Problem 1 (15 points)
Find the interval of convergence for the power series:

(a) $$\sum_{n=0}^{\infty} \frac{(x-3)^n}{3^n}$$

Using the ratio test on the absolute value of the terms in the series we compute (I've left off the first algebraic steps) $\lim_{n\to\infty} \frac{|x-3|}{3} = \frac{|x-3|}{3}$ and solve to see what values for x make this less than 1: $\frac{|x-3|}{3} < 1$ gives $-3 < x-3 < 3$ or $0 < x < 6$. So the interval of convergence is from 0 to 6 except that we don't yet know what happens at the end points. At $x = 0$ and at $x = 6$ the series becomes respectively

$$\sum_{n=0}^{\infty} (-1)^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} 1^n.$$

Both of those diverge, so the interval of convergence is $0 < x < 6$ or $(0, 6)$.

(Another way to do this one: The series is actually geometric with ratio $\frac{x-3}{3}$ and we know that will converge only when the ratio is between ± 1. This leads to the same conclusion.)

(b) $$\sum_{n=0}^{\infty} \frac{2^n x^n}{(n+3)!}$$

We follow the same procedure and get for the limiting ratio $\lim_{n\to\infty} \frac{2|x|}{n+3}$. This gives 0 for any x, so this series converges absolutely for all x, i.e. the interval of convergence is the whole real line.

(c) $$\sum_{n=0}^{\infty} \frac{n!\, x^n}{(1000)^n}$$

This time the ratio test leads us to $\lim_{n\to\infty} \frac{(n+1)|x|}{1000}$ which is infinite for any fixed $x \neq 0$. Hence this series converges <u>only</u> for $x = 0$.

Problem 2 (13 points)
Find an equation for the conic section that:

1. Has foci at $(0, 13)$ and $(0, -13)$.
2. Crosses the y-axis at $(0, 5)$ and $(0, -5)$.
3. Does not cross the x-axis at all.

Since the curve has its foci on the y-axis, one just as far above the origin as the other is below, the center is at the origin. If an ellipse or circle had its center there, it would cross both axes, but we are given that it does not cross the x-axis. So it has to be a parabola or hyperbola. But a parabola would have only one focus. So the curve must be a hyperbola.

A hyperbola with center at the origin and foci on the y-axis has an equation of the form $-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ for some constants a and b. The places it crosses the y-axis will be at $(0, \pm a)$, so $a = 5$. The distance from the center to a focus is c, where $c^2 = a^2 + b^2$, and in this case $c = 13$. So $13^2 = 5^2 + b^2$, $b^2 = 169 - 25 = 144$, $b = 12$. Thus the equation is

$$-\frac{x^2}{144} + \frac{y^2}{25} = 1.$$

Problem 3 (14 points)
Consider the equation

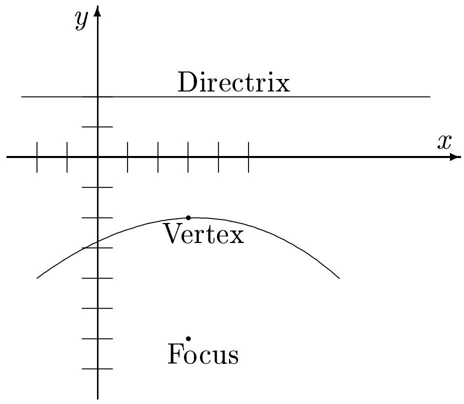
$$x^2 - 6x + 16y + 41 = 0.$$

(a) What kind of curve does this represent (circle, ellipse, parabola, or hyperbola)?

(b) Where should the center (h, k) of the coordinate system be moved so as to put this curve in "standard position"?

(c) Sketch the curve. Show both original and translated axes, and label significant items.

This is a parabola: It has x squared but y only to the first power, and it does not have an xy term. (If it did have an xy term the rotation of axes might change what we had for x^2 and for y^2.) Or, one can calculate the discriminant, $B^2 - 4AC$, where both B and C are zero: This gives zero and we know that implies the curve is a parabola. Or, one can answer this part after doing the rest of the problem.
Completing the square, $(x-3)^2 + 16y = -32$. Putting this in the standard form we used in class, $16(y+2) = -(x-3)^2$ or $y + 2 = -\frac{1}{16}(x-3)^2$. Thus the center is at $x = 3$ and $y = -2$. The parameter p we used is such that $\frac{1}{4p} = \frac{1}{16}$ so $p = 4$. Thus the focus is 4 units on one side of the center and the directrix is 4 units on the other side. We now have the parabola in the form $y' = -\frac{1}{4p}(x')^2$ where $y' = y + 2$ and $x' = x - 3$. This will open downward along the y' axis, so the focus will be 4 units below the center and the directrix will be the horizontal line 4 units above the center. The vertex, at the center, will be at $(3, -2)$. The focus is at $(3, -6)$. The directrix is the line $y = 2$.



Problem 4 (14 points)
The Maclaurin series for $\ln(1+x)$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

(a) Show that this <u>is</u> the Maclaurin series for $\ln(x+1)$. We have seen in class at least two quite different ways to get this particular series.

(b) Use some number of initial terms of that series to approximate $\ln(1.1)$ to within ± 0.00003. Be sure (a) to show, using calculus methods, that you have used enough terms (Do not just calculate $\ln(1.1)$ on a calculator and compare it to your result!) and (b) to show the actual answer, the sum of those terms.

