k(y, x)}$, and thus it commutes with the operator (2.3). The invariant differential operators are also seen to be normal.

We have not up to now put any restrictions on our point-pair invariants $k(x, y)$, but always only assumed that the kernel and the function that the operator acted on were such that the integral also existed if absolute values were taken of the integrands.

It is now time to impose conditions that will enable us to make definite statements about the absolute convergence of the series on the right-hand side of (2.2) and also about the behavior of $K(x, y; \chi)$.

We make the following assumptions :

$k(x, y)$ should have a majorant,

$k_1(x, y)$ such that (a) $\int_S k_1(x, y)\, dy < \infty$, (b) $k_1(x, y)$ is of regular growth; that is to say, there should exist positive constants δ and A such that for all x and y,†

† One can relax this, and permit kernels with, for instance, a singularity at $x = y$ by requiring (b′) to be fulfilled only if the smallest geodesic distance $d(x, y)$ exceeds some fixed number.

$$k_1(x, y) \leqslant A \int\limits_{d(y,y')<\delta} k_1(x, y')\, dy', \tag{b'}$$

where $d(y, y')$ denotes the smallest geodesic distance between y and y'. Under these assumptions the above series for $K(x, y; \chi)$ converges absolutely for x and y in S, and uniformly if x and y are in some compact subregion of S.‡

We also make the assumption that the fundamental domain $\mathscr{D}$ of Γ in S is compact. Then $K(x, y\,; \chi)$ will be uniformly bounded for x and y in $\mathscr{D}$ (and therefore also for all x and y in S). Therefore also the expression

$$\int\limits_{\mathscr{D}}\int\limits_{\mathscr{D}} \sigma\,(K(x, y\,; \chi)\, \overline{K(x, y\,; \chi)}')\, dx\, dy$$

is finite ($\overline{K}'$ denotes the conjugate transposed of the matrix K) so that the integral operator is of the Hilbert-Schmidt class, and the classical methods from the theory of integral equations can be applied.

Consider now the functions $F(x)$ satisfying (2.1) which are eigenfunctions of our fundamental operators D_i for $i = 1, 2, \ldots, l$. We can then show from the preceding results about our integral and differential operators, that there exist an orthonormal system of eigenfunctions $F_i(x)$, which is complete in our Hilbert space, and such that if we write

$$D_j\, F_i(x) = \lambda_j^i\, F_i(x) \tag{2.4}$$

for $j = 1, 2, \ldots, l$; the l-tuples $\lambda^i = (\lambda_1^i, \lambda_2^i, \ldots, \lambda_l^i)$ have no finite point of accumulation in l-dimensional space. The completeness in particular follows from the easily established fact that the system of all admissible kernels $K(x, y; \chi)$ is complete.

About the eigenvalues, the l-tuples λ^i, one could at once make statements based upon the fact that if the kernel of an integral

‡ Thus in particular $K\,(x, y\,; \chi)$ is continuous if $k\,(x, y)$ is.

operator is Hermitian (which is the case if $k(x, y) = \overline{k(y, x)}$), the eigenvalues $h(\lambda^i)$ must be real ; also, by looking at the differential operators, if we have chosen the fundamental operators self-adjoint, as one always can, the λ_j^i for $j = 1, 2, \ldots, l$ have to be real, and for the elliptic ones the sign of the eigenvalue is also given. In terms of the corresponding normed rotationally symmetric eigenfunctions, it follows that $\omega_{\lambda^i}(x, y) = \overline{\omega_{\lambda^i}(y, x)}$, and $|\omega_{\lambda^i}(x, y)| \leqslant 1$ for all x and y in S.

Formally we have the expansion of $K(x, y\,; \chi)$ in terms of the eigenfunctions F_i,

$$\sum_{M \in \Gamma} \chi(M)\, k(x, My) = \sum_i h(\lambda^i)\, F_i(x)\, \bar{F}_i{}'(y).^{\S} \tag{2.5}$$

The absolute convergence of the right-hand side and the equality of the two sides could be proved under suitable additional assumptions about $k(x, y)$. However, since the eigenfunctions themselves occur in (2.5), our attention here will instead centre on the trace of the integral operators, where the eigenfunctions do not anymore occur.

We may formally compute the trace of the integral operator in two ways, namely on the one hand as

$$\sum_i h(\lambda^i), \tag{2.6}$$

and on the other hand as

$$\int_{\mathscr{D}} \sigma(K(x, x\,; \chi))\, dx = \sum_{M \in \Gamma} \sigma(\chi(M)) \int_{\mathscr{D}} k(x, Mx)\, dx. \tag{2.7}$$

We leave aside for the moment the question whether the series (2.6) is convergent or only summable in some sense and also the

§ This formula in the case $\chi(M)$ identically 1, can be used for estimation of the number of points Mx in large regions with rotational symmetry about the point y.

question whether the sum is actually equal to the expression (2.7), and turn our attention first to the latter expression. Under our assumption on $\mathscr{D}$ and $k(x, y)$ the series on the right-hand side of (2.7) actually is absolutely convergent, even when we take absolute values under the integral signs.

We shall rearrange the series on the right-hand side of (2.7) by combining the terms in a suitable way. For this purpose we introduce some notations.

Two elements M_1 and M_2 in Γ are said to be conjugate within Γ if there exists $M_3 \in \Gamma$ such that $M_1 = M_3 M_2 M_3^{-1}$; we call the class of all elements in Γ which are conjugate to a given M the conjugate class of M in Γ, and denote it by the symbol $\{M\}_\Gamma$. The subgroup of Γ formed by the elements which commute with M we call Γ_M and denote its elements by N_M. Similarly we define conjugacy within G, and denote by $\{m\}_G$ the class of all elements in G conjugate to an m in G. Clearly $\{M\}_\Gamma$ is contained in $\{M\}_G$. Also the subgroup of G formed by the elements of G which commute with m we call G_m and denote its elements by n_m; clearly Γ_M is contained in G_M.

