Final Exam   December 16, 2003   ANSWERS

<u>Problem 1</u>   (17 points)
Let $f(x) = 3x^2 - 4x + 4$, $a = -1$, and $b = 4$. (These are used throughout the rest of this problem.)

(a) The mean value theorem for derivatives tells us that there is some c in the interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
Find a number c that does what that theorem guarantees.

<u>Answer:</u>
We compute $f(b) = f(4) = 3 \times 4^2 - 4 \times 4 + 4 = 36$, and similarly $f(a) = 11$. Hence $(f(b) - f(a))/(b - a) = \frac{25}{5} = 5$. Now we have to find a number c in the interval $(-1, 4)$ such that $f'(c) = 5$. That gives us the equation $6c - 4 = 5$, $c = \frac{3}{2}$.

(b) The mean value theorem for integrals tells us that there is some d in the interval (a, b) such that $f(d)$ is the same as the average value of the function f on the interval $[a, b]$.

  (i) Find the average value of f on $[a, b]$.
  <u>Answer:</u>
  The average or mean value of f on the interval is

  $$\frac{1}{4 - (-1)} \int_{-1}^{4} (3x^2 - 4x + 4)\, dx = \frac{1}{5} \left[x^3 - 2x^2 + 4x \right]_{-1}^{4} = \frac{1}{5}(48 - (-7)) = 11.$$

  (ii) Find a number d that does what this theorem guarantees.
  <u>Answer:</u>
  We need to solve $3d^2 - 4d + 4 = 11$. We find the values $d = -1$ and $d = \frac{7}{3}$ using the quadratic formula. Only one of the roots, $\frac{7}{3}$, is in the open interval $(-1, 4)$, so we have to choose $d = \frac{7}{3}$. (The other choice was the value observed above, $f(a) = 11$, and we are not supposed to use an endpoint here.)

<u>Problem 2</u>   (17 points)
Find the general solution to the differential equation

$$x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}.$$

(It was announced during the exam that you could assume $x > 0$.)
<u>Answer:</u>
First divide through by x. Now we have $\frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^3}$. This is in the form for a first order linear differential equation as we studied it, with $P(x) = \frac{3}{x}$ and $Q(x) = \frac{\sin x}{x^3}$. We next compute $\int P(x)\, dx = \int \frac{3\, dx}{x} = 3\ln|x| + C$ and ignore the $+C$ for this purpose. This gives us the integrating factor $I(x) = e^{3\ln x} = (e^{\ln x})^3 = x^3$.
Now we can write out the solution $y = \frac{1}{I(x)} \int I(x)\, Q(x)\, dx = x^{-3} \int \sin(x)\, dx = x^{-3}(-\cos(x) + C)$. You can multiply out that last expression if you wish. It is essential that the x^{-3} stay multiplied on C, though: C does not stand alone here!

<u>Problem 3</u>   (16 points)
Let $f(x) = 3x - 2$. Use the $\epsilon - \delta$ definition of limit to show $\lim_{x \to 4} f(x) = 10$.

Answer:

Suppose we are given a number $\epsilon > 0$. We need a "recipe" that finds a number $\delta > 0$ such that, whenever x is closer than δ to 4 (but not equal to it), we are assured that $f(x)$ is closer than ϵ to 10. We write out both those conditions as $0 < |x - 4| < \delta$ and $|f(x) - 10| < \epsilon$: The second, rewritten to use our specific function $f(x)$, is $|3x - 2 - 10| < \epsilon$, or $|3x - 12| < \epsilon$, which we can rewrite as $3|x - 4| < \epsilon$ or $|x - 4| < \frac{\epsilon}{3}$. Comparing these it seems like a good idea to try $\delta = \frac{\epsilon}{3}$. Now we need to check that this choice of δ "works". You might be able to work backward, checking that all of the steps that led you to this choice of δ were reversible, to establish this. Here is a more direct version: If $0 < |x - 4| < \delta = \frac{\epsilon}{3}$, then $3|x - 4| < \epsilon$ so $|3x - 12| < \epsilon$ and so $|(3x - 2) - 10| < \epsilon$ or in other words $|f(x) - 10| < \epsilon$, which is what we had to show.

Problem 4   (20 points)

Take the indicated derivatives:

(a) $\dfrac{dy}{dx}$, for $y = \ln(\sin(x) + 2)$.

