Mathematics 222, Lecture 2 (Wilson) Your Name: ____________________________________

Exam 2 April 25, 2002

- **Write your answers to the seven problems in the spaces provided. If you must continue an answer somewhere other than immediately after the problem statement, be sure (a) to tell where to look for the answer, and (b) to label the answer wherever it winds up. In any case, be sure to circle your final answer to each problem.**
- **On the other side of this sheet there is a collection of facts and formulas.**
- **Wherever applicable, leave your answers in exact forms (using π, e, $\sqrt{3}$, $\ln(2)$, and similar numbers) rather than using decimal approximations.**
- **You may refer to notes you have brought in on one sheet of paper or two index cards, as announced in class.**
- **There is scratch paper on the back of the last page.**

BE SURE TO SHOW YOUR WORK, AND EXPLAIN WHAT YOU DID. YOU MAY RECEIVE REDUCED OR ZERO CREDIT FOR UNSUBSTANTIATED ANSWERS. ("I did it on my calculator" and "I used a formula from the book" are not sufficient substantiation...)

Problem	Points	Score
1	15	
2	13	
3	14	
4	14	
5	15	
6	15	
7	14	
TOTAL	100	

Some formulas, identities, and numeric values you might find useful:

Values of trig functions:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	—

Trig facts:

1. $\tan\theta = \frac{\sin\theta}{\cos\theta}$
2. $\sec\theta = \frac{1}{\cos\theta}$
3. $\sin^2\theta + \cos^2\theta = 1$
4. $\sec^2\theta = \tan^2\theta + 1$
5. $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
6. $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
7. $\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$
8. $\sin^2 x = \frac{1}{2}(1-\cos 2x)$
9. $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

Derivative formulas:

1. $\frac{d}{dx}\tan x = \sec^2 x$
2. $\frac{d}{dx}\sec x = \sec x\,\tan x$
3. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
4. $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
5. $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$
6. $\frac{d}{dx}\ln x = \frac{1}{x}$
7. $\frac{d}{dx}e^x = e^x$

Integral formulas:

1. $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C$, if $n \neq -1$
2. $\int \frac{1}{u}\,du = \ln|u| + C$
3. $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$
4. $\int \frac{du}{1+u^2} = \tan^{-1} u + C$
5. $\int \sec(u)\,du = \ln|\sec(u)+\tan(u)| + C$
6. $\int u\,dv = uv - \int v\,du$

Algebra formulas:

1. $\ln(xy) = \ln(x) + \ln(y)$
2. $a^{x+y} = a^x a^y$
3. $a^b = e^{b\ln a}$

Problem 1 (15 points)
Find the interval of convergence for the power series:

(a) $$\sum_{n=0}^{\infty} \frac{(x-3)^n}{3^n}$$

(b) $$\sum_{n=0}^{\infty} \frac{2^n\, x^n}{(n+3)!}$$

(c) $$\sum_{n=0}^{\infty} \frac{n!\, x^n}{(1000)^n}$$

Problem 2 (13 points)
Find an equation for the conic section that:

1. Has foci at $(0, 13)$ and $(0, -13)$.
2. Crosses the y-axis at $(0, 5)$ and $(0, -5)$.
3. Does not cross the x-axis at all.

You might wish to sketch the graph of this curve for your own purposes, but grading will be based on your reasoning and the equation you produce.