We now group together the terms on the right-hand side of (2.7), where M belongs to the same conjugacy class in Γ. The factor $\sigma(\chi(M))$ has the same value for all elements M belonging to the same conjugacy class in Γ. Therefore we consider the sum

$$\sum_{M \in \{M_0\}_\Gamma} \int_{\mathscr{D}} k(x, Mx)\, dx. \tag{2.8}$$

The terms here are of the form

$$\int_{\mathscr{D}} k(x, M_1^{-1} M_0 M_1 x)\, dx = \int_{\mathscr{D}} k(M_1 x, M_0 M_1 x)\, dx$$

$$= \int_{M_1 \mathscr{D}} k(x, M_0 x)\, dx,$$

with $M_1\mathscr{D}$ denoting the image of $\mathscr{D}$ under the transformation M_1. Two M_1 give the same $M_1^{-1}M_0M_1$, if and only if they differ on the left by an element of Γ_{M_0}. Thus the expression (2.8) becomes

$$\int_{\mathscr{D}_{M_0}} k(x, M_0\, x)\, dx,$$

where the domain of integration is given by $\mathscr{D}_{M_0} = \sum^*_{M\in\Gamma} M\mathscr{D}$, Σ^* indicating that the summation is carried over a complete set of elements M such that no two differ on the left by an element of Γ_{M_0}. It is easily seen that $\mathscr{D}_{M_0}$ is actually a fundamental domain of the discontinuous group Γ_{M_0} in S. Thus we may rewrite the right-hand side of (2.7) as

$$\sum_{\{M\}_\Gamma} \sigma(\chi\,(M)\,) \int_{\mathscr{D}_M} k(x, Mx)\, dx, \tag{2.9}$$

where the summation is extended over one representative for each conjugacy class in Γ. We shall transform the expression

$$\int_{\mathscr{D}_M} k(x, M\, x)\, dx$$

still further. We introduce on G_M with elements n_M the Haar measure dn_M which is invariant with respect to multiplication on the right. We construct some function $p(x)$ which is everywhere on S real and non-negative, and for which

$$\int_{G_M} p(n_M x)\, dn_M = 1, \quad \text{for all } x \text{ in } S.$$

This can be done by constructing first a function $q(x) \geqslant 0$, everywhere on S, for which the integral

$$\int_{G_M} q(n_M x)\, dn_M = q_1(x)$$

exists and is positive for every x in S. This can be done for instance by defining

$$q(x) = \begin{cases} 1 + \rho(x) - d(x), & \text{for } d(x) < 1 + \rho(x), \\ 0, & \text{for } d(x) \geqslant 1 + \rho(x), \end{cases}$$

where $d(x)$ denotes the smallest geodesic distance from x to some fixed point x_0, and $\rho(x) = \min_{n_M \in G_M} d(n_M x)$. Then $p(x) = q(x)/q_1(x)$ is seen to satisfy the above requirements. The group Γ_M acting on the right of G_M is discontinuous and we may denote by G_M/Γ_M a fundamental domain of Γ_M in G_M; we then get

$$\int_{\mathscr{D}_M} k(x, Mx)\, dx$$

$$= \int_{G_M} \int_{\mathscr{D}_M} k(x, Mx)\, p(n_M x)\, dx\, dn_M$$

$$= \sum_{N_M \in \Gamma_M} \int_{G_M/\Gamma_M} \int_{\mathscr{D}_M} k(x, Mx)\, p(n_M\, N_M\, x)\, dx\, dn_M$$

$$= \sum_{N_M \in \Gamma_M} \int_{G_M/\Gamma_M} \int_{\mathscr{D}_M} k(N_M\, x, M\, N_M\, x)\, p\, (n_M\, N_M x)\, dx\, dn_M$$

$$= \sum_{N_M \in \Gamma_M} \int_{G_M/\Gamma_M} \int_{N_M \mathscr{D}_M} k(x, Mx)\, p(n_M x)\, dx\, dn_M$$

$$= \int_{G_M/\Gamma_M} \int_{S} k(x, M\, x)\, p\, (n_M x)\, dx\, dn_M$$

$$= \int_{G_M/\Gamma_M} \int_{S} k(n_M x, M n_M x)\, p(n_M x)\, dx\, dn_M$$

$$= \int_{G_M/\Gamma_M} \int_{S} k(x, Mx)\, p(x)\, dx\, dn_M$$

$$= \int_{G_M/\Gamma_M} dn_M \int_{S} k(x, Mx)\, p(x)\, dx,$$

where we repeatedly have used the fact that $k(x, Mx)$ is invariant under the group G_M, that the measure dn_M is right-invariant, that the measure dx is invariant, and also that

$$G_M = \sum_{N_M \in \Gamma_M} (G_M/\Gamma_M).\, N_M \text{ and } S = \sum_{N_M \in \Gamma_M} N_M\, D_M.$$

Writing now

$$\int\limits_{G_M/\Gamma M} dn_M = \mu(G_M/\Gamma_M),$$

this factor measures the volume of the fundamental domain of Γ_M in G_M, and does not in any way depend on $k(x, y)$. For the other factor we write

$$\int\limits_S k(x, Mx)\, p(x)\, dx = g(\{ M \}_G),$$

and observe that this factor only depends on $k(x, y)$ and on the conjugacy class $\{ M \}_G$ of M in G. Combining our results we may now write

$$\int\limits_{\mathscr{D}} \sigma(K(x, x; \chi))\, dx = \sum_{\{M\}_\Gamma} \sigma(\chi\, (M))\, \mu(G_M/\Gamma_M)\, g(\{M\}_G). \quad (2.10)$$

We now turn to the question 'when and in what sense are the two expressions (2.6) and (2.10) equal?' We can at first say that the series (2.6) converges absolutely and is equal to (2.10) if $k(x, y)$ can be written in the form

$$\int\limits_S k_1(x, z)\, \overline{k_1(y, z)}\, dz, \quad (2.11)$$

where k_1 is a point-pair invariant satisfying our conditions (a) and (b). From this we get next that the same conclusion holds if k can be written in the form

$$\int\limits_S k_1(x, z)\, k_2(z, y)\, dz, \quad (2.11')$$

if k_1 and k_2 both satisfy the conditions (a) and (b), since (2.11′) can be written as a linear combination of expressions of the form (2.11).

Introducing the notation, for $\epsilon > 0$,

$$\kappa_\epsilon(x, y) = \begin{cases} C_\epsilon, & \text{for } d(x, y) < \epsilon, \\ 0, & \text{for } d(x, y) \geqslant \epsilon, \end{cases}$$

where $d(x, y)$ is the smallest geodesic distance between x and y, and where C_ϵ is a constant depending on ϵ, chosen such that

$$\int_S \kappa_\epsilon(x, y)\, dy = 1,$$

(the integral clearly is independent of x), $\kappa_\epsilon(x, y)$ is a point-pair invariant satisfying (a) and (b). Writing

$$\theta_\epsilon(\lambda) = \int_S \kappa_\epsilon(x, y)\, \omega_\lambda(y, x)\, dy,$$

we have

$$\lim_{\epsilon \to 0} \theta_\epsilon(\lambda) = 1,$$

and that for the λ^i, in addition $|\theta(\lambda^i)| \leqslant 1$.†

Now let $k(x, y)$ satisfy (a) and (b) and in addition be continuous; considering the class

$$k_\epsilon(x, y) = \int_S \kappa_\epsilon(x, z)\, k(z, y)\, dz,$$

for $0 < \epsilon < 1$, we get that the class k_ϵ satisfy our conditions (a) and (b) uniformly, and that $\lim_{\epsilon \to 0} k_\epsilon(x, y) = k(x, y)$, uniformly for x and y in any compact subregion of S. Using this we can show that the " trace formula "

$$\sum_i h(\lambda^i) = \sum_{\{M\}_\Gamma} \sigma(\chi(M))\, \mu(G_M/\Gamma_M)\, g(\{M\}_G) \tag{2.12}$$

† With equality only if $\omega_{\lambda^i}(x, y) = 1$.

is valid if we give the left-hand side the interpretation

$$\lim_{\epsilon \to 0} \sum_i h(\lambda^i)\, \theta_\epsilon(\lambda^i).$$

In particular (2.12) holds whenever k satisfies (a) and (b) and is continuous and the left-hand side of (2.12) converges absolutely.