Answer:

$$\frac{dy}{dx} = \frac{1}{\sin(x) + 2}\cos(x) = \frac{\cos(x)}{\sin(x) + 2}.$$

(b) $\dfrac{d^2 f}{dx^2}$, for $f(x) = 3e^{2x} + 5x^3$.

Answer:
First calculate the first derivative: $\frac{df}{dx} = 3e^{2x}(2) + 5(3x^2) = 6e^{2x} + 15x^2$. Now take the second derivative, $\frac{d^2 f}{dx^2} = 6e^{2x}(2) + 30x = 12e^{2x} + 30x$.

(c) $D_x f$, for $f(x) = \dfrac{1}{x}\arcsin(x^2 - 2)$.

Answer:
We need to use the product rule as well as know the derivative of the inverse sine:

$$D_x f = D_x\left(\frac{1}{x}\right)\arcsin(x^2 - 2) + \frac{1}{x}D_x(\arcsin(x^2 - 2))$$

$$= \frac{-1}{x^2}\arcsin(x^2 - 2) + \frac{1}{x}\frac{1}{\sqrt{1 - (x^2 - 2)^2}}(2x)$$

$$= -\frac{\arcsin(x^2 - 2)}{x^2} + \frac{2}{\sqrt{-x^4 + 4x^2 - 3}}.$$

(d) $f'(x)$, for $f(x) = \displaystyle\int_2^{x^2 - 1} \tan^{-1}(t)\, dt$.

Answer:
We use the first fundamental theorem of calculus together with the chain rule:

$$f'(x) = \frac{df}{d(x^2 - 1)} \times \frac{d(x^2 - 1)}{dx} = \tan^{-1}(x^2 - 1) \times (2x) = 2x\tan^{-1}(x^2 - 1).$$

<u>Problem 5</u>   (14 points)
Let $f(x) = e^x$.

(a) Find an equation for the tangent line to the graph of $f(x)$ at $x = 1$.

<u>Answer:</u>
Since $f'(x) = e^x$, the slope of the tangent line is $e^1 = e$. The line goes through $(1, f(1)) = (1, e)$, so the line can be written as $y - e = e(x - 1)$. This form is OK, but if we rewrite it as $y = e + e(x - 1) = ex$ it is set up for part (b).

(b) Use a tangent line approximation (also known as a linear approximation) to estimate $e^{\frac{1}{2}}$.

<u>Answer:</u>
The last form of the answer to (a) is already $f(1) + f'(1)(x-1)$ and so is the expression for linear approximation at 1. Using $x = \frac{1}{2}$ we get $e^{\frac{1}{2}} \approx e + e(\frac{1}{2} - 1) = e(1 - \frac{1}{2}) = \frac{e}{2}$.

(c) By thinking about the graph of the shape of the graph of e^x, tell whether you think your estimate in (b) will be higher or lower than the actual value of $e^{\frac{1}{2}}$. Explain your reasoning.

<u>Answer:</u>
The graph of e^x is concave upward ($f''(x) = e^x$ is positive), so the tangent line will stay below the curve except at the one point where it touches it. Hence the linear approximation, using a point on the tangent line instead of one on the curve, will give a result <u>lower</u> than the actual value.

<u>Problem 6</u>   (14 points)
Let $f(x) = x^2 - 2$.

(a) Set up a Riemann sum approximating $\int_1^5 f(x)\,dx$ which uses a partition of $[1, 5]$ into 4 equal subintervals and evaluation of f at the right end of each subinterval. Show the numbers to be added up explicitly, e.g. $\frac{5}{4}$, without variables in them, i.e. not as $\frac{1}{2}x_i$ or anything like that. You should add them up, thus arriving at an approximate value for the integral.

<u>Answer:</u>
Dividing $[1, 5]$ into four equal pieces we get the subintervals $[1, 2]$, $[2, 3]$, $[3, 4]$, and $[4, 5]$. Each has width $\Delta x = 1$. The right ends of the pieces are 2, 3, 4, and 5. Thus our sum is $f(2) \times 1 + f(3) \times 1 + f(4) \times 1 + f(5) \times 1 = 2 + 7 + 14 + 23 = 46$.

(b) Evaluate $\int_1^5 f(x)\,dx$ using the second Fundamental Theorem of Calculus.

<u>Answer:</u>

$$\int_1^5 f(x)\,dx = \int_1^5 (x^2 - 2)dx = \left[\frac{x^3}{3} - 2x\right]_1^5 = \left(\frac{125}{3} - 10\right) - \left(\frac{1}{3} - 2\right) = 33\frac{1}{3}.$$

(c) The answer you got in (b) is either less than or more than the answer you got in (a). Explain why, making reference to the direction the graph of f is curving and the choice of where to evaluate $f(x)$ in part (a).

