Problem 1
Set up an integral to compute the area inside one leaf of the 5-leafed rose $r = 3\sin(5\theta)$. Be sure to make clear how you establish the limits of integration.
You do not have to evaluate this integral.
ANSWER: When $\theta = 0$, $r = 3\sin(0) = 0$. As θ increases, r first increases and then decreases, reaching 0 again when $3\sin(5\theta)$ is next 0. That will be when $5\theta = \pi$, so $\theta = \frac{\pi}{5}$. Hence the area is

$$\int_0^{\frac{\pi}{5}} \frac{1}{2} r^2\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{5}} 9\sin^2(5\theta)\, d\theta = \frac{9}{2}\int_0^{\frac{\pi}{5}} \sin^2(5\theta)\, d\theta.$$

(You were not required to evaluate this integral, but if you do you should get $\frac{9}{20}\pi$.)

Problem 2
Evaluate the integrals:

(a) $$\int e^{2x} \sin(3x)\, dx$$

ANSWER: Use integration by parts. You can choose u and dv in several ways and have the problem work out: I will use $u = \sin(3x)$ and $dv = e^{2x} dx$. Then we have $du = 3\cos(3x)$ and $v = \frac{1}{2}e^{2x}$. Hence the integral becomes

$$\int e^{2x} \sin(3x)\, dx = \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{2}\int e^{2x}\cos(3x)\, dx.$$

We now apply parts again, using $u = \cos(3x)$ and $dv = e^{2x}\, dx$, to work out this second integral. We have $du = -3\sin(3x)$ and $v = \frac{1}{2}e^{2x}$, so continuing the equalities above we get

$$\int e^{2x} \sin(3x)\, dx = \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{2}\left[\frac{1}{2}e^{2x}\cos(3x) + \frac{3}{2}\int e^{2x}\sin(3x)\, dx\right]$$

$$= \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{4}e^{2x}\cos(3x) - \frac{9}{4}\int e^{2x}\sin(3x)\, dx.$$

That last integral is the same as the one on the left: Add the last term to both sides and combine to get

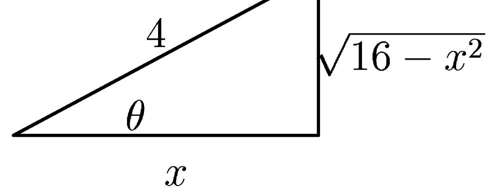
$$\frac{13}{4}\int e^{2x} \sin(3x)\, dx = \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{4}e^{2x}\cos(3x).$$

Now divide by $\frac{13}{4}$ and also remember this is an indefinite integral, to get

$$\int e^{2x} \sin(3x)\, dx = \frac{2}{13}e^{2x}\sin(3x) - \frac{3}{13}e^{2x}\cos(3x) + C.$$

(b) $$\int \frac{x^2\, dx}{\sqrt{16 - x^2}}$$

trig subsitution. If I draw a right triangle and label one acute angle θ, and label the sides as shown to the right, we can read off the relationships $x = 4\cos\theta$ and $\sqrt{16-x^2} = 4\sin\theta$, and hence $dx = -4\sin\theta\, d\theta$.



Putting these into the integral we get

$$\int \frac{x^2\, dx}{\sqrt{16-x^2}} = \int \frac{16\cos^2\theta}{4\sin\theta}(-4\sin\theta\, d\theta)$$

$$= -16\int \cos^2\theta\, d\theta = -8\int (1+\cos(2\theta))\, d\theta = -8\theta - 4\sin(2\theta) + C.$$

Now we need to substitute back in terms of the original variable x. We can use $\sin(2\theta) = \sin(\theta+\theta) = 2\sin(\theta)\cos(\theta)$ from the formulas included with the exam. In one form the answer is $-8\cos^{-1}\frac{x}{4} - \frac{x\sqrt{16-x^2}}{2} + C$: There are other, equivalent, forms.