Various types of sufficient conditions for absolute convergence can be given,‡ for instance that $K(x, y; \chi)$ have partial derivatives up to the order $[n/2] + 1$, which is the case if $k(x, y)$ has partial derivatives up to this order which are such that (2.2) can be differentiated term by term and the resulting series converges absolutely.

The trace formula (2.12) may be used on the one hand to investigate the distribution of the l-tuples λ^i and on the other hand also to investigate the distribution of the conjugate classes $\{M\}_\Gamma$, the latter in the following sense : The conjugate classes in G can be characterized by a certain number of numerical parameters and so with each $\{M\}_\Gamma$ can be associated the numerical parameters that characterize $\{M\}_G$; it is the distribution of these numerical parameters that can be investigated by means of (2.12).

We shall mention briefly a certain generalization of (2.12) which is of interest in connection with the so-called Hecke-operator for the classical modular group and their analogues.

Let us have given in connection with our group Γ and the representation $\chi(M)$, a subset Γ^* of elements M^* of G with the following properties : The set Γ^* (it does not need to be a group) and the

‡ Actually in the case of a particular G and S, the more convenient such conditions are those that can be expressed in terms of $h(\lambda)$ only. This involves expressing $k(x, y)$ in the form $\int h(\lambda)\, \omega_\lambda(x, y) d\lambda$ where $d\lambda$ is a certain measure, and seeing what properties of $h(\lambda)$ are sufficient to ensure that $k(x, y)$ is continuous and satisfies (a) and (b), then determining enough about the asymptotic distribution of the $\lambda^{(i)}$ to see what additional condition should be imposed to ensure the absolute convergence of $\sum_i h(\lambda^i)$.

elements M^*, are such that with M^* the inverse M^{*-1} is also in Γ^*, further for M^* in Γ^* and M in Γ the element M^*M is also in Γ^*, and there should be a finite set of "left-representatives" M_1^*, M_2^*, $\ldots$, M_κ^* such that $\Gamma^* = \sum_{i=1}^{\kappa} M_i^* \Gamma$, or otherwise expressed, every M^* can in a unique way be represented as $M_i^* M$ with $M \in \Gamma$. Further let there be associated with each M^* a ν by ν matrix $\chi(M^*)$ (not necessarily unitary) such that $\chi(M^*M) = \chi(M^*)\chi(M)$ for M^* in Γ^* and M in Γ, and such that

$$\chi(M^{*-1}) = \overline{\chi(M^*)}'.$$

Defining now the operator T^* by

$$T^*F(x) = \sum_{i=1}^{\kappa} \chi(M_i^*)\, F(M_i^{*-1}x), \tag{2.13}$$

one establishes that $T^*F(x)$ again satisfies (2.1). T^* is seen to be self-adjoint in our Hilbert space and further to commute with our invariant integral operators (2.3) and with the fundamental differential operators. Therefore our complete orthonormal system of eigenfunctions $F_i(x)$ may be chosen such that they are also eigenfunctions of T^*; writing then

$$T^*F_i(x) = \lambda_*^i F_i(x),$$

it can be shown by multiplying the T^* with an operator of the form (2.3), which gives us an integral operator with the kernel

$$K^*(x, y; \chi) = \sum_{M^* \in \Gamma^*} \chi(M^*)\, k(x, M^*y),$$

and computing the trace of this integral operator in a similar way that

$$\sum_i h(\lambda^i)\, \lambda_*^i = \sum_{\{M^*\}_\Gamma} \sigma(\chi(M^*))\, \mu(G_{M^*}/\Gamma_{M^*})\, g(\{M^*\}_G), \tag{2.14}$$

where the conjugacy classes $\{M^*\}_\Gamma$ are defined by conjugacy with respect to Γ (that is M_1^* and M_2^* belong to the same conjugacy class, if and only if there exists $M \in \Gamma$ such that $M_1{}^* = MM_2{}^* M^{-1}$), and Γ_{M^*} is the subgroup of Γ that commute with M^*. What was said about the validity of (2.12) also holds for (2.14).

If for some l-tuple Λ it happens that $\omega_\Lambda(x, y)$ satisfies the condition

$$\int_S |\omega_\Lambda(x, y)| \, dy < \infty,$$

then one can show that $\omega_\Lambda(x, y)$ satisfies both our conditions (a) and (b), and it can therefore be used as a $k(x, y)$ in our trace formula. Since it is seen that for the $h(\lambda)$ corresponding to $\omega_\Lambda(x, y)$ one has $h(\lambda) = 0$ for $\lambda \neq \Lambda$, and

$$h(\Lambda) = \int_S |\omega_\Lambda (x, y)|^2 \, dy,$$

we get on the left-hand side of the formula (2.12) simply $N(\Lambda)\, h(\Lambda)$, where $N(\Lambda)$ is the number of the l-tuples λ^i that are equal to Λ. I conjecture, but have only so far been able to verify this conjecture for special types of spaces, that in this case $g\,\{M\}_G$† $= 0$ for all M which do not belong to some compact subgroup of G so that (as one easily establishes) the number of terms on the right-hand side which are not zero is finite. This would imply that one gets a finite expression for $N(\Lambda)$. As will be indicated later this has interesting applications to the problem of determining the number of linearly independent regular analytic automorphic forms of a given dimension, in one or more complex variables.‡

We have so far assumed that the fundamental domain $\mathscr{D}$ of our group Γ is compact. If we relax this condition and only require that D have finite volume, the situation changes somewhat. While the kernel $K(x, y; \chi)$ will behave as before as long as at least one

† Of course the special g that is derived from $\omega_\Lambda(x, y)$.

‡ Similar remarks apply to formula (2.14), which is of interest for the theory of Hecke-operators, as applied to the analytic modular forms.

of the points x and y is restricted to a compact subregion of $\mathscr{D}$ (or of S for that matter), the kernel may exhibit a singular behavior as both points tend simultaneously towards the " non-compact boundary " of $\mathscr{D}$, such that the integral

$$\int_{\mathscr{D}} \int_{\mathscr{D}} \sigma(K(x, y; \chi) \, \overline{K(x, y; \chi)}') \, dx \, dy \tag{2.15}$$

does not exist. If, as it may happen for some χ, the kernels K behave well enough at the " non-compact boundary " for (2.15) to exist, the situation is not significantly changed, the spectrum of l-tuples λ for which there are eigenfunctions F is still discrete and the eigenfunctions are in our Hilbert space, and one may in specific cases by showing special care with the transformations M that leave some " part " of the " non-compact boundary " fixed (namely by grouping together those that have the same Γ_M), prove a trace formula that is not essentially different in form from (2.12), only that some terms on the right-hand side will no longer correspond to a single conjugacy class $\{M\}_\Gamma$, but to an aggregate of conjugacy classes.

