Mathematics 234, Fall 2004 Lecture 3 (Wilson)

Second Midterm Exam November 15, 2004 ANSWERS

Problem 1

(a) Set up but do not evaluate an integral in spherical coordinates to compute the volume of the part of the sphere of radius 3, centered at the origin, where z is non-negative. (I.e. the region in the sphere $x^2 + y^2 + z^2 \leq 9$ which is above the xy-plane.)

Answer:

$$\int_0^{2\pi} \int_0^{\pi/2} \int_o^3 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

(b) Evaluate the triple integral $\int_0^{\pi} \int_0^{2\theta} \int_0^{\sqrt{4-r^2}} 2\, r\, z\; dz\; dr\; d\theta$

Answer:

$$\int_0^{\pi} \int_0^{2\theta} \int_0^{\sqrt{4-r^2}} 2\, r\, z\; dz\; dr\; d\theta = \int_0^{\pi} \int_0^{2\theta} (4r - r^3)\, dr\, d\theta = \int_0^{\pi} (8\theta^2 - 4\theta^4)\, d\theta = \frac{8}{3}\pi^3 - \frac{4}{5}\pi^5.$$

Problem 2

The volume of a cylindrical can of radius r and height h is $V(r, h) = \pi\, r^2\, h$.

A certain can is intended to have a radius of $r = 2$ inches and a height of $h = 5$ inches. Due to manufacturing tolerances, the radius changes to $r = 2.01$ inches and the height changes to $h = 4.98$ inches.

Use partial derivatives to approximate how much the volume of the can changes from its designed value.

Answer: $V_r = 2\pi r h$, which at $(2, 5)$ gives 20π. $V_h = \pi r^2$, which at $(2, 5)$ gives 4π. The change in r is $\Delta r = 0.01$ and the change in h is $\Delta h = -0.02$. Thus the change in V is approximately $0.01 \times 20\pi + (-0.02) \times 4\pi = \pi(0.2 - 0.08) = 0.12\pi$. (The actual change is 0.119698π.)

Problem 3

Find an equation for the tangent plane to the surface $x^3 + 3xyz + 2y^3 - z^3 = -15$ at the point $(1, -1, 2)$.

Answer: Letting $f(x, y, z) = x^3 + 3xyz + 2y^3 - z^3$, we have $\nabla f = (3x^2 + 3yz)\vec{\imath} + (3xz + 6y^2)\vec{\jmath} + (3xy - 3z^2)\vec{k}$. At $(1, -1, 2)$ that gives $-3\vec{\imath} + 12\vec{\jmath} - 15\vec{k}$. Since the vector from $(1, -1, 2)$ to any point in the tangent plane must be perpendicular to that gradient vector, we can write an equation for the plane through $(1, -1, 2)$ perpendicular to ∇f as $-3(x - 1) + 12(y + 1) - 15(z - 2) = 0$, which can be simplified to $x - 4y + 5z = 15$.

Problem 4

(a) Compute $\iiint\limits_R (xz + 2z)\ dV$, where R is the region in the first octant bounded by $x^2 + z^2 = 4$ and $x + y = 2$.

Answer: (There are various orders in which to set this up, here is one example. All should result in the same numeric answer.)

$$\iiint\limits_R (xz + 2z)\ dV = \int_0^2 \int_0^{2-x} \int_0^{\sqrt{4-x^2}} (xz + 2z)\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} (-\frac{1}{2}x^3 - x^2 + 2x + 4)\, dy\, dx = \int_0^2 (\frac{1}{2}x^4 - 4x^2 + 8)\, dx$$

$$= \left[\frac{1}{10}x^5 - \frac{4}{3}x^3 + 8x\right]_0^2 = \frac{128}{15}.$$

(b) Convert the integral $\displaystyle\int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2 + y^2 + 2x)\, dy\, dx$ to polar coordinates.

You do not have to evaluate this integral.

Answer: The region of integration is the half of the circle of radius 2, with center at the origin, that is on the left of the y-axis. In polar coordinates we get

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^2 r(r^2 + 2r\cos(\theta))\, dr\, d\theta.$$

(The function $f(x, y) = x^2 + y^2 + 2x$ has some symmetries but not all. Integrating over another region, rotated about the origin, would not generally give the same answer, and is not an equivalent answer.)

Problem 5

Find and identify all relative maxima, relative minima, and saddle points, for the function $f(x, y) = x^3 + y^2 - 6x^2 + y - 1$.

Answer: $f_x = 3x^2 - 12x$ and $f_y = 2y + 1$. Setting those equal to zero we get $3x(x - 4) = 0$, so $x = 0$ or $x = 4$, and $y = -\frac{1}{2}$. There are no singular or boundary points so we examine the points $(0, -\frac{1}{2})$ and $(4, -\frac{1}{2})$. Using the second derivative test, $f_{xx} = 6x - 12$: At $(0, -\frac{1}{2})$ that gives -12 and at $(4, -\frac{1}{2})$ it produces 12. More simply, $f_{yy} = 2$. And $f_{xy} = 0$. Thus $D = f_{xx}f_{yy} - f_{xy}^2$ is -24 at $(0, -\frac{1}{2})$, so that is a saddle point, and D gives 24 at $(4, -\frac{1}{2})$, so that is either a relative maximum or a relative minimum: Since $f_{xx} = 12 > 0$, it is a relative minimum.