(c) Based on a calculus result, is your answer larger or smaller than the actual value of $\ln(1.1)$?

The easier way to show that is the Maclaurin series is to use a geometric series and term-by-term integration: The derivative of $f(x) = \ln(1+x)$ is $\frac{1}{1+x}$. That looks like the sum of the geometric series $1 - x + x^2 - x^3 + x^4 + \cdots$ with initial term $a = 1$ and ratio $r = -x$. Since that gave the derivative of what we were after, the series we want can be had by integrating this one. This produces $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ as given. To use this series for $\ln(1+x)$ to compute $\ln(1.1)$ we need to use $x = 0.1$ so that $1+x = 1.1$. At this positive value for x the series is an alternating series, so we can use the alternating series estimation theorem both to decide on the number of terms and to find whether our approximation is large or small. To get the remainder less than 0.00003 in size we need to use terms until the first term we omit is smaller than that. That occurs when $\frac{(0.1)^{n+1}}{n+1} < 0.00003$. If we try $n = 2$ we get $0.000333\ldots$ which is not small enough, but if we try $n = 3$ we get 0.000025 which is. So we use for our approximation the terms through the third degree, $\ln(1.1) \approx (.1) - \frac{(.1)^2}{2} + \frac{(.1)^3}{3} = 0.095333\ldots$. Since the first term we omitted, $-\frac{(0.1)^4}{4}$, is negative, the sum of the whole series (and hence the actual value of $\ln(1.1)$ is smaller than this approximation: Thus our approximation is too large.

Problem 5 (15 points)
Find all solutions of the differential equation

$$y'' + 3y' - 4y = 2e^x.$$

(a) Find all solutions of $y'' + 3y' - 4y = 0$:

The characteristic equation is $r^2 + 3r - 4 = 0$. The roots are $r = -4$ and $r = 1$. Hence the solutions to the homogeneous equations are $y_h = C_1 e^{-4x} + C_2 e^x$.

(b) Find a solution of $y'' + 3y' - 4y = 2e^x$:

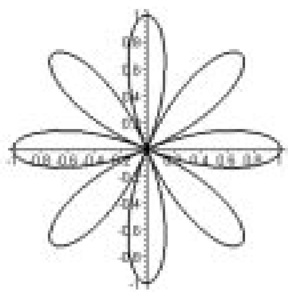
Using the method of undetermined coefficients, we look at the right hand side $2e^x$ and see a multiple of e^{nx} where $n = 1$. Since 1 is a single root of the characteristic equation, we use Cxe^x as our prototype solution y_p. Then $y_p' = C(e^x + xe^x)$ and $y_p'' = C(2e^x + xe^x)$. Putting these into the equation, $C(2e^x + xe^x) + 3C(e^x + xe^x) - 4Cxe^x = 2e^x$. The terms in xe^x cancel out. The rest give us $5Ce^x = 2e^x$, so $C = \frac{2}{5}$. Hence our solution is $y_p = \frac{2}{5}xe^x$.

(c) Your answer to the original problem:

For this we just combine the answers to (a) and (b) to get $y = C_1 e^{-4x} + C_2 e^x + \frac{2}{5}xe^x$.

Problem 6 (15 points)

Find the area inside one leaf of the 8-leaved rose $r = \cos(4\theta)$. Be sure to show how you find the limits of integration: Do not just look at the picture and guess! (Do go ahead and evaluate the integral and find the area. The integral should be one you can do, perhaps getting help from the formulae given earlier in this exam.)



The area inside any leaf is the same as any other: I will use the one extending along the positive x-axis. To find the upper and lower boundaries of this leaf we solve the equation $r = 0$, i.e. $\cos 4\theta = 0$. Since the cosine is zero at $\frac{\pi}{2}$ and $-\frac{\pi}{2}$, the first θ values on each side of zero that make $\cos 4\theta = 0$ are $\theta = \frac{\pi}{8}$ and $\theta = -\frac{\pi}{8}$. Thus we calculate the area in one leaf as

$$\frac{1}{2}\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \cos^2 4\theta \, d\theta = \frac{1}{2}\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \frac{1}{2}(1 + \cos 8\theta)\, d\theta = \frac{1}{4}\left[\theta + \frac{1}{8}\sin 8\theta\right]_{-\frac{\pi}{8}}^{\frac{\pi}{8}} = \frac{\pi}{16}.$$

Problem 7 (14 points)
Suppose a polar curve $r = f(\theta)$ has the property that, at every point (r, θ) on the curve, the angle ψ between the tangent line and the radius vector satisfies $\tan(\psi) = r\,\theta$. Suppose in addition that when $\theta = 1$ (radian), $r = 1$. What is $f(\theta)$?

We know that $\tan\psi = \frac{r}{\frac{dr}{d\theta}}$, so we have $\frac{r}{\frac{dr}{d\theta}} = r\theta$. Rearranging, $dr = \frac{d\theta}{\theta}$.

Integrating we get $r = \ln\theta + C$. Putting in the condition that $r(1) = 1$ gives $1 = \ln 1 + C$ so $C = 1$. Hence $r = f(\theta) = \ln\theta + 1$.