If however χ is such that (2.15) does not exist, there are in general continuous spectra (which may even be multi-dimensional) besides the discrete spectrum. In some of the simpler cases, where these continuous spectra have been studied, it is possible to remove them by replacing the kernel $K(x, \; y \; ; \; \chi)$ with a modified kernel which retains only the eigenfunctions from the discrete spectrum and with unchanged eigenvalues $h(\lambda^i)$, the computation of the trace of this modified integral operator leads then to a trace formula, which however besides terms of the type occurring on the righthand side of (2.12) will contain terms of a radically new nature.

3. We shall in the following give some explicit illustrations of the formulas in the case of some simpler spaces S and groups G satisfying our conditions.

First we consider the case when S is the hyperbolic plane for which we use the model represented by the upper complex half-plane $z = x + iy$, $y > 0$, with the metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$. Our group G may be taken as the group formed by all motions $mz = \frac{az+b}{cz+d}$ where $ad - bc = 1$, a, d, b, and c real.† The Laplacian corresponding to the metric $y^2\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is the only fundamental operator, and the point-pair invariants are seen to be all of the form

$$k(z, z') = k\left(\frac{|z - z'|^2}{yy'}\right).$$

A representative set of eigenfunctions is given by y^s since

$$y^2\Delta\, y^s = \lambda\, y^s$$

with $\lambda = -s(1-s)$. Writing $s = \frac{1}{2} + ir$, we shall use for convenience the r instead of the λ as parameter. The connection between $k(z, z')$ and $h(r)$ is given by the relations

$$\left.\begin{aligned} &\int_w^\infty \frac{k(t)}{\sqrt{(t-w)}}\,dt = Q(w), \quad k(t) = -\frac{1}{\pi}\int_t^\infty \frac{dQ(w)}{\sqrt{(w-t)}}, \\ &\qquad\qquad Q(e^u + e^{-u} - 2) = g(u), \\ &h(r) = \int_{-\infty}^{\infty} e^{iru}\, g(u)\, du, \quad g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iru}\, h(r)\, dr. \end{aligned}\right\} \quad (3.1)$$

Regarding now $h(r)$ as the primary function, we see that if $h(r)$ satisfies the conditions§:

(1) $h(r) = h(-r)$,

(2) $h(r)$ is regular analytic in a strip $|\mathrm{Im}\, r| < \frac{1}{2} + \epsilon$, where $\epsilon > 0$, and

§ This is not the full group of isometries, since this also contains the elements $\frac{a\bar{z}+b}{c\bar{z}+d}$, with $ad - bc = -1$. However, we shall for simplicity assume that our discontinuous group Γ has only true motions as elements.

§ The conditions (2) and (3) could be somewhat weakened.

(3) $h(r) = O((1 + |r|^2)^{-1-\epsilon}$ in this strip;

then k will exist and satisfy our conditions (a) and (b).

The elements of G can, as it is well known, be divided into four types, of which the first consists only of the identity element, while the others are respectively the hyperbolic, the elliptic and the parabolic elements. For a hyperbolic element m there is always a representative of the conjugacy class $\{m\}_G$ of the form $z \to \rho z$, where ρ is real and > 1. We call ρ the norm of m, and also the norm of the hyperbolic conjugacy class $\{m\}_G$ and denote it by $N\{m\}$, leaving the subscript G out. An elliptic element has always one (and only one) fixed point in the space and represents a rotation of the plane around this point, by an angle which we may count positive in the counter-clockwise direction; we call this the rotation angle of the elliptic element and also of the elliptic conjugacy class in G represented by the element. Finally if an element is parabolic it belongs to one of the two parabolic conjugacy classes represented by $z \to z + 1$ and $z \to z - 1$ respectively.

In Γ we shall call a hyperbolic element P primitive, if it is not a power with exponent > 1 of any other element in the group Γ, correspondingly we say that the conjugacy class $\{P\}_\Gamma$ is primitive. For the elliptic elements of Γ, those with the same fixed point form a finite group generated by a single element, and the one that has the smallest positive rotation angle we call primitive and denote it by R and call the corresponding class a primitive elliptic conjugacy class $\{R\}_\Gamma$ in Γ. Finally a parabolic element of Γ which is not a power with exponent > 1 of any other element in Γ, and which belongs to the first of the two parabolic conjugacy classes in G, we call a primitive parabolic element of Γ and, denoting it by S, the corresponding class $\{S\}_\Gamma$ a primitive parabolic class. It should be mentioned that if the area of the fundamental domain $\mathscr{D}$ of Γ is finite, that is to say

$$A(\mathscr{D}) = \int_{\mathscr{D}} \frac{dx\,dy}{y^2} < \infty,$$

there are only a finite number of elliptic and primitive parabolic conjugacy classes in Γ, and if $\mathscr{D}$ is compact there are no parabolic ones. The primitive hyperbolic classes $\{P\}_\Gamma$ on the other hand are always present in infinite number.

Assuming first that $\mathscr{D}$ is compact, the trace formula takes the form

$$\sum_i h(r_i) = \frac{A(\mathscr{D})\,\nu}{2\pi} \int_{-\infty}^{\infty} r\,\frac{e^{\pi r} - e^{-\pi r}}{e^{\pi r} + e^{-\pi r}}\, h(r)\, dr + $$

$$+ \sum_{\{R\}_\Gamma} \sum_{k=1}^{m-1} \frac{\sigma(\chi^k(R))}{M \sin k\pi/m} \int_{-\infty}^{\infty} \frac{e^{-2\pi r k/m}}{1 + e^{2-\pi r}}\, h(r)\, dr +$$

$$+ 2 \sum_{\{P\}_\Gamma} \sum_{k=1}^{\infty} \frac{\sigma(\chi^k(P)) \log N\{P\}}{(N\{P\})^{k/2} - (N\{P\})^{-k/2}}\, g(k \log N\{P\}). \quad (3.2)$$

Here the r_i are the values for which there is a solution of the equation

$$y^2 \Delta F(z) = \lambda F(z), \quad \lambda = -(\tfrac{1}{4} + r^2)$$

with $F(z)$ in our Hilbert space ; since we count both values of r that give the same λ (and if $\lambda = -\frac{1}{4}$, $r = 0$ with double multiplicity) our formula actually represents twice the trace of the integral operator. $A(\mathscr{D})$ is the area of the fundamental domain. $m = m(R)$ represents the order of the primitive elliptic element R, and the summations $\sum\limits_R$ and $\sum\limits_P$ are taken over one representative from each primitive elliptic and each primitive hyperbolic class respectively. The r_i have to be such that $\frac{1}{4} + r^2$ is real and non-negative, so that the r_i are either real, or they are purely imaginary with absolute value $\leqslant \frac{1}{2}$.† The formula (3.2) can now on the one hand be used for determining the asymptotic distribution of the r_i, and on the other hand the asymptotic distribution of the norms of the primitive hyperbolic classes in Γ. Under our assumptions on $h(r)$ all infinite series occurring in (3.2) converge absolutely.

