RESEARCH STATEMENT

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My research interests are broad in *commutative algebra*, *algebraic geometry*, and *arithmetic geometry*. More specifically, my work has focused on syzygies of algebraic varieties, Brill-Noether theory of curves, stable maps, moduli spaces and cycles of curves and K3 surfaces, as well as their arithmetic applications.

The theory of syzygies of algebraic varieties explores the connection between the geometry of projective varieties and the algebra of their defining equations. This field already boasts a wealth of elegant results, particularly in the study of curves. Brill-Noether theory of curves and vector bundles on K3 surfaces have become classics in the field of study. Among many intriguing discoveries, Green's Conjecture 1 on the syzygies of canonical curves stands out as particularly attention-grabbing. My work towards the conjecture in positive characteristic (Project 1, 2) combines various techniques of vector bundles and deformation on moduli spaces of K3 surfaces in positive characteristic. In another direction, I am studying Brill-Noether theory of curves as general covers of genus 2 curves (Project 3), which not only holds significance on its own but may also verify Green's conjecture for those curves. Furthermore, my ongoing work (Project 4 with Daniele Agostini and Jinhyung Park) aims to address Gross-Popescu's conjecture on syzygies of a $(1, n)$-polarized abelian surface, by studying syzygies of curves on it.

A brief synopsis of my various projects is listed here, followed with detailed expansion in the rest of this statement.

Project 1. *Green's Conjecture on syzygies of canonical curves in positive characteristic.*

The major breakthrough of this conjecture over $\mathbb{C}$ for generic curves was first established by Voisin [Voi02], [Voi05], followed by an essential simplification by Kemeny [Kem20b]. Among many proofs towards this conjecture in positive characteristic, my work [Wei23] presents an alternate geometric proof, elaborated in Proposition 1, Proposition 2 and Theorem 2.

Project 2. *Geometric Syzygy Conjecture of canonical curves in positive characteristic.*

The conjecture is seen as a refined one of Green's conjecture. Generic curves over $\mathbb{C}$ are resolved by [Kem24]. A special case (Theorem 8, Theorem 4) in positive characteristic is studied in my paper [Wei23]. This ongoing work would benefit from studying rational nodal curves on K3 surfaces in positive characteristic.

Project 3. *Pencils on general covers of genus 2 curves and Green's conjecture for such curves.*

This ongoing work aims to study Brill-Noether theory and Green's conjecture for a general cover of a genus 2 curve. I wish to leverage degeneration methods of covers of genus 2 curves with Kontsevich moduli space of stable maps to study the pencils, and proceed by Aprodu's linear growth theorem to deal with Green's conjecture.

Project 4. *Gross-Popescu Conjecture on $(1, n)$-polarized abelian surfaces.*

The conjecture claims a striking relation between the syzygies of a general abelian surface and the positivity of the polarization. This is an ongoing work with Daniele Agostini and Jinhyung Park.

1. Syzygies of Algebraic Curves

The theory of syzygies, initiated by James Joseph Sylvester and David Hilbert in the late 1800s, gained renewed momentum through the work of David Eisenbud, Mark Green, Robert Lazarsfeld, Claire Voisin, and many others since the 1980s. This field sits at the intersection of algebraic geometry and commutative algebra and remains one of the most actively studied topics among geometers and algebraists today.

For a *projectively normal* variety $X \subseteq \mathbb{P}^r_{\mathbb{F}}$ embedded by the complete linear series of a very ample line bundle L, the homogeneous coordinate ring of X, $S_X := S/I_X \simeq \Gamma_X(L) = \oplus_{n\geq 0} H^0(X, L^{\otimes n})$ is a module over $S := \mathbb{F}[x_0, \cdots, x_r] \simeq \mathrm{Sym}\ H^0(X, L)$. One follows Hilbert in considering a minimal free resolution of $\Gamma_X(L)$ as an S-module, which demonstrates its delicate algebraic structures.