Answer:
The graph of f is a parabola opening upward, symmetric about the y-axis, and is increasing and concave upward throughout the interval $[1, 5]$. Picking the right end of each subinterval in (a) picks a value for f higher than anywhere else in the subinterval and guarantees that the piece of the Riemann sum corresponding to that subinterval is definitely greater than the actual area under the curve in that subinterval. Hence that Riemann sum is guaranteed to be bigger than the integral, consistent with our answers $46 > 33\frac{1}{3}$.

Problem 7   (18 points)
Let $f(x) = 3x^4 - 20x^3 + 36x^2 + 7$.

(a) Find all critical points of $f(x)$ on the interval $[-1, 4]$.

Answer:
As usual the endpoints -1 and 4 are critical points. The function f, given by a polynomial, is differentiable everywhere so there are no singular points. Now we have to find the stationary points, where $f'(x) = 0$. Taking the derivative, $f'(x) = 12x^3 - 60x^2 + 72x = 12(x^3 - 5x^2 + 6x)$, so we are looking for the x values that make $x^3 - 5x^2 + 6x = 0$. We can factor x out of this, $x(x^2 - 5x + 6) = 0$, so the values we are looking for are $x = 0$ and the solutions of $x^2 - 5x + 6 = 0$. You could use the quadratic formula, but this last expression factors nicely and we have $(x - 3)(x - 2) = 0$. Thus the stationary points are $x = 0$, $x = 2$, and $x = 3$. All together, the critical points are -1, 0, 2, 3, and 4.

(b) For each critical point in the <u>open</u> interval $(-1, 4)$, tell if it is a local maximum, a local minimum, or neither.

Answer:
We only have to consider 0, 2, and 3. We can use the second derivative test: $f''(x) = 36x^2 - 120x + 72$. Thus $f''(0) = 72$, $f''(2) = -24$, and $f''(3) = 36$. Hence there is a local minimum at $x = 0$, a local maximum at $x = 2$, and a local minimum at $x = 3$.

(c) Find all points of inflection of the graph of $f(x)$. (Remember that our text defines a point of inflection to be the point on the graph, i.e. it has two coordinates!)

Answer:
A point of inflection will have $f''(x) = 0$: Solving $36x^2 - 120x + 72 = 0$ using the quadratic formula we get $x = \frac{1}{3}(5 + \sqrt{7}) \approx 2.55$, and $x = \frac{1}{3}(5 - \sqrt{7}) \approx 0.785$. You can check that f' changes sign at each of those, so they do correspond to points of inflection. The values of f at those points are about 36.32 and 20.64, so the actual points of inflection are approximately $(2.55, 36.32)$ and $(0.785, 20.64)$.

Problem 8   (21 points)
Evaluate the integrals:

(a) $\displaystyle \int \frac{\sin(x)}{1 + \cos(x)} \, dx$

Answer:
Let $u = 1 + \cos(x)$. Then $du = -\sin(x) \, dx$. The integral becomes $-\int \frac{du}{u} = -\ln|u| + C$, or $-\ln|1 + \cos(x)| + C$. Since $\cos(x)$ is never less than -1, $1 + \cos(x)$ is always at least 0, so you can then drop the absolute value signs.

(b) $\quad \int_0^{\ln(2)} x\, e^{3x^2}\, dx$

Answer:
Let $u = 3x^2$. Then $du = 6x\, dx$, or $x\, dx = \frac{1}{6}\, du$. The end points on the integral change: When $x = 0$, $u = 0$, but when $x = \ln(2)$, $u = 3(\ln(2))^2$. We can rewrite this last number as $(\ln 2)(3 \ln 2)$. Thus the integral becomes

$$\frac{1}{6} \int_0^{3(\ln(2))^2} e^u\, du = \frac{1}{6} \left[e^u \right]_0^{(\ln 2)(3 \ln 2)} = \frac{1}{6} \left(2^{3 \ln(2)} - 1 \right) \approx 0.5377.$$

(c) $\quad \int \cos(5x)\, dx$

Answer:
Let $u = 5x$ so $dx = \frac{1}{5}\, du$. The integral is then $\frac{1}{5} \int \cos(u)\, du = \frac{1}{5} \sin(u) + C = \frac{1}{5} \sin(5x) + C$.