† These latter could of course only occur in finite number, but one can show that their number for suitable Γ and χ may become arbitrarily large, although it can be shown to be less than a certain constant times $\nu\, A(\mathscr{D})$.

(3.2) has a rather striking analogy to certain formulas that arise in analytic number theory from the zeta - and L-functions of algebraic number fields. This leads us to introduce the function defined by

$$Z_\Gamma(s;\chi) = \prod_{\{P\}_\Gamma} \prod_{k=0}^{\infty} |E_\nu - \chi(P)\,(N\{P\})^{-s-k}|\,, \qquad (3.3)$$

for real part of $s > 1$, when the product converges absolutely. E_ν is here the ν by ν identity matrix, and $|\ldots|$ denotes the determinant.

From (3.2) one can derive the following facts about this analytic function of s :

(A). $Z_\Gamma(s;\chi)$ is an integral function of s of order 2, except in the case when the genus of the fundamental domain $\mathscr{D}$ is zero, in this case there may be a pole at $s = 0$ of order at most ν.‡

(B). $Z_\Gamma(s;\chi)$ has "trivial" zeros at the integers $-k$ for $k \geqslant 0$, whose multiplicity can be explicitly given in terms of k, ν, $A(\mathscr{D})$ (or the genus of the fundamental domain if one prefers), and the $m(R)$, the orders of the primitive elliptic classes, and the traces $\sigma(\chi^i(R))$ for $i = 1, 2, \ldots m(R) - 1$. In the particular case that there are no elliptic classes in Γ one has that the multiplicity of the trivial zero at $-k$ is $(2k+1)(2p-2)$, where p is the genus§ (which is in this case always > 1).

(C). $Z_\Gamma(s;\chi)$ satisfies a functional equation which relates the value of $Z_\Gamma(1-s;\chi)$ to that of $Z_\Gamma(s;\chi)$. The form of this functional equation depends on the quantities ν, $A(\mathscr{D})$, and the orders $m(R)$ of the primitive elliptic classes and the traces $\sigma(\chi^i(R))$ for $i = 1, 2, \ldots,$ $m(R) - 1$. In the particular case that there are no elliptic classes in Γ this functional equation has the form

$$Z_\Gamma(1-s;\chi) = Z_\Gamma(s;\chi)\exp\left\{-\nu A(\mathscr{D})\int_0^{s-\frac{1}{2}} v \operatorname{tg} \pi v \, dv\right\}. \qquad (3.4)$$

‡ If one assumes the representation $\chi(M)$ to be irreducible, this pole only occurs for $\chi(M)$ identically equal to 1, and is then a simple pole.

§ In this particular case $p-1 = \dfrac{A\,(D)}{4\pi}$.

(D). The zeros of $Z_\Gamma(s, \chi)$ which are not mentioned under (B), are the numbers $\frac{1}{2} + ir_i$, and have thus real part equal to $\frac{1}{2}$, with the possible exception of a finite number of zeros that are real and lie in the interval $0 \leqslant s \leqslant 1$.

As one sees from (D) the analog of the Riemann hypothesis is true for our $Z_\Gamma(s; \chi)$ with the slight modification that real zeros may occur in the interval $0 \leqslant s \leqslant 1$.

If we only require $A(\mathscr{D})$ to be finite, there will, if $\mathscr{D}$ is not compact, always be at least one primitive parabolic class $\{S\}_\Gamma$. If $\{S_i\}_\Gamma$ for $i = 1, 2, \ldots, \kappa$, are the different primitive parabolic classes in Γ, the situation will depend on the matrices $\chi(S_i)$; if $\chi(S_i)$ has μ_i eigenvalues equal to 1, we say that χ is singular of degree μ_i with respect to the class $\{S_i\}_\Gamma$, and singular of degree $\mu = \sum_{i=1}^{\kappa} \mu_i$ with respect to Γ. If $\mu = 0$, that is if χ is non-singular with respect to Γ, the situation is only slightly altered from the compact case. The spectrum is still discrete and in our trace-formula (3.2), will occur on the right-hand side the new term

$$- 2\, g(0) \sum_{i=1}^{\kappa} \log \| E_\nu - \chi(S_i) \| . \tag{3.5}$$

This new term does not essentially alter the statements (A), (B), (C) and (D) about $Z_\Gamma(s; \chi)$. If $\mu \geqslant 1$ however the situation is very much altered, in that we have then for our eigenvalue problem, besides the discrete spectrum, also a continuous spectrum of multiplicity μ. As mentioned in the previous section we have then first to investigate the eigenfunctions in the continuous spectra and then to remove their contribution to the kernel K and develop a trace formula for the modified kernel. As a description of the general case is rather complicated, we shall here only briefly indicate the results in the simplest case when there is only one parabolic class $\{S\}_\Gamma$ with respect to which χ is singular, and further that χ is one dimensional, so that $\chi(S) = 1$.

We may assume for simplicity that one representative S of the class $\{S\}_\Gamma$ is $Sz = z + 1$. Forming for real part of s greater than 1 the function

$$\left.\begin{aligned} E(z, s; \chi) &= \sum_{M \in S/\Gamma} \overline{\chi(M)}\,(\operatorname{Im} M z)^s = \sum_{M \in S/\Gamma} \overline{\chi(M)}\,\frac{y^s}{|cz + d|^{2s}} \\ M z &= \frac{a z + b}{c z + b}, \end{aligned}\right\} \tag{3.6}$$

where $M \in S/\Gamma$ means that M runs over a complete set of elements of Γ that do not differ by a power of S on the left, one establishes that this series is absolutely convergent for $\sigma > 1$, $s = \sigma + it$. Further one has

$$E(Mz, s; \chi) = \chi(M)\, E(z, s; \chi),$$

for M in Γ, and

$$y^2 \Delta\, E(z, s; \chi) = -s(1 - s)\, E(z, s; \chi).$$

It can then be proved that $E(z, s; \chi)$ is a meromorphic function of s in the whole s-plane, and that the poles are all in the region $\sigma < \frac{1}{2}$, with the possible exception of a finite number of simple poles which are real and lie in the interval $\frac{1}{2} < s \leqslant 1$; these poles are independent of z, and $E(z, s; \chi)$ may be written as a quotient of two integral functions in s, each of which is at most of order 2 and where the denominator is independent of z. Further $E(z, s; \chi)$ satisfies a functional equation, which may be described as follows:

We write for $\sigma > 1$,

$$\phi(s, \chi) = \frac{\pi^{\frac{1}{2}}\,\Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_{c \neq 0} \sum_{0 \leqslant d < |c|} \frac{\overline{\chi(M)}}{|c|^{2s}}; \tag{3.7}$$

then one can show that $\phi(s, \chi)$ is meromorphic in the whole s-plane and regular for $\sigma \geqslant \frac{1}{2}$ with the possible exception of a finite number of simple poles in the interval $\frac{1}{2} < s \leqslant 1$, and can be written as a quotient of two integral functions at most of order 2. Further one has the functional equation

$$\phi(s, \chi)\, \phi(1-s, \chi) = 1.^{\dagger} \tag{3.8}$$

Then the functional equation of $E(z, s; \chi)$ is

$$E(z, s; \chi) = \phi(s, \chi)\; E(z, 1-s; \chi). \tag{3.9}$$

Forming now the kernel

$$H(z, z'; \chi) = \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r)\, E(z, \tfrac{1}{2} + ir; \chi)\, \overline{E(z', \tfrac{1}{2} + ir; \chi)}\, dr, \tag{3.10}$$

one can show that the kernel

$$K^*(z, z'; \chi) = K(z, z'; \chi) - H(z, z'; \chi), \tag{3.11}$$

where

$$K(z, z'; \chi) = \sum_{M \in \Gamma} \chi(M)\, k(z, Mz'),$$

has the property that it retains only the discrete spectrum (that is all eigenfunctions which are not in our Hilbert space are annihilated by the integral operator with kernel K^*), and this is retained with unchanged eigenvalues $h(r_i)$. The evaluation of the trace of this modified integral operator then gives us a trace formula which differs from the earlier in that on the right-hand side we have the new terms

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} h(r)\, \frac{\phi'}{\phi}\, (\tfrac{1}{2} + ir, \chi)\, dr - \frac{1}{\pi} \int_{-\infty}^{\infty} h(r)\, \frac{\Gamma'}{\Gamma}\, (1 + ir)\, dr -$$

$$- 2 \log 2.\; g(0) + \tfrac{1}{2}\, (1 - \phi(\tfrac{1}{2}, \chi)\,)\; h(0). \tag{3.12}$$

These new terms make a rather drastic change in the nature of $Z_\Gamma(s, \chi)$, in particular $Z_\Gamma(s; \chi)$ will have simple poles at $s = -1/2$, $-3/2$, $-5/2, \ldots$; because of the second term in (3.12), the last term produces a simple pole at $s = \frac{1}{2}$ if $\phi(\frac{1}{2}, \chi) = -1$ (this pole may however be cancelled by a zero if one or more of the r_i equals zero), so that $Z_\Gamma(s; \chi)$ is no longer an integral function. Furthermore in addition to the non-trivial zeros at the points $\frac{1}{2} + ir_i$, namely wherever $\phi(s, \chi)$ has a pole in the region $\sigma < \frac{1}{2}, Z_\Gamma(s, \chi)$ will have a

† Since the coefficients of the Dirichlet series of (3.7) are actually real, this implies $|\phi(\frac{1}{2} + ir, \chi)| = 1$.

a zero of the same multiplicity. The functional equation is also correspondingly modified in that besides simple factors also the function $\phi(s, \chi)$ occurs in it.

In the general case one has a system of μ series like (3.6), with similar properties : when s is replaced by $1 - s$, this system transforms by a matrix $\phi(s, \chi)$ whose elements are of a similar nature as (3.7) ; the determinant of this matrix will essentially then play the role that $\phi(s, \chi)$ does in the former case.

In the 3-dimensional hyperbolic space, the situation is similar but in some respects simpler. One can introduce also there a $Z_\Gamma(s; \chi)$‡ which although it will be a function of order 3, has a functional equation which is essentially simpler than in the case of the hyperbolic plane. For general n-dimensional hyperbolic space the explicit computations are somewhat complicated by the fact that the groups Γ_M and G_M now may not always be abelian when M is different from the identity element; this complicates the form of the trace formula, which however is always in a certain sense simpler when n is odd than when n is even. The non-compact case with finite volume of $\mathscr{D}$ can in all these cases be treated satisfactorily.

For groups acting simultaneously on the product of a finite number of such spaces,§ the situation can also be handled even in the non-compact case as long as the " non-compact boundaries " of $\mathscr{D}$ are point-like.

For other higher dimensional spaces, as for instance the space of positive definite, n by n symmetric matrices with determinant 1, the situation, for $n > 2$, is not so simple. The continuous spectra that may occur in the non-compact case at present cannot be handled properly. One will also here try to obtain them by analytic continuation of certain Dirichlet series, like we did for the hyperbolic plane; only these Dirichlet series are more complicated and in the case of spectra that have a dimension > 1, they are Dirichlet series in several

‡ Defined by a somewhat more complicated product than (3.3).

§ Like the so-called Hilbert group acting on a product of hyperbolic planes.

variables. This problem of analytic continuation cannot be handled at present, except for special groups that arise from arithmetic, where one may be able to utilize this to effect the continuation. As an example could be mentioned the case when $n = 3$ for the above space, and the group Γ is the group of 3 by 3 matrices with determinant 1 and integral rational elements, and χ identical to 1; when one is led to consider the series

$$\zeta_Y(s, s') = \sum_{X'Z=0} (X' Y X)^{-s} (Z' Y^{-1} Z)^{-s'},$$

where the summation is carried over all pairs of column vectors X and Z, with integral rational components which satisfy the conditions $X'X > 0$, $Z'Z > 0$, $X'Z = 0$. The series converges absolutely for $\sigma > 1, \sigma' > 1$, where $s = \sigma + it$, $s' = \sigma' + it'$. One can in this case show that

$$(s-1)\ (s'-1)\ (s + s'-3/2)\ \zeta(2s + 2s'-1)\ \zeta_Y(s, s'),$$

where $\zeta(2s + 2s' - 1)$ is the ordinary Riemann zeta-function, is an integral function in the two complex arguments s and s'. Further if one writes

$$\xi_Y(s, s') \doteq \pi^{-2s-2s'}\ \Gamma(s)\ \Gamma(s')\ \Gamma(s + s' - \tfrac{1}{2})\ \zeta(2s + 2s'-1)\ \zeta_Y(s, s'),$$

then the function $\xi_Y(s, s')$ remains invariant by replacing (s, s') by any of the following pairs of complex arguments $(s + s' - 1/2, 1-s')$, $(1 - s, s + s' - 1/2)$, $(3/2 - s - s', s)$, $(s', 3/2 - s - s')$ and $(1 - s', 1 - s)$, so that it has a larger number of functional equations than the zeta-functions in one variable. It should be noted that the group under which $\xi_Y(s, s')$ is invariant is isomorphic to the permutation group of 3 elements, as the three quantities $4s + 2s' - 3$, $2s' - 2s$, $-4s' - 2s + 3$, undergo permutations. $\xi_Y(1/2 + it, 1/2 + it')$ is here connected with the two-dimensional continuous spectrum. Besides this there is a denumerably infinite sequence of Dirichlet series in one complex variable that are connected with one-dimensional continuous spectra.