$$0 \to \cdots \to \underbrace{\bigoplus_j S(-i-j)^{\beta_{i,i+j}}}_{i\text{-th syzygy}} \to \cdots \to \underbrace{\bigoplus_j S(-1-j)^{\beta_{1,1+j}}}_{\text{1st syzygy}} \to S \to \Gamma_X(L) \to 0$$

For example, the 1-st syzygy space describes the number and the degree of minimal generators of the ideal I_X; the 2-nd syzygy space describes those of their relations. Higher syzygy space describes higher relations. Those $\beta_{p,p+q}(S_X) = \dim_{\mathbb{F}} \operatorname{Tor}_p(S_X, \mathbb{F})_{p+q}$ are called graded Betti numbers. It can also be computed by Koszul cohomology

$$\bigwedge^{p+1} H^0(X,L) \otimes H^0(X, L^{\otimes q-1}) \to \bigwedge^{p} H^0(X,L) \otimes H^0(X, L^{\otimes q}) \to \bigwedge^{p-1} H^0(X,L) \otimes H^0(X, L^{\otimes q+1}).$$

We have the equality $\beta_{p,p+q} = \dim K_{p,q}(X,L) =: b_{p,q}$.

A notion of property (N_p) is developed by Green and Lazarsfeld to distinguish the Betti tables $(b_{p,q})$ having the simplest possible resolution for a number of steps.

Definition 1 (property $(\boldsymbol{N_p})$). *One says that a polarized variety (X,L) satisfies property (N_p) if it is projectively normal and $K_{j,q}(X,L) = 0$ for $j \le p$ and $q \ge 2$.*

In particular, (N_0) stands for projectively normality, (N_1) means homogeneous ideal being generated by quadrics, (N_p) means all higher syzygies having only linear strands at least until p-th steps.

The study of syzygies of curves, initiated by Noether and Petri, was revolutionised by the work of Mark Green, and have inspired many beautiful ideas through, in particular, Green's conjecture in the mid 1980s.

1.1. **Green's Conjecture on Syzygies of Canonical Curves.** For a smooth non-hyperelliptic curve C of genus g, its canonical bundle ω_C gives a projectively normal (N_0) embedding $C \hookrightarrow \mathbb{P}^{g-1}$ by Noether's theorem. Petri observed property (N_1) for the ideal of a canonical curve under certain conditions.

Theorem 1 (Petri). [ACGH85, III §3] *Let C be a nonhyperelliptic projective curve of genus $g \ge 4$. Consider its canonical embedding $C \hookrightarrow \mathbb{P}^{g-1}$, C is cut out by quadrics except when C is trigonal or isomorphic to a plane quintic.*

The exceptions in the theorems are those curves with Clifford Index 1. It was Green's insight to relate syzygies of canonical curves with the existence of their special linear series, in the guise of the Clifford Index. Precisely, the Clifford index of C is defined as follows

$$\operatorname{Cliff}(C) := \min\{\deg(A) - 2h^0(A) + 2 \mid A \in \operatorname{Pic}(C), h^0(A) \ge 2, h^1(A) \ge 2\}.$$

Conjecture 1 (Green [Gre84c]). *Let C be a smooth non-hyperelliptic curve of genus g, let $K_{p,q}(C,\omega_C)$ denote the (p,q)-th Koszul cohomology group, then*

$$K_{p,2}(C,\omega_C) = 0 \iff p < \operatorname{Cliff}(C).$$

or equivalently, (C,ω_C) has property (N_p) if and only if $p < \operatorname{Cliff}(C)$.

For a general smooth curve of genus g, the sharpness of the Brill-Noether Theorem implies that its Clifford index $\operatorname{Cliff}(C) = \lfloor \frac{g-1}{2} \rfloor$. For even genus $g = 2k$ or odd genus $g = 2k+1$, a single vanishing

$$K_{k,1}(C,\omega_C) = 0$$

suffices to prove Green's Conjecture for a general smooth curve.

