Final Exam May 7, 2006

ANSWERS

Problem 1 (24 points)
Find the derivative $D_x y$ for:

(a) $y = \sin(x)\,\cos(x)$

ANSWER: We need to use the product rule: $D_x y = \sin(x) D_x \cos(x) + \cos(x) D_x \sin(x) = -\sin^2(x) + \cos^2(x)$.

(b) $y = \dfrac{x^3}{\sin(x)}$

ANSWER: We could either use the quotient rule or rewrite the function as $y = x^3(\sin(x))^{-1}$. Using the quotient rule we get $D_x y = \frac{3x^2 \sin(x) - x^3 \cos(x)}{\sin^2(x)}$.

(c) $y = \sin^{-1}(x^2)$ (If you prefer the other notation, $y = \arcsin(x^2)$)

ANSWER: We need the formula $D_x\left(\sin^{-1}(u)\right) = \frac{1}{\sqrt{1-u^2}} D_x u$. Then $D_x y = \frac{1}{\sqrt{1-x^4}} \times 2x = \frac{2x}{\sqrt{1-x^4}}$.

(d) $y = \ln[(2-x)\,(1+x^3)]$.

ANSWER: The way this is written we would have to combine the product rule with the chain rule and the derivative of the logarithm. But using rules for logarithms we can simplify it to $y = \ln(2-x) + \ln(1+x^3)$ and then $D_x y = \frac{1}{2-x} \times (-1) + \frac{1}{1+x^3} \times (3x^2) = -\frac{1}{2-x} + \frac{3x^2}{1+x^3}$.

Problem 2 (18 points)
Find an equation for the tangent line to the graph of $y = \cos^{-1}(2x)$ (or in other notation $y = \arccos(2x)$, at the point $(\frac{1}{4}, \frac{\pi}{3})$. which could be rewritten as $y = -\frac{4}{\sqrt{3}}x + \frac{1}{\sqrt{3}} + \frac{\pi}{3}$.
ANSWER: We could remember a formula for the derivative of the inverse cosine, but it is perhaps easier to remember that $\arccos(2x) + \arcsin(2x) = \frac{\pi}{2}$ so $y = \frac{\pi}{2} - \arcsin(2x)$ and then use the derivative of the inverse sine. We have $y' = -\frac{1}{\sqrt{1-4x^2}} \times 2 = \frac{-2}{\sqrt{1-4x^2}}$. So the slope at $x = \frac{1}{4}$ will be $y'(\frac{1}{4}) = -\frac{2}{\sqrt{\frac{3}{4}}} = -\frac{4}{\sqrt{3}}$. Hence the tangent line can be written as $y - \frac{\pi}{3} = -\frac{4}{\sqrt{3}}\left(x - \frac{1}{4}\right)$ which could be rewritten as $y = -\frac{4}{\sqrt{3}}x + \frac{1}{\sqrt{3}} + \frac{\pi}{3}$.

Problem 3 (24 points)
Evaluate the integrals:

(a) $\displaystyle\int x^2 \cos(x^3+5)\,dx$.

ANSWER: Since the derivative of x^3+5 is x^2 it looks useful to let $u = x^3+5$. Then $du = 3x^2\,dx$, so the integral becomes $\frac{1}{3}\displaystyle\int \cos(u)\,du = \frac{1}{3}\sin(u) + C = \frac{1}{3}\sin(x^3+5) + C$.

(b) $\displaystyle\int \frac{dx}{4+x^2}$.

ANSWER: This looks a lot like an integral that would yield an arctangent. If we rewrite the denominator as $4(1+(\frac{x}{2})^2$ and then let $u = \frac{x}{2}$ so that $du = \frac{1}{2}\,dx$, the integral becomes $\frac{1}{4} \times 2 \displaystyle\int \frac{du}{1+u^2} = \frac{1}{2}\arctan(u) + C = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$.

Problem 4 (20 points)
The Fundamental Theorems of Calculus tell us relationships between the derivative and the integral. Using words, equations, pictures, whatever you find helpful, explain what these theorems say.
(I won't worry about which one you call the First theorem or the Second, different books don't even agree on that. There is no one right answer to this question! Try to describe what is going on, what the relationships between derivative and integral are that are so important.)