Similar Dirichlet series in up to $(n - 1)$ complex variables, can be defined for general n, by looking at the definite forms in $(n - 1)$

variables that can be represented by the quadratic form with matrix Y, then the $(n-2)$ forms that can be represented by the $(n-1)$ form, and so on down to a form in one variable, and forming a product of the determinants of the $(n-1), (n-2), \ldots, 1$ form raised to complex exponents $-s_{n-1}, -s_{n-2}, \ldots, -s_1$ respectively, and summing over all such " descending " series of forms that are inequivalent in a certain sense. In the case $n = 3$ this would lead to a function which differs only by a simple factor (which is independent of Y) from $\zeta_Y(s_1, s_2)$ as it was defined above. The general study of these series has not yet been undertaken, but it is conceivable that it may prove of value for the theory of quadratic forms.

4. We shall finally give some applications to more classical problems. We go back to the hyperbolic plane $z = x + iy$, $y > 0$, and add a third coordinate ϕ, where we will identify ϕ and $\phi + 2\pi$. On this space consisting now of points (z, ϕ), we take the following group G with elements m_α, where m is a real matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant 1, and α a real number, and let it act on the space (z, ϕ) in the way that

$$m_\alpha(z, \phi) = \left(\frac{a\,z + b}{c\,z + d},\ \phi + \arg\,(cz + d) + \alpha \right).$$

Further we define μ such that

$$\mu(z, \phi) = (-\bar{z}, -\phi).$$

One then establishes that the two differential forms

$$\frac{dx^2 + dy^2}{y^2} \text{ and } d\phi - \frac{dx}{2y},$$

have the property that they both are invariant under G, the first one is also invariant under μ whereas the second only changes sign. We may therefore take for instance

$$ds^2 = \frac{dx^2 + dy^2}{y^2} + \left(d\phi - \frac{dx}{2y} \right)^2$$

as our invariant metric. We have two fundamental differential operators in this case, namely $\frac{\partial}{\partial\phi}$ and the Laplacian derived from the metric. The point-pair invariants are seen to be all functions of the two real arguments

$$\frac{|z-\bar{z}'|^2}{4\,yy'} \text{ and } \phi - \phi' + \arg\frac{z-\bar{z}'}{2i},$$

where (z, ϕ) and (z', ϕ') are the two points.

If we now have a group Γ which is discontinuous in the hyperbolic plane, it is seen that the group $\bar{\Gamma}$ obtained from Γ by, for each transformation $Mz = \frac{az+b}{cz+d}$ in Γ, counting both $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $-M = \begin{pmatrix} -a, & -b \\ -c, & -d \end{pmatrix}$ as different elements of $\bar{\Gamma}$, has the property that when $\bar{\Gamma}$ acts on our space (z, ϕ) in the way

$$M(z, \phi) = \left(\frac{a\,z+b}{c\,z+d},\ \phi + \arg\,(c\,z+d)\right),$$

the group $\bar{\Gamma}$ is discontinuous in this space, and if the fundamental domain $\mathscr{D}$ in the hyperbolic plane of Γ is compact, then so is the fundamental domain $\bar{\mathscr{D}}$ of $\bar{\Gamma}$, and if $\mathscr{D}$ has finite area $\bar{\mathscr{D}}$ has finite volume. The converse is also true.

Similarly a representation χ of Γ can be extended to $\bar{\Gamma}$ by letting both M and $-M$ correspond to the same χ as the transformation Mz in Γ. There may however also be other representations χ of $\bar{\Gamma}$ where the two elements M and $-M$ correspond to different χ.

If we have such a representation $\chi(M)$ of Γ one now sees that the eigenfunctions of our operators, because of the presence of the fundamental operator $\frac{\partial}{\partial\phi}$ and the identification of (z, ϕ) and $(z, \phi + 2\pi)$ must be of the form $e^{-ik\phi}$ times a function of the point z, where k is an integer. The eigenfunctions $F(z, \phi)$ which satisfy the relation

$$F(M(z, \phi)) = \chi(M)\, F(z, \phi)$$

can therefore be written in the form

$$F(z, \phi) = y^{k/2}\, F(z)\, e^{-ik\phi}, \tag{4.1}$$

and the former relation takes the form

$$F(Mz) = \chi(M)\, (cz + d)^k\, F(z). \tag{4.2}$$

Thus we see that we have the same type of transformation law as (in the case of one-dimensional χ) is known from the theory of the analytic automorphic forms.†

Instead of studying the general eigenfunction, and the general form of the trace-formula, which can be carried out without serious difficulties, we shall here only study a particular type that is associated with certain eigenfunctions with rotational symmetry which have the property that they satisfy our conditions (a) and (b) and so can be used as point-pair invariants in forming our kernels K.

It can be established that

$$\omega_k\, (z, \phi\,; z', \phi') = \frac{(y\, y')^{k/2}}{\{(z - \bar{z}')/2i\}^k}\, e^{-ik(\phi - \phi')}, \tag{4.3}$$

for any integer k is an eigenfunction in (z, ϕ) which is a point-pair invariant in the two points (z, ϕ) and (z', ϕ'); further that for $k > 2$ the conditions (a) and (b) are satisfied. As a consequence the integral operator with the kernel

$$K_k\, (z, \phi\,; z', \phi'\,; \chi) = \sum_{M \in \bar{\Gamma}} \chi(M)\, \omega_k\, \{z, \phi\,; M(z', \phi')\}, \tag{4.4}$$

can be shown to have only eigenfunctions§ of the form (4.1) where $F(z)$ is an analytic function of z regular in the interior of the upper half plane and satisfying the condition that

† We get here only integral dimensions, k; if one wants to study arbitrary real dimension, one has to give up the identification $(z, \phi) = (z, \phi + 2\pi)$, also $\bar{\Gamma}$ has to be defined in a different way.