Following Voisin's framework of proof of generic Green's conjecture over $\mathbb{C}$ ([Voi02], [Voi05]), let (X,L) be a polarized K3 surface of degree $2g-2$. For $\operatorname{char}(\mathbb{F}) \ne 2$, the hyperplane restriction theorem [Gre84c] implies

$$K_{p,1}(C,\omega_C) = K_{p,1}(X,L), \quad \forall\, 0 \le p \le 2g-3.$$

whenever C is a smooth and connected (e.g. general) hyperplane section of L on X.

Although Green's conjecture fails over char p field for general curves for small prime p (See [Sch86] and [Muk10]), it holds true once lower bound for p is assumed. Among many proofs ([AFP+19], [RS22], [Par22]) towards this conjecture in positive characteristic, my work [Wei23] presents an alternate geometric proof. Adapting Kemeny's approach [Kem20a] using secant bundles over a K3 surface and homological computation, with deformation arguments on the moduli space of K3 surfaces, I am able to prove the generic Green's conjecture in $\operatorname{char} p$.

Theorem 2 (W)**.** [Wei23, Thm 1.2] *Let $\mathbb{F}$ be any algebraically closed field with* $\mathrm{char}(\mathbb{F}) = p > 0$*: Let C be a general curve over $\mathbb{F}$ of even genus $g = 2k$ or odd genus $g = 2k+1$ for $g \geq 4$. Assume $p \geq \frac{g+4}{2}$. Then $K_{k,1}(C, \omega_C) = 0$. In particular, C satisfies Green's conjecture.*

1.2. **Geometric Syzygy Conjecture on Syzygies of Canonical Curves.** After Petri's theorem 1 on the ideals, being generated by quadrics, of a canonical curve with Clifford index at least 2, Andreotti-Mayer [AM67] and Arbarello-Harris [AH81] proved that those quadrics generating the ideal can be chosen of rank at most four, provided the curve is general. This was extended to arbitrary curves by Green [Gre84c].

For higher linear syzygis, we have a notion of *rank* to generalize the rank of a quadric. The following conjecture by Schreyer unifies and generalizes both Voisin's Theorem on the generic Green's Conjecture and the Andreotti-Mayer-Arbarello-Harris Theorem:

Conjecture 2 (Geometric Syzygy Conjecture)**.** *For a general smooth curve C of genus g, all linear syzygy spaces $K_{p,1}(C, \omega_C)$ are spanned by syzygies of minimal rank $p+1$.*

Precisely, for a non-hyperelliptic curve C of genus $g \geq 5$, the first linear syzygy space $K_{1,1}(C, \omega_C)$, representing space of quadrics, is generated by rank two syzygies of the form

$$\alpha \in K_{1,1}(C, \omega_C; \mathrm{H}^0(A)), \quad A \in W^1_{g-1}(C).$$

Over $\mathbb{C}$, Kemeny [Kem24, Thm 0.2] proved the Geometric Syzygy Conjecture. The theorem is shown via the technique of *projection of syzygies*, which provides an inductive way to prove results on syzygies of smooth curves from the last syzygy space. The starting point is the following:

Theorem 3 (Kemeny)**.** [Kem20b, Cor 0.4] *Let C be a general curve of even genus $g = 2k \geq 4$. Then the last syzygy space $K_{k-1,1}(C, \omega_C)$ is generated by syzygies of lowest possible rank k. More precisely, $K_{k-1,1}(C, \omega_C)$ is generated by rank k syzygies*

$$\alpha \in K_{k-1,1}(C, \omega_C; \mathrm{H}^0(\omega_C \otimes A^{-1})), \quad A \in W^1_{k+1}(C).$$

Adapting Kemeny's approach on vector bundles on K3 surfaces, with deformation arguments on moduli space of K3 surfaces, I proved its counterpart in positive characteristic with an assumption on $\mathrm{char}\, p$.

Theorem 4 (W)**.** [Wei23, Thm 1.3] *Fix an algebraically closed field $\mathbb{F}$ with* $\mathrm{char}(\mathbb{F}) = p > 0$*. Let C be a general curve over $\mathbb{F}$ of even genus $g = 2k$ for $k \geq 2$. Assume $p > 2k$. The result of Theorem 3 holds.*

It is worth mentioning that the inductive way Kemeny carries out relies on a non-trivial fact about the existence of rational nodal curves on complex K3 surfaces, developed by Chen [Che02] and many other people. If one wants to proceed the story in full generality in positive characteristic, studying rational nodal curves on K3 surfaces in positive characteristic benefits the induction process.