Problem 3

Use Simpson's (Parabolic) Rule to evaluate the integral $\int_1^9 (x^2+1)\, dx$, using 4 subintervals.

ANSWER: If we go from 1 to 9 in four steps the intermediate points will be 3, 5, and 7, with a step width $h = 2$. The function x^2+1 evaluated at 1, 3, 5, 7, and 9 respectively gives 2, 10, 26, 50, and 82. Combining these in the formula for Simpson's rule we get

$$\frac{2}{3}[2 + 4\times 10 + 2\times 26 + 4\times 50 + 82] = \frac{2}{3}\times 376 = \frac{752}{3} = 250.6666\ldots.$$

Problem 4

For the differential equation $y'' - 6y' + 10y = 0$, we can express the set of all solutions as $y = e^{3x}(C_1\cos x + C_2\sin x)$. The first part of this problem asks you find that solution. Hence clearly you will get no credit for simply writing down the solution, credit will come from showing how to find the solution, but at the same time you can check your work and also be sure of proceeding to the last part of the problem.

Part I: Find all solutions of $y'' - 6y' + 10y = 0$.

- What is the characteristic (associated) polynomial for this differential equation?

  ANSWER: $r^2 - 6r + 10 = 0$

- What are the roots of the characteristic polynomial?

  ANSWER: $r = \frac{6\pm\sqrt{36-40}}{2} = 3 \pm i$

- What role do the roots you found play in writing out the solution $y = e^{3x}(C_1\cos x + C_2\sin x)$? Show where each number in the solutions came from as a part of the roots.

  ANSWER: When we have complex conjugate roots $a \pm bi$, our original solution form would have produced a combination of exponential functions with complex exponents. We found this could be rewritten as e^{ax} times the combination $C_1\cos bx + C_2\sin bx$. Since in this case we have $a = 3$ and $b = 1$, we get the combination shown.

ANSWER: Using the table for undetermined coefficients, with the right hand side a multiple of e^{4x}, we try the particular solution $y_p = Ce^{4x}$ where C has to be determined. Then $y_p' = 4Ce^{4x}$ and $y_p'' = 16Ce^{4x}$. Putting these in the equation and factoring out e^{4x} we get $e^{4x}(16C - 24C + 10C) = 4e^{4x}$ so $2C = 4$ and $C = 2$. Thus our particular solution is $y_p = 2e^{4x}$. Combining that with the general solution for the homogeneous equation found above we get $y(x) = e^{3x}(C_1 \cos x + C_2 \sin x) + 2e^{4x}$.

Part III: Find the solution of $\quad y'' - 6y' + 10y = 4e^{4x} \quad$ satisfying $\quad y(0) = 5 \quad$ and $\quad y'(0) = 16$.

ANSWER: We use the general solution found in Part II and determine the correct choices for C_1 and C_2. Since we will need $y'(0)$, we take the derivative to get $y'(x) = 3e^{3x}(C_1 \cos x + C_2 \sin x) + e^{3x}(-C_1 \sin x + C_2 \cos x) + 8e^{4x}$. We can now calculate both y and y' at 0: $y(0) = C_1 + 2$, which must equal 5, so $C_1 = 3$. $y'(0) = 3C_1 + C_2 + 8$, so $16 = 9 + C_2 + 8$ and $C_2 = -1$. Hence the solution is $y(x) = e^{3x}(3\cos x - \sin x) + 2e^{4x}$.

Problem 5

Use Taylor's theorem to estimate the error if we approximate $e^{\frac{1}{2}}$ using the terms of the Maclaurin series for e^x through the term $\frac{x^6}{6!}$. Your estimate should take into account the value $\frac{1}{2}$ at which we are applying this.

You may use the fact $e^{\frac{1}{2}} < 2$.