§ That is corresponding to an eigenvalue different from zero,

$$y^{k/2} F(z)$$

is uniformly bounded throughout this region, and every such eigenfunction corresponds to the same eigenvalue given by

$$\int_S |\omega_k (z, \phi ; z', \phi')|^2 \, d(z, \phi)$$

where the integral is taken over our whole (z, ϕ)-space and $d(z, \phi)$ is the invariant element of volume. The trace-formula for this particular kernel gives us then the number N_k of regular analytic forms $F(z)$ satisfying (4.2)‡ as a finite expression depending on k, ν, the area $A(\mathscr{D})$ of the fundamental domain, the elliptic primitive classes $\{R\}_\Gamma$ and the eigenvalues of the χ's that correspond to them, and the primitive parabolic classes and the eigenvalues of the χ's that correspond to them. The hyperbolic classes give no contribution at all. For $k = 2$ it is possible to obtain a similar result by replacing ω_2 with $\omega_2 \left(\dfrac{(y\, y')^{1/2}}{|(z - \bar{z}')/2i|} \right)^\delta$ where $\delta > 0$, and in the trace formula for this kernel letting δ tend to zero.

If we consider the classical modular group Γ with elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $ad - bc = 1$, and a, d, b, c are rational integers, and the representation χ identical to 1, Hecke has introduced certain operators T_n for each positive integer n and studied their action on the regular modular forms, in connection with his theory about Dirichlet series with functional equations (of a certain type) and Euler products.

These T_n are of the type (2.13), associated with the set of transformations $M^{(n)} = \dfrac{1}{\sqrt{n}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c, d are rational integers with $ad - bc = n$, in the way that T^* was associated with the set M^*. The generalized trace formula (2.14) gives then applied to the

‡ If D is not compact, but has finite area, the condition that $y^{k/2} F(z)$ is uniformly bounded, implies that we are only counting the so-called cusp-forms here,

point-pair invariant eigenfunction ω_k for $k > 2$,[§] the following formula for the trace of the Hecke operator T_n acting on the space of cusp-forms of dimension $-k$.

$$\sigma_k(T_n) = -\frac{1}{2}\sum_{-2\sqrt{n}<m<2\sqrt{n}} H(4n-m^2)\,\frac{\eta_m^{k-1}-\bar{\eta}_m^{k-1}}{\eta_m-\bar{\eta}_m} - \\ -\sum_{\substack{d|n \\ d\leqslant\sqrt{n}}}{}' d^{k-1} + \delta(\sqrt{n})\,\frac{k-1}{12}\,n^{k/2-1}. \quad (4.5)$$

Here the $H(d)$ denotes the number of inequivalent positive definite forms $ax^2+bxy+cy^2$ with $4ac-b^2=d$, counted in the usual way that a form equivalent to $a(x^2+y^2)$ is counted with the weight $\frac{1}{2}$ and one equivalent to $a(x^2+xy+y^2)$ with weight 1/3. Further

$$\eta_m = \frac{m + i\,(4n-m^2)^{\frac{1}{2}}}{2}.$$

$\delta(x)$ is defined as 1 if x is an integer and zero otherwise, and Σ' means that if $d=\sqrt{n}$ the corresponding term is counted with weight $\frac{1}{2}$. For $k=2$ one can again by a limit process arrive at a similar formula which however will contain one new term, and turns out (since there are no cusp forms of dimension -2 for the modular group, so that $\sigma_2(T_n)=0$), to be identical with the so-called Kronecker class number relation. For $k = 4$, 6, 8, 10 and 14 there are again no cusp forms, so that the left-hand side of (4.5) is zero, which gives five new class number relations, while for $k=12$, for instance the left-hand side is identical to the number theoretical function $\tau(n)$ of Ramanujan, so that one gets an explicit (admittedly rather complicated) formula for this. While the results about the number of regular analytic forms of a given dimension $-k$ and representation χ of $\bar{\Gamma}$ are classical[†] and previously were derived from the Riemann-Roch formula, the evaluation of the trace of

§ k will here be even, since with χ identical to one there are no non-vanishing functions satisfying (4.2), for k odd, since the left-hand side remains the same by replacing M by $-M$ whereas the right-hand side changes sign.

† Although as far as I know only the case of one-dimensional χ occurs in the literature.

the Hecke operator has not yet been accomplished by other means. From our point of view these expressions are finite elementary cases of the general trace formulas (2.12) and (2.14).

It is of interest to note that the method sketched above carries immediately over to the analytic automorphic forms in higher dimensional spaces, as for instance a product space formed by a finite number of hyperbolic planes or the general symplectic space, which can all be handled in a similar way without any essential difficulties occurring as long as the discontinuous group Γ has compact fundamental domain. For the symplectic space for instance, one can introduce in a similar way as before a space (Z, ϕ) and define the group G acting on the space with elements M_α, where the symplectic matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with $MZ = (AZ + B)(CZ + D)^{-1}$, and we define

$$M_\alpha(Z, \phi) = (MZ,\ \phi + \arg |CZ + D| + \alpha),$$

and as before

$$\mu(Z, \phi) = (-\bar{Z}, -\phi).$$

In this space again the point-pair invariant of the two points (Z, ϕ) and (Z^*, ϕ^*) which has the form

$$\omega_k(Z, \phi; Z^*, \phi^*) = \frac{|Y|^{k/2}\,|Y^*|^{k/2}}{\left|\dfrac{Z - \bar{Z}^*}{2i}\right|^k} e^{-ik(\phi - \phi^*)},$$

where $Z = X + iY$, is an eigenfunction for every integer k and for k positive and large enough‡ it will again satisfy our requirements (a) and (b).

We shall finally briefly indicate the most general result that we at present can obtain along these lines. Let there be in our space S a sequence of l-tuples $\lambda^{(k)}$, $k = 1, 2, 3 \ldots$, with the property that we have the relation

‡ If we consider the symplectic space of dimension $n^2 + n$, this takes place for $k > 2n$.

$$\omega_{\lambda(k)}(x, y) = (\omega_{\lambda(1)}(x, y))^k$$

for all positive integers k, where as before $\omega_\lambda(x, y)$ denotes the eigenfunction in x that corresponds to the l-tuple λ, and has rotational symmetry around the point y and is normed so as to take the value 1 for $x = y$. Further assume that for k sufficiently large and positive,

$$\int_S |\omega_{\lambda(k)}(x, y)|\, dy < \infty;$$

then $\omega_{\lambda(k)}(x, y)$ can be seen to satisfy both conditions (a) and (b). If we now have a discontinuous group Γ whose fundamental domain is compact, with a representation by unitary matrices χ, in our space, and denote the number of eigenfunctions corresponding to the eigenvalue $\lambda^{(k)}$ by N_k, then one can show that for k sufficiently large, N_k is given by a finite expression,

$$N_k = P_0(k) + \Sigma\, \epsilon_i^k P_i(k), \tag{4.6}$$

where P_0 is a polynomial and the P_i certain polynomials in general of lower degree† and the ϵ_i are certain roots of unity, such that if q_i is the smallest positive integer for which $\epsilon_i^{q_i} = 1$, the number q_i divides the order of some element§ in Γ which is of finite order.

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† The only case when some of them can be of the same degree as P_0 is when Γ contains other elements than the identity which commute with the whole group G.

§ Different from the identity.