Question 1. *For a general K3 surface (X, L) in positive characteristic, does there exist a nodal rational curve in $|L|$? Is every rational curve nodal?*

2. Gross-Popescu's Conjecture on Syzygies of Abelian Surfaces

We work over $\mathbb{C}$. Let (A, L) be a general $(1, n)$-polarized abelian surface. Igor Reider [Rei88] and Arnaud Beauville proved that L is base-point free if and only if $n \geq 3$, and L is very ample if and only if $n \geq 5$, thus giving an embedding $A \subseteq \mathbb{P}(H^0(A, L)^\vee)$, we aim to study its property (N_p).

Robert Lazarsfeld and Fuentes García prove he projective normality (N_0) of (A, L) when $n \geq 7$. See Daniele Agostini's notes [Ago17] for its history and a unified proof. Gross and Popescu [GP98] studied the homogeneous ideal of a general $(1, n)$-polarized abelian surface (A, L) with $n \geq 10$ and proved it is generated by quadrics (N_1). Towards the end of the paper, they formulated a conjecture

Conjecture 3 ([GP98, p.375])**.** *A general $(1, n)$-polarized abelian surface (A, L) with $n \geq 10$, has property (N_p) for $0 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor - 4$.*

The initial attempts to address the conjecture involve studying the local positivity of the polarization. This includes *Seshadri constants* $\epsilon(A, L)$, developed in [LPP11] by Lazarsfeld, Pareschi and Popa; the theory of *infinitesimal Newton-Okounkov bodies* by Küronya and Lozovanu in [KL15]; the *basepoint-freeness threshold* introduced in [JP20] by Jiang and Pareschi; and *Bridgeland stability conditions* by Rojas in [Roj22].

The best result they deliver is the following: for a general $(1, n)$-polarized abelian surface (A, L), it has property (N_p) for
$$n > (p+2)^2 \iff 0 \le p < \sqrt{n} - 2.$$

Our ideas avoid any discussion on the local positivity. Instead, we use a Lefschetz hyperplane section of a surface to relate their Koszul cohomology groups, whose machinery were developed by Green ([Gre84a], [Gre84b]). This process benefits from Green's conjecture on canonical curves, which relies on the study of Brill-Noether theory of curves on an abelian surface.

More specifically, hyperplane section delivers the following sequence for $H^1(A, \mathcal{O}_A) \neq 0$:
$$0 \to H^0(A, \mathcal{O}_A) \to H^0(A, L) \xrightarrow{\psi} H^0(C, \omega_C) \to H^1(A, \mathcal{O}_A) \to 0$$
Let $W := \operatorname{im}(\psi) \subseteq H^0(C, \omega_C)$, we are able to reduce the condition $K_{p,2}(A, L) = 0$ to the following two conditions for any curve $C \in |L|$ (discovered independently by Agostini and me)

(i) $K_{p,2}(C, \omega_C; W) = 0$

(ii) the map $K_{p+1,1}(C, \omega_C; W) \to \wedge^{p+1} W \otimes H^1(A, \mathcal{O}_A)$ is surjective.

My current idea is to investigate the above two conditions for a *binary elliptic curve* $C \in |L|$ coming as a cyclic cover of order n onto a degenerated genus 2 curve (i.e. union of two elliptic curves). In particular, the curve C is glued from two elliptic curves via sets of points forming cyclic groups of order n. To prove (i), on one hand, I wish to leverage Aprodu's linear growth theorem on such singular binary curves while studying theory of pencils on it. On the other hand, I wish to directly study minimal free resolutions of such binary curve through a double structure on elliptic curves. For (ii), Agostini proved for one degenerated cover with $n+1$ irreducible components. It is hopeful that a similar computation can be carried out for degenerated cover with 2 irreducible components.