ANSWER: The Taylor remainder term for e^x and $n = 6$ will be $R_6(x) = \frac{e^c}{7!}x^7$, where c is some number between 0 and x. (I have simplified from $(x-a)^7$ to x^7 because we are using a Maclaurin series, and also used the fact that all derivatives of e^x, and the 7^{th} in particular, are just e^x.) We need to evaluate $R_6(x)$ for $x = \frac{1}{2}$, so we have $\frac{e^c}{7!}\left(\frac{1}{2}\right)^7$. We don't know just what c is but e^x is an increasing function, so e^c for $0 \le c \le \frac{1}{2}$ will be at most $e^{\frac{1}{2}}$. We know that is less than 2. Hence we can say

$$R_6(x) < \frac{2 \times (\frac{1}{2})^7}{7!} = \frac{2}{128 \times 5040} = \frac{1}{322560} \approx 3.1 \times 10^{-6}.$$

(As a check, if you sum the terms $1 + \frac{1}{2} + \frac{(1/2)^2}{2!} + \cdots + \frac{(1/2)^6}{6!}$ you get about 1.64871962. My calculator says $e^{1/2}$ is about 1.6487212707, for a difference of about 1.652×10^{-6}, so the error is within the bound we found.)

Problem 6

For the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-3x)^n}{n}$ determine the interval of convergence (convergence set).

ANSWER: If we use the ratio test on the of the terms in this series, the ratio of successive terms is

$$\frac{\frac{(-3x)^{n+1}}{n+1}}{\frac{(-3x)^n}{n}} = \frac{n}{n+1}(-3x).$$

We want the limit of the absolute value of that ratio, as $n \to \infty$, which is $\rho = |3x|$ since $\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 1$. Hence the series will converge, absolutely, whenever $|3x| < 1$, i.e. whenever $-\frac{1}{3} < x < \frac{1}{3}$. Next we need to check what happens at the endpoints $x = -\frac{1}{3}$ and $x = \frac{1}{3}$. At $x = -\frac{1}{3}$ the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(1)^n}{n}$ which is the harmonic series, so it diverges. At $x = \frac{1}{3}$ the series becomes

$\sum_{n=1} \frac{}{n}$, the alternating harmonic series, so it converges. Hence the set of x values for which the series converges is $-\frac{1}{3} < x \le \frac{1}{3}$, or expressed as an interval it is $(-\frac{1}{3}, \frac{1}{3}]$.

Problem 7

The equation $25x^2 + 9y^2 = 225$ describes a conic section. For this curve, find:

(a) where it crosses the x-axis, if it does at all (coordinates of point(s))

ANSWER: First divide through the equation by 225. We get $\frac{x^2}{9} + \frac{y^2}{25} = 1$, so the curve is an ellipse with its long axis vertical and its foci on the y-axis. It will cross the x-axis where $y = 0$, i.e. where $x^2 = 9$, $x = \pm 3$, so the coordinates are $(-3, 0)$ and $(3, 0)$.

(We have usually used the letters a and b to denote the quantities which here have the values $\sqrt{25} = 5$ and $\sqrt{9} = 3$, respectively, and I will use those letters below.)

(b) where it crosses the y-axis, if it does at all (coordinates of point(s))

ANSWER: Using the modified equation to see where the curve crosses the y-axis, where $x = 0$, we get $y^2 = 25$ so $y = \pm 5$ and the points are $(0, 5)$ and $(0, -5)$.

(c) its focus or foci (coordinates of point(s))

ANSWER: The foci are on the y-axis at $(0, \pm c)$, where $c = \sqrt{a^2 - b^2} = \sqrt{25 - 9} = 4$. Thus the foci are at $(0, 4)$ and $(0, -4)$.

(d) its eccentricity (a number)

ANSWER: The eccentricity is $\frac{c}{a} = \frac{4}{5}$.

Problem 8

Let $\vec{u} = 2\vec{\imath} + 2\vec{\jmath} - \vec{k}$ and $\vec{v} = 3\vec{\jmath} + 4\vec{k}$ for all parts of this problem.