Problem 9   (15 points)
Find the area of the region in the plane which is bounded by (i) $y = e^{\frac{x}{2}}$, (ii) $y = 2x$, (iii) $x = 1$, and (iv) $x = 4$.
Answer:
In the interval $[1, 4]$, the straight line $y = 2x$ stays above the curve $y = e^{\frac{x}{2}}$. Hence the area can be computed as

$$\int_1^4 \left(2x - e^{\frac{x}{2}} \right) = \left[x^2 - 2e^{\frac{x}{2}} \right]_1^4 = (16 - 2e^2) - (1 - 2e^{\frac{1}{2}}) \approx 3.519.$$

Problem 10   (16 points)
A Petri dish contains a colony of bacteria. If $f(t)$ gives the amount of the colony at time t, the rate of growth of f is proportional to the value of f itself.
An experimenter observes that there are 0.2 grams of bacteria five hours into an experiment, and 0.8 grams after the experiment has been going for 15 hours.

(a) How many grams were there at the start of the experiment?

Answer:
Because we are told that the growth rate is proportional to the value of f, we know $f(t) = Ce^{kt}$ for some constants C and k. We measure time t in hours since the experiment started, and we are given that $f(5) = 0.2$ and $f(15) = 0.8$. These give us equations $Ce^{5k} = 0.2$ and $Ce^{15k} = 0.8$. If we divide the second by the first we get $e^{15k - 5k} = \frac{0.8}{0.2} = 4$, $e^{10k} = 4$, so $10k = \ln(4)$ and $k = \frac{1}{10} \ln(4)$. Now use this in one of the two equations. I will use the first: $Ce^{\frac{1}{2} \ln(4)} = 0.2$, so $C = (0.2)e^{-\frac{1}{2} \ln(4)} = (0.2)\sqrt{\frac{1}{4}} = 0.1$.

Now $F(0) = C$, so the initial amount was 0.1 grams.

(b) How many grams $f(t)$ will there be for an arbitrary time t, assuming the same conditions continue to hold? (Your answer should be a function involving t, not the value of t at some specific time t you have chosen!)

Answer:
The way I did part (a) we already have this, $f(t) = Ce^{kt} = 0.1\, e^{\frac{t}{10} \ln(4)} = (0.1)\, 4^{t/10}$ or various other forms you can transform it into using laws of exponents and logarithms.

It is possible to derive C in a different way, deducing from the given data that the population doubles every five hours, in which case you would not yet have this function worked out and would at this stage have to do the work I put in (a).

<u>Problem 11</u>   (16 points)
An object is being moved against a resisting force. It starts at a certain position, and for the first three feet it is moved it encounters a fixed resisting force of 25 pounds. After that the force starts changing. For the next three feet the force is $22 + x$ pounds, where x is the distance the object has moved from its starting position.
When the object has been moved six feet, (the first three with the constant force and the last three with the varying force) how much work has been done?
<u>Answer:</u>
We break this into two parts: The work done in the first three feet, plus the work done in the remaining three feet. We can set up each as an integral: The first part becomes $\int_0^3 (25)\, dx$, and the second is $\int_3^6 (22+x)\, dx$. Evaluating the first integral we get $25 [x]_0^3 = 75$ foot pounds, while the second gives $\left[22x + \frac{x^2}{2}\right]_3^6 = (132+18) - (66 + \frac{9}{2}) = 79\frac{1}{2}$ foot pounds. Adding these we get $154\frac{1}{2}$ foot pounds.
You can also avoid doing the first integral since the force on that interval is constant: The work on that part must be force $\times$ distance, $3 \times 25 = 75$ foot pounds.

<u>Problem 12</u>   (16 points)
Use an integral to find the volume of the solid generated by rotating about the y-axis the region bounded by $y = x$, $x = 0$, and $y = 2$.
<u>Answer:</u>
The solid that results is a cone, its point at the origin and the top two units up, circular with radius 2. So we could compute the volume as $\frac{1}{3}\pi \times 2^2 \times 2 = \frac{8}{3}\pi$. But we are explicitly told to use an integral!
This can be done either with disks or shells. Using disks, corresponding to cutting the part of the y-axis from $y = 0$ to $y = 2$ into small increments, we get

$$\int_0^2 \pi y^2 \, dy = \pi \left[\frac{y^3}{3}\right]_0^2 = \frac{8}{3}\pi.$$

(Fortunately, this is the same answer we got the other way!)