3. Brill-Noether Theory of Curves

Let $\mathbb{F} = \overline{\mathbb{F}}$ be an algebraically closed field. The existence of an embedding of a curve $C \subseteq \mathbb{P}^r$ of degree d is equivalent to the existence of very ample line bundle $A \in \operatorname{Pic}^d(C)$ with $h^0(A) \ge r+1$. Brill-Noether theory concerns those special divisors on a smooth curve C. For given integers r, d, define Brill-Noether loci
$$W^r_d(C) := \{A \in \operatorname{Pic}^d(C) \,|\, h^0(A) \ge r+1\} \subseteq \operatorname{Pic}^d(C).$$
It contains an open subvariety $G^r_d(C) := \{(V, A) \,|\, V \subseteq \mathrm{H}^0(A), \dim(V) = r+1, A \in W^r_d(C)\}$. The expected dimension of $W^r_d(C)$ is given by Brill-Noether number $\rho(r, d, g) = g - (r+1)(r-d+g)$.

Theorem 5 (Brill-Noether-Petri, Griffiths-Harris)**.** *Let C be a general smooth curve over $\mathbb{F} = \overline{\mathbb{F}}$, then $W^r_d(C) \neq \varnothing$ if and only if $\rho(r, d, g) \ge 0$. Moreover,* $\dim W^r_d(C) = \rho(r, d, g)$ *whenever empty.*

While the theorem is originially proved by degeneration of curves, Lazarsfeld [Laz86] discovered an elegant proof without degeneration through K3 surfaces when $\mathbb{F} = \mathbb{C}$.

3.1. Brill-Noether theory through K3 surfaces.

When $\mathbb{F} = \mathbb{C}$, Lazarsfeld [Laz86] first introduced a vector bundle $E = E_{C,A}$ associated to a pair (C, A) on a complex K3 surface X with $\operatorname{Pic}(X) = \mathbb{Z}[L]$, where $C \in |L|$ is a smooth curve of genus g lying on X, and A a base-point free line bundle on C with $h^0(A) = r+1$ and $\deg A = d$. $E^\vee$ fits into an exact sequence
$$0 \to E^\vee \to \mathrm{H}^0(C, A) \otimes \mathcal{O}_X \to i_* A \to 0,$$
where $i : C \hookrightarrow X$ is the inclusion. Dualizing it gives
$$0 \to \mathrm{H}^0(C, A)^* \otimes \mathcal{O}_X \to E \to i_*(\omega_C \otimes A^{-1}) \to 0. \tag{1}$$
E is a vector bundle of rank $r+1$, and $c_1(E) = \mathcal{O}_X(C)$, $c_2(E) = d$; $h^0(E) = h^0(A) + h^1(A)$, $h^1(E) = h^2(E) = 0$, $h^0(E^\vee) = h^1(E^\vee) = 0$. Moreover $\chi(E \otimes E^\vee) = 2h^0(E \otimes E^\vee) - h^1(E \otimes E^\vee) = 2 - 2\rho(r, d, g)$.

The assumption $\operatorname{Pic}(X) = \mathbb{Z}[L]$ guarantees $h^0(E \otimes E^*) = 1$, which implies $\rho(r, d, g) \ge 0$. Furthermore, [Laz86] shows a general curve $C \in |L|$ satisfies Petri's condition via generic smooth theorem over $\mathbb{C}$. These deduce Theorem 5 withough degeneration.

Question 2. *Does the same elegant story apply verbatim to positive characteristic field?*

A naive answer is NO! Simply because the assumption $\operatorname{Pic}(X) = \mathbb{Z}[L]$ fails easily. In fact, Tate conjecture implies that any K3 surface over $\overline{\mathbb{F}}_p$ has even Picard number [Huy16, §XVII. Cor 2.9]. However, a speical case when $\rho(r, d, g) = 0$ is proved under the assumption that a K3 surface X is general in its moduli.