(a) One of the Fundamental Theorems of Calculus:

ANSWER: The derivative of a function defined by an integral is the integrand: More specifically if we let $F(x)$ be defined as the integral from some point a to x of a given function $f(t)$, then at any particular x value the rate at which F is changing is just the value of f at that x value.

(b) The other Fundamental Theorem of Calculus:

ANSWER: If we want to evaluate a definite integral and we can find some antiderivative of the function being integrated, we can evaluate that antiderivative at the upper and lower end points on the integral and subtract and that will give the same value that the definition of the definite integral as a limit of sums would give.

Problem 5 (16 points)
Find the area between the curves $y = x^2$ and $y = 2x - x^2$.
ANSWER: The two curves meet where $x^2 = 2x - x^2$, $2x^2 = 2x$, $x^2 = x$, $x(x-1) = 0$, i.e. where $x = 0$ or $x = 1$. To the left of $x = 0$ the graph of $y = x^2$ lies above $y = 2x - x^2$, and also to the right of $x = 1$. So the only finite region between the curves is the region where $0 \leq x \leq 1$, and on that interval the $y = x^2$ curve is the lower one. So the area we want can be computed as $\int_0^1 \left(2x - x^2 - x^2\right)\, dx$
$= \int_0^1 (2x - 2x^2)\, dx = 2\int_0^1 (x - x^2)\, dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left[(\frac{1}{2} - \frac{1}{3}) - (0)\right] = 2 \times \frac{1}{6} = \frac{1}{3}$.

Problem 6 (18 points)
Find all solutions of the equation $\dfrac{dy}{dx} + \dfrac{1}{x}y = 2e^{x^2}$.
You may assume $x > 0$.
ANSWER: This is set up for us as a 1^{st}-order linear differential equation, $\frac{dy}{dx} + P(x)y = Q(x)$, where $P(x) = \frac{1}{x}$ and $Q(x) = 2e^{x^2}$. Calculating $\int P(x)\, dx = \int \dfrac{dx}{x} = \ln x\ (+C)$ where we ignore the constant, we get the integrating factor $e^{lnx} = x$. Now we can multiply the original equation by that integrating factor and get $x\frac{dy}{dx} + y = 2xe^{x^2}$: The left side is exactly the derivative of xy with respect to x, so if we integrate we get $xy = \int 2xe^{x^2}\, dx$. Letting $u = x^2$ so $du = 2x\, dx$ we get $xy = e^{x^2} + C$. So $y = \frac{1}{x}\left(e^{x^2} + C\right)$.

Problem 7 (18 points)
Let $y = 2x^3 + 3x^2 - 36x - 4$.

(a) On which intervals of real numbers is this an increasing function? A decreasing function? Give reasons based on calculus, not just looking at the graph on a calculator.

ANSWER: We can tell where the function is increasing or decreasing by examining the sign of its derivative. $y' = 6x^2 + 6x - 36$. That is a continuous function (being a polynomial) at all x, so it can change sign only where the derivative is zero. Setting it equal to zero, and factoring, we have $6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x - 2)(x + 3) = 0$, so either $x - 2 = 0$, $x = 2$, or $x + 3 = 0$, $x = -3$. Calculating y' at $x = -4$ we get $6 \times (-4)^2 + 6 \times (-4) - 36 = 36 > 0$: Since $y' > 0$ at -4 and y' can only change sign at -3 and 2, y' must be positive at every x to the left of -3, i.e. the function is increasing on $(-\infty, -3)$. Similarly calculating y' at $x = 0$ we get -36, so y' is negative and hence y is increasing everywhere on $(-3, 2)$. Lastly, y' at $x = 3$ is $6 \times 3^2 + 6 \times 3 - 36 = 36$, so y' is positive, and y is increasing, on $(2, \infty)$.

(b) On which intervals of real number is the graph of this function concave upward? Concave downward? Give reasons based on calculus, not just looking at the graph on a calculator.

ANSWER: We use the second derivative $y'' = 12x + 6$ to determine concavity. That is zero at $x = -\frac{1}{2}$. For any $x < -\frac{1}{2}$, $12x$ is more negative than -6 so $y'' < 0$, and the graph is concave downward. For any $x > -\frac{1}{2}$ the second derivative is positive and the graph is concave upward. So the graph is concave downward on $(-\infty, -\frac{1}{2})$ and concave upward on $(-\frac{1}{2}, \infty)$.