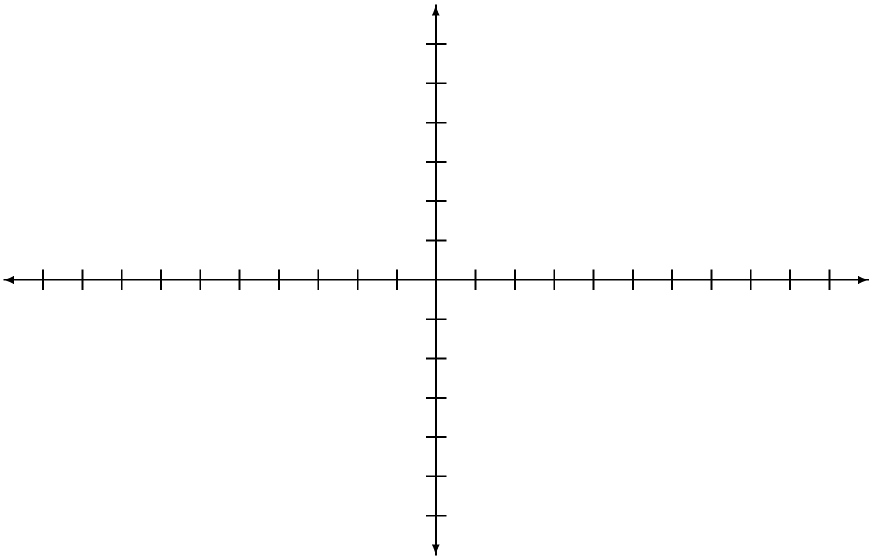
Problem 3 (14 points)
Consider the equation

$$x^2 - 6x + 16y + 41 = 0.$$

(a) What kind of curve does this represent (circle, ellipse, parabola, or hyperbola)?

(b) Where should the center (h, k) of the coordinate system be moved so as to put this curve in "standard position"?

(c) Sketch the curve on the axes below. Show both original and translated axes, and label significant items. (What items are significant depends on what the curve is. For a circle, show the center and radius. For an ellipse, show the foci and the ends of the major and minor axes. For a parabola, show the focus and directrix and vertex. For a hyperbola, show the foci and vertices and asymptotes.)



Problem 4 (14 points)
The Maclaurin series for $\ln(1+x)$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

(a) Show that this <u>is</u> the Maclaurin series for $\ln(x+1)$. We have seen in class at least two quite different ways to get this particular series.

(b) Use some number of initial terms of that series to approximate $\ln(1.1)$ to within ± 0.00003. Be sure (a) to show, using calculus methods, that you have used enough terms (Do not just calculate $\ln(1.1)$ on a calculator and compare it to your result!) and (b) to show the actual answer, the sum of those terms.

(c) Based on a calculus result, is your answer larger or smaller than the actual value of $\ln(1.1)$?

Problem 5 (15 points)
Find all solutions of the differential equation

$$y'' + 3y' - 4y = 2e^x.$$

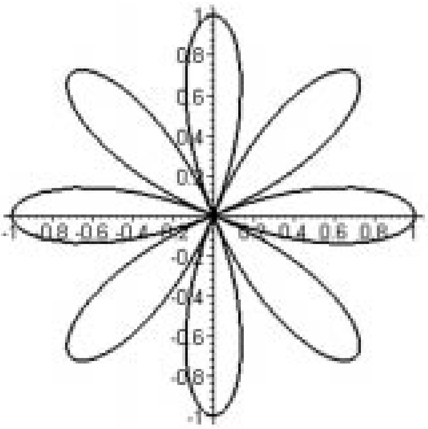
(a) Find all solutions of $y'' + 3y' - 4y = 0$:

(b) Find a solution of $y'' + 3y' - 4y = 2e^x$:

(c) Your answer to the original problem:

Problem 6 (15 points)

Find the area inside one leaf of the 8-leaved rose $r = \cos(4\theta)$. Be sure to show how you find the limits of integration: Do not just look at the picture and guess! (Do go ahead and evaluate the integral and find the area. The integral should be one you can do, perhaps getting help from the formulae given earlier in this exam.)



Problem 7 (14 points)
Suppose a polar curve $r = f(\theta)$ has the property that, at every point (r, θ) on the curve, the angle ψ between the tangent line and the radius vector satisfies $\tan(\psi) = r\,\theta$. Suppose in addition that when $\theta = 1$ (radian), $r = 1$. What is $f(\theta)$?
(The answer is $r = f(\theta) = \ln(\theta) + 1$. Your task is to show that this must be the case, not just to show that this function has the properties given above.)

Scratch Paper