Proposition 1 (W). [Wei23, Prop 1.1] *Fix an algebraically closed field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$. Let (X, L) be a general member of any component of the moduli space of primitively polarized K3 surfaces over $\mathbb{F}$ of genus g for $g \geq 3$. Let r, d be such integers that the Brill-Noether number $\rho(r, d, g) = 0$. Consider the Brill-Noether loci $W^r_d(C)$ for a general smooth curve $C \in |L|$, then each $A \in W^r_d(C)$ has $h^0(A) = r + 1$ and A base-point free. Moreover, $\dim W^r_d(C) = 0$.*

Proving Theorem 5 in positive characteristic in full generality without degeneration requires more investigation. In particular, I hope to use deformation argument on moduli space of K3 surfaces to establish Petri's condition for a general smooth curve C on a general K3 surface.

3.2. **Brill-Noether theory of covers of curves.** The following type of questions has been circulating for a while.

Question 3. *Given a finite étale/branched cover $f : C \to D$ of two smooth curves, how to analyze Brill-Noether theory of C through the geometry of D?*

When the target curve D is $\mathbb{P}^1$, the lowest degree of such map is called the gonality of the curve. Larson [Lar21], Larson-Larson-Vogt [LLV20] and Cook-Powell-Jensen [CPJ22a], [CPJ22b] study a refined Brill-Noether theory of a general k-gonal curve through the splitting type of vector bundles over $\mathbb{P}^1$.

When the target curve is a an elliptic curve, Kemeny [Kem20a] and Knutsen-Lelli-Chiesa [KLC24] study pencils ($r = 1$) of a general cover of elliptic curves through investigating Kontsevich moduli space of stable maps.

The story becomes more complicated when the target curve has higher genus. One of my ongoing work is to study the case when D is a genus 2 curve. I'm studying pencils on such curve C from a general cover of D by degenerating D into union of two elliptic curves and investigating the related Kontsevich moduli space of stable maps.

3.3. **Green's Conjecture on covers of curves.** Green's conjecture in this fashion is initiated by Marian Aprodu and Gavril Farkas [AF12]. Kemeny [Kem20a] shows a general cover of elliptic curves satisfies Green's conjecture by applying the following Aprodu's linear growth theorem.

Theorem 6 ([Apr05, Thm 2])**.** *Let C be any k-gonal smooth curve of genus g with $k < \lfloor \frac{g}{2} \rfloor + 2$ satisfies the following linear growth theorem*

$$\dim W^1_{k+n}(C) \leq n \quad \forall 0 \leq n \leq g + 2 - 2k.$$

Then $\mathrm{Cliff}(C) = k - 2$ and C verifies Green's conjecture.

If one can develop the theory of pencils on a general cover of a genus 2 curve, it is hopeful that Green's conjecture can be established using a similar approach.

4. Moduli Space of K3 surfaces (in positive characteristic)

The theory of moduli space over $\mathbb{C}$ using the Global Torelli Theorem of K3 surfaces and the period map of the second cohomology has been initiated by Pjateckiĭ-Šapiro and Šafarevič in [PŠŠ71], [PŠŠ73]. Even more, the surjectivity of the period map guarantees the existence of K3 sufaces with prescribed polarized K3 lattice. This simple fact plays a crucial role in Voisin's proof of generic Green's conjecture, using K3 surfaces with certain lattice [Voi02], [Voi05].

However, both the Global Torelli Theorem and surjectivity of period map fails in positive characteristic. The closest set-up for Lazarsfeld-Mukai bundle on a K3 surface with Picard number 1 is presented by Ogus' theorem:

Theorem 7. [Ogu79] *Let $\mathbb{F}$ be an algebraically closed field with $\mathrm{char}(\mathbb{F}) = p > 0$. Let $(\mathcal{X}/\mathcal{T}, \mathcal{L})$ be a versal $\mathbb{F}$-deformation of a primitively polarized K3 surface (X, L) of degree $2d$, with $(L) \subseteq \mathrm{Pic}(X)$ a direct summand. Then the geometric generic fiber $(X_{\overline{\mathcal{T}}}, \mathcal{L}_{\overline{\mathcal{T}}})$ is ordinary with degree $2d$, and $\mathrm{Pic}(X_{\overline{\mathcal{T}}})$ is generated by $\mathcal{L}_{\overline{\mathcal{T}}}$.*

This theorem combined with Proposition 1 leads to a construction of a Lazarsfeld-Mukai bundle on a general K3 surfaces in positive characteristic, using deformation. The following theorem implies a special case (Theorem 4) of Geometric Syzygy Conjecture 2 and generic Green's conjecture for even genus curves [Wei23].