(a) Find a vector of unit length in the direction of $\vec{v}$.
(Express your answer as a combination of $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$.)

ANSWER: We calculate the length of $\vec{v}$ as $|\vec{v}| = \sqrt{0 + 9 + 16} = 5$. Then the unit vector in the same direction as $\vec{v}$ is $\frac{1}{5}\vec{v} = 0\vec{\imath} + \frac{3}{5}\vec{\jmath} + \frac{4}{5}\vec{k}$.

(b) Find $\cos(\theta)$ where θ is the angle between $\vec{u}$ and $\vec{v}$.

ANSWER: We can find $\cos\theta$ as $\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{0+6-4}{3 \times 5} = \frac{2}{15}$.

(c) Find the scalar projection of $\vec{u}$ upon $\vec{v}$.

ANSWER: The scalar projection is $|\vec{u}|\cos\theta = 3 \times \frac{2}{15} = \frac{2}{5}$.

(d) Find the vector projection of $\vec{u}$ upon $\vec{v}$.

ANSWER: The vector projection has magnitude the scalar projection we just computed (since that was positive) and direction the same as the unit vector we found in part (a), so we can get the vector projection as $\frac{2}{5}(0\vec{\imath} + \frac{3}{5}\vec{\jmath} + \frac{4}{5}\vec{k}) = 0\vec{\imath} + \frac{6}{25}\vec{\jmath} + \frac{8}{25}\vec{k}$.

Problem 9

$5x - 3y + 2z = 8$.

ANSWER: We need to find a vector $\vec{n}$ perpendicular to the plane we have to describe. But that plane is parallel to $5x - 3y + 2z = 8$, so any vector perpendicular to one is perpendicular to the other, and we can read off that $\vec{n} = 5\vec{\imath} - 3\vec{\jmath} + 2\vec{k}$ is perpendicular to the plane given in the problem, so we use that. Now we can restate the problem: Find an equation for the plane through $(2, 5, -6)$ that is perpendicular to $5\vec{\imath} - 3\vec{\jmath} + 2\vec{k}$. In this form we can just write down the equation as $5(x-2) - 3(y-5) + 2(z+6) = 0$, or $5x - 3y + 2z = -17$.

(b) Find equations for the line which passes through $(2, 5, -6)$ and is perpendicular to both of the planes in (a). You may express these in parametric or symmetric form.

ANSWER: We already know that $\vec{n} = 5\vec{\imath} - 3\vec{\jmath} + 2\vec{k}$ gives the correct direction for the line. Hence we can restate the problem as: Find equations for the line in the direction of $5\vec{\imath} - 3\vec{\jmath} + 2\vec{k}$ through the point $(2, 5, -6)$. In parametric form we can write this as $x = 2 + 5t, = 5 - 3t$, and $z = -6 + 2t$. To express these in symmetric form, solve each for t and set the results equal: $\frac{x-2}{5} = \frac{y-5}{-3} = \frac{z+6}{2}$.

Problem 10

For each series, tell whether it converges or diverges. Be sure to give reasons for your answers.

(a) $$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

ANSWER: (For these series there is generally more than one way to justify the answer. I have given one that I think is among the easiest.)

After some algebraic simplifying, the ratio of successive terms for this series is $\frac{n+1}{n}\frac{3^n}{3^{n+1}}$: Taking the limit as $n \to \infty$ we get $\frac{1}{3}$, which is less than 1, so this series converges by the ratio test.

(b) $$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$$

ANSWER: The denominator is $n^{1/3}$ so this is a p-series. Since $p = \frac{1}{3}$ is less than 1, the series diverges.

For each series, tell whether it diverges, converges absolutely, or converges conditionally. Be sure to give reasons for your answers.

(c) $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{1000n + 1,000,000}$$

ANSWER: As $n \to \infty$, the individual terms of this series approach $\frac{1}{1000}$. Since that is not zero, the series must diverge (the n^{th} term test).