(c) Where does this graph have point(s) of inflection?

ANSWER: The only point of inflection is where the second derivative changes sign, which we found in (b) to be at $x = -\frac{1}{2}$. The corresponding point on the graph has $y = 2x^3 + 3x^2 - 36x - 4 = 2 \times (-\frac{1}{8}) + 3 \times \frac{1}{4} - 36 \times (-\frac{1}{2}) - 4 = -21\frac{1}{2}$, i.e. the actual point is $(-\frac{1}{2}, -21\frac{1}{2})$.

Problem 8 (20 points)
Let R be the region between the curves $y = x$ and $y = x^2$.
If we rotate R about the x-axis, what is the volume of the resulting solid?
ANSWER: We can set this up either with slices across the axis of rotation, "washers", or slices parallel to the axis, "shells". The curves meet at $(0, 0)$ and $(1, 1)$.
Using washers: The inner radius of the washer is $y = x^2$ and the outer radius is $y = x$. So the area of one side of a washer is $\pi(x^2 - (x^2)^2) = \pi(x^2 - x^4$. Hence the volume we want can be computed as $\pi \int_0^1 (x^2 - x^4)dx = \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi(\frac{1}{3} - \frac{1}{5}) = \frac{2\pi}{15}$.
Using shells: The width of a shell from left to right extends from $x = y$ to $x = \sqrt{y}$, a distance $\sqrt{y} - y$. The radius of a shell coming from a slice at height y is y, so if the shell has thickness Δy it contributes $2\pi y(\sqrt{y} - y)\Delta y$ to the total volume. Thus we can evaluate the integral $2\pi \int_0^1 y\,(\sqrt{y} - y)\,dy$ which gives the same result.

Problem 9 (22 points)
Evaluate the integrals:

(a) $\displaystyle\int_1^2 \frac{e^{3/x}}{x^2}\,dx$.

ANSWER: The derivative of $\frac{3}{x}$ is, up to a constant multiple, $\frac{1}{x^2}$, so we try $u = \frac{3}{x}$. Then $du = -\frac{3}{x^2}dx$. Now at $x = 1$, $u = 3$, and at $x = 2$, $u = \frac{3}{2}$, so the integral becomes $-\frac{1}{3}\int_3^{\frac{3}{2}} e^u\,du$ $= -\frac{1}{3}\left[e^u\right]_3^{\frac{3}{2}} = \frac{1}{3}\left(e^3 - e^{\frac{3}{2}}\right)$

(b) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^3(2x)\,\cos(2x)\,dx$.

ANSWER: Again we note that one part, $\cos 2x$, is almost the derivative of what is inside the other (cubing) part, so we let $u = \cos 2x$ and then $du = -2\sin(2x)dx$. When $x = 0$, $u = \cos(0) = 1$, and when $x = \frac{\pi}{4}$, $u = \cos\frac{\pi}{2} = 0$. So the integral becomes $-\frac{1}{2}\int_1^0 u^3du = -\frac{1}{2}\left[\frac{u^4}{4}\right]_1^0 = -\frac{1}{2}(0 - \frac{1}{4}) = \frac{1}{8}$.

Problem 10 (20 points)
Let $y = 2x^3 + 3x^2 - 36x - 4$.
Find and identify any/all local/global maxima and minima for this function.
ANSWER: In problem 7 we found $y' = 6x^2 + 6x - 36$ with zeros only at -3 and $+2$: The derivative exists anywhere, and the domain has no endpoints, so the only critical points are $x = -3$ and $x = 2$. In problem 7 we also found the graph changes from increasing to decreasing at $x = -3$, so there is at least a local maximum there, by the first derivative test. And we found the graph changes from decreasing to increasing at $x = 2$, so there must be at least a local minimum there. Since the function was increasing on $(-\infty, -3)$, i.e. decreasing if we went off to the left, with the x^3 term making it go arbitrarily far in the negative direction, there is no global minimum. Likewise the function increases without bound to the right, so there is no global maximum. Thus there are only two points to consider. At $x = -3$, the function takes the value 77, and at $x = 2$ it takes the value -48: Pulling the results together, there is a local but not global maximum at $(-3, 77)$ and there is a local but not global minimum at $(2, -48)$.