Theorem 8 (W)**.** [Wei23, Thm 1.1] *Fix an algebraically closed field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = p > 0$. Let (X, L) be a general member in any component of the moduli space of primitively polarized K3 surfaces of even genus $g = 2k \geq 8$ over $\mathbb{F}$. Assume $p \geq k + 2 = \frac{g+4}{2}$. Let E be the rank two Lazarsfeld-Mukai bundle. For any nonzero $s \in \mathrm{H}^0(E)$, the space $K_{k-1,1}(X, L; \mathrm{H}^0(L \otimes I_{Z(s)}))$ is a one-dimensional subspace of $K_{k-1,1}(X, L)$. The morphism*

$$\begin{array}{rccc} \psi: & \mathbb{P}(\mathrm{H}^0(E)) & \to & \mathbb{P}(K_{k-1,1}(X, L)) \\ & [s] & \mapsto & [K_{k-1,1}(X, L; \mathrm{H}^0(L \otimes I_{Z(s)}))] \end{array}$$

is the Veronese embedding of degree $k - 2$.

In particular, ψ induces a natural isomorphism $\mathrm{Sym}^{k-2}\mathrm{H}^0(X, E) \simeq K_{k-1,1}(X, L)$.

In order to prove generic Green's conjecture for curves of odd genus g, following Voisin's framework and Kemeny's simplification, the existence of K3 surface X with $\mathrm{Pic}(X) = \mathbb{Z}[L] \oplus \mathbb{Z}[R]$ and $L^2 = 2g - 2$, $R^2 = -2$, $L \cdot R = 2$ is highly favored. This is possible by deforming a Kummer surface X from abelian surface A in positive characteristic, fixing an ample line class L and another rational class R in $\mathrm{Pic}(X)$ such that $L^2 = 2d$, $R^2 = -2$ and $L \cdot R = 2$.

Proposition 2 (W)**.** [Wei23, §5.1] *Let $\mathbb{F}$ be an algebraically closed field with $\mathrm{char}(\mathbb{F}) = p > 0$. Then the universal $\mathbb{F}$-deformation $(\mathcal{X}; \mathcal{L}, \mathcal{R}) \to \mathcal{T}$ of the above Kummer surface $(X; L, R)$ has dimension 18 and the geometric generic fiber $X_{\overline{\mathcal{T}}}$ is ordinary, with $\mathrm{Pic}(X_{\overline{\mathcal{T}}}) = \mathbb{Z}[L_{\overline{\mathcal{T}}}] \oplus \mathbb{Z}[R_{\overline{\mathcal{T}}}]$.*

5. Future Research Interest

Here, I briefly summarize some projects I am considering for future research. The ground field is assumed to be $\mathbb{C}$, while an interest in arithmetic field can be explored.

5.1. Prym-Green Conjecture.

The conjecture predicts the resolution of a *paracanonical* curve

$$\phi_{\omega_C \otimes \eta} : C \hookrightarrow \mathbb{P}^{g-2},$$

where C is a smooth algebraic curve with genus g and $\eta \in \mathrm{Pic}^0(C)[\ell]$ an ℓ-torsion line bundle.

The conjecture over $\mathbb{C}$ has been proved for all *odd* genus g when $\ell = 2$ [KF16] using *Nikulin surfaces*, or $\ell \geq \sqrt{\frac{g+2}{2}}$ [KF17] using *elliptic K3 surfaces*.