(d) $$\sum_{n=1}^{\infty} (-1)^n \frac{5}{n}$$

ANSWER: This series is the same as $5 \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ if that series converges. But that series amounts to the alternating harmonic series, which does converge, so the original series converges. (You can also easily use the Alternating Series test.) However, if we make all the terms positive, we get something equivalent to the harmonic series and that diverges. Hence this series converges conditionally.

Suppose the position of an object is represented by the vector $\vec{r}(t) = \cos(2t)\vec{\imath} - 3t\vec{\jmath} + \sin(2t)\vec{k}$ at any time t.

(a) Where is the object (coordinates of a point) at time $t = \frac{\pi}{2}$?

ANSWER: The vector $\vec{r}(\frac{\pi}{2})$ will give the position, $\vec{r}(\frac{\pi}{2}) = \cos\pi\,\vec{\imath} - \frac{3\pi}{2}\,\vec{\jmath} + \sin\pi\,\vec{k} = -\vec{\imath} - \frac{3\pi}{2}\,\vec{\jmath} + 0\vec{k}$, as a vector emanating from the origin. The point in space that indicates is $(-1, -\frac{3\pi}{2}, 0)$.

(b) What is the velocity (vector) of the object at time $t = \frac{\pi}{2}$?

ANSWER: We take the derivative and evaluate at that value of t: $\vec{r}'(t) = -2\sin(2t)\vec{\imath} - 3\vec{\jmath} + 2\cos(2t)\vec{k}$, so $\vec{r}'(\frac{\pi}{2}) = 0\vec{\imath} - 3\vec{\jmath} - 2\vec{k}$.

(c) What is the acceleration (vector) of the object at time $t = \frac{\pi}{2}$?

ANSWER: Now go on to the second derivative: $\vec{r}''(t) = -4\cos(2t)\vec{\imath} + 0\vec{\jmath} - 4\sin(2t)\vec{k}$, so $\vec{r}''(\frac{\pi}{2}) = 4\vec{\imath} + 0\vec{\jmath} + 0\vec{k}$.

(d) Find equations in symmetric form for the tangent line to the path of this object at $t = \frac{\pi}{2}$.

ANSWER: We know from (b) the direction is given by $\vec{r}'(\frac{\pi}{2}) = 0\vec{\imath} - 3\vec{\jmath} - 2\vec{k}$, and from (a) the line goes through $(-1, -\frac{3\pi}{2}, 0)$. We are asked for symmetric form: It may be easier to think of the line in parametric form first and then convert. We have $x = -1 + 0t$, $y = -\frac{3\pi}{2} - 3t$, and $z = 0 - 2t$, so $\frac{x+1}{0} = \frac{y+\frac{3\pi}{2}}{-3} = \frac{z-0}{-2}$.

Problem 12

(a) Evaluate the integral $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$.

ANSWER: This is an improper integral: The denominator goes to zero at each end of the interval. Hence we must evaluate it using limits. We can pick any point between -1 and 1, I will use 0, to break this into two pieces: Rewriting the problem as

$$\lim_{a\to -1^+} \int_a^0 \frac{1}{\sqrt{1-x^2}}\,dx + \lim_{b\to 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}}\,dx,$$

and using $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$, we get

$$\lim_{a\to -1^+} [\arcsin(0) - \arcsin(a)] + \lim_{b\to 1^-} [\arcsin(b) - \arcsin(0)] = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi.$$

(b) Evaluate the integral $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$.

Hint: This is much easier if you use some geometry...

ANSWER: This will be just the area under the semicircle of radius 1, centered at the origin, extending from $(-1, 0)$ to $(1, 0)$ and crossing the y-axis at $(0, 1)$. That is one half of a circular disk of radius 1, so the answer is $\frac{1}{2}\left(\pi\, 1^2\right) = \frac{\pi}{2}$.

You can do this with much more work using a trig substitution.