Problem 6

Set up but do not evaluate an integral to compute the surface area for the section of the sphere $x^2 + y^2 + z^2 = 4$ above the xy-plane and inside the vertical cylinder $x^2 + (y-1)^2 = 1$.

Answer: We can take the sphere above the xy-plane to be the graph of $z = \sqrt{4 - x^2 - y^2}$. Then $f_x = \frac{-x}{\sqrt{4-x^2-y^2}}$ and $f_y = \frac{-y}{\sqrt{4-x^2-y^2}}$. Hence what we need to integrate is

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2} + 1} = \sqrt{\frac{4}{4 - x^2 - y^2}}.$$

The region of integration is the circle in the xy-plane with center $(0, 1)$ and radius 1. We can then set up the integral as

$$\int_{-1}^{1} \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} \sqrt{\frac{4}{4 - x^2 - y^2}} \, dy \, dx.$$

Problem 7

A thin plate covers the triangular region in the plane whose edges are the x-axis, the line $y = 2x$, and the line $x = 1$.

The density of this plate is given by the function $\delta(x, y) = 6x + 6y + 6$.

(a) Use an integral to evaluate the mass of this plate.

Answer: We integrate $\delta(x, y)$ through the triangular region,

$$\text{mass} = \int_0^1 \int_0^{2x} (6x + 6y + 6) \, dy \, dx = 14 \text{ mass units, unspecified.}$$

(b) Find the moment M_x of the plate about the x-axis.

Answer:

$$M_x = \int_0^1 \int_0^{2x} y(6x + 6y + 6) \, dy \, dx = 11.$$

(c) Find the moment M_y of the plate about the y-axis.

Answer:

$$M_y = \int_0^1 \int_0^{2x} x(6x + 6y + 6) \, dy \, dx = 10.$$

(d) Find the coordinates $(\overline{x}, \overline{y})$ of the center of mass of this plate.

Answer:

$\overline{x} = \frac{M_y}{mass} = \frac{10}{14} = \frac{5}{7}$, and $\overline{y} = \frac{M_x}{mass} = \frac{11}{14}$.

(e) Find the moment of inertia (second moment) of this plate about the x-axis, I_x.

Answer:

$$I_x = \int_0^1 \int_0^{2x} y^2(6x + 6y + 6) \, dy \, dx = 12.$$

(f) Find the radius of gyration of this plate about the x-axis.

Answer:

$$\rho = \sqrt{\frac{I_x}{mass}} = \sqrt{\frac{12}{14}} = \sqrt{\frac{6}{7}}.$$

Problem 8

Find the largest and smallest values of $f(x,y) = x^2 + y$ subject to $x^2 + y^2 = 4$.

Answer: We use Lagrange Multipliers, seeking points (x,y) where $\nabla f(x,y)$ is parallel to $\nabla g(x,y)$, for $g(x,y) = x^2 + y^2 - 4$. $\nabla f = 2x\vec{\imath} + \vec{\jmath}$ and $\nabla g = 2x\vec{\imath} + 2y\vec{\jmath}$. Where those are parallel, $\nabla f = \lambda \nabla g$ for some constant λ. That makes $2x\vec{\imath} + \vec{\jmath} = 2x\lambda\vec{\imath} + 2y\lambda\vec{\jmath}$ so $2x = 2x\lambda$ and $1 = 2y\lambda$. From the first of those equations we have $2x(1-\lambda) = 0$ so either $x = 0$ or $\lambda = 1$. If $\lambda = 1$ the second equation gives $2y = 1$ or $y = \frac{1}{2}$. So either $x = 0$ or $y = \frac{1}{2}$. Using the equation $g(x,y) = 0$, i.e. $x^2 + y^2 = 4$, if $x = 0$ then $y = \pm 2$, while if $y = \frac{1}{2}$ we have $x = \pm\frac{\sqrt{15}}{2}$. Thus we have four points to consider, $(0,2)$, $(0,-2)$, $(\frac{\sqrt{15}}{2}, \frac{1}{2})$, and $(-\frac{\sqrt{15}}{2}, \frac{1}{2})$.

We can evaluate f at each of these four points: $f(0,2) = 2$, $f(0,-2) = -2$, $f(\frac{\sqrt{15}}{2}, \frac{1}{2}) = 4\frac{1}{4}$, and $f(-\frac{\sqrt{15}}{2}, \frac{1}{2}) = 4\frac{1}{4}$. Comparing the results, we see the largest value for f is $4\frac{1}{4}$, taken at both $(\frac{\sqrt{15}}{2}, \frac{1}{2})$ and $(-\frac{\sqrt{15}}{2}, \frac{1}{2})$. while the smallest value is -2 which is taken only at $(0,-2)$.