The existence of such K3 surfaces over $\mathbb{C}$ is guaranteed by the surjectivity of the period map, which fails in char p. To prove the same statement in char p using similar methods, It would be natural to study the existence and deformation of those K3 surfaces in char p.

For a general curve C of *even* genus $g = 2i + 6$, Prym-Green Conjecture amounts to the vanishing statement of Koszul cohomological group

$$K_{i,2}(C, \omega_C \otimes \eta) = K_{i+1,1}(C, \omega_C \otimes \eta).$$

However, computations using degeneration and computer algebra tools seem to suggest two possible mysterious exceptions in level 2 and genus $g = 8, 16$. It is tempting to believe the counterexamples can be extrapolated to higher genus, possibly of high divisibility by 2. [CFVV16] first proved the failure of the conjecture in genus $g = 8$ with level 2, while it is still unknown for $g = 16$.

My interest is to seek another proof of the failure of the conjecture in genus $g = 8$ with level 2, using scant bundle approaches. Hopefully, this method can cast some lights on the mysterious shadow of higher genus cases.

5.2. Ceresa cycles and Green-Griffith invariants.

Let C be a smooth algebraic curve of genus $g \geq 2$. For each point $x \in C$, the Abel-Jacobi maps $\mu_x : C \to J(C)$ via $p \mapsto [p] - [x]$. Denote C^- the image of C under the involution $(-1)_{J(C)} : J(C) \to J(C)$. The *Ceresa cycle* is the class $\mathfrak{z}(C) := [C] - [C^-]$ in the group of 1-cycles on $J(C)$ modulo algebraic equivalence (independent of the choice of x). It is well-known that it is always homologically trivial. Obviously, the Ceresa cycle of hyperelliptic curve is always trivial. Ceresa [Cer83] proved that if C is a general curve of genus $g \geq 3$, its Ceresa cycle is not algebraically trivial. People are interested in exhibiting explicit examples of high genus curve when its Ceresa cycle is not algebraically trivial. For example, Fermat curve $F(p) : \{x^p + y^p = z^p\}$ is proved to have Ceresa cycle of infinite order when $p = 4$ [Blo84] and when $p > 7$ prime [EM21]. My interest lies in posing such questions for smooth curves on K3 surfaces.

Question 4. *Is the Ceresa cycle of curves on K3 surfaces algebraically trivial? If not, is it of infinite order or torsion?*

One standard approach is to study the associated *normal function* on $\mathcal{M}_g$ as a holomorphic section of $J(\wedge_0^3\mathbb{H})$ denoting the family of intermediate Jacobians $H_3(J(C),\mathbb{C})/([C]\times H_1(J(C)))$

$$\begin{tikzcd} J(\wedge_0^3\mathbb{H}) \arrow[r] & \mathcal{M}_g \arrow[l, bend right, "\nu"'] \end{tikzcd}$$

where $\mathbb{H}=H_1(C,\mathbb{Z})$. The *rank* of such a normal function at a moduli point $[C]$ is the rank of the tangent map of ν restricted to the fiber of ν at $[C]$. For example, when $g=2$, ν is identically 0 and thus has rank 0; when $g=3$, the rank of ν is exactly 1 at every point of the hyperelliptic locus [Hai24, Cor 3]. As one might expect, ν has maximum possible rank at a general point on the moduli $\mathcal{M}_g$ for any $g\geq 3$, [Hai24, Thm 1], [GZ24], . More generally, the rank of the normal function of Ceresa cycles gives a stratification on $\mathcal{M}_g$. It is natural to ask

Question 5. *How does this stratification interplay with the stratification by gonality/Clifford index?*

Green-Griffith invariant $\overline{\delta_C}(\nu)$ is an obstruction for a normal function ν being locally constant at the moduli point $[C]$, it defines a Koszul-Green cohomology class in $\mathrm{Gr}_F^0(\mathcal{V}\otimes\Omega^1_{\mathcal{M}_g})$. For any non-hyperelliptic curve C of genus 3, [CP95, Thm 4.2.4] proves $\overline{\delta_C}(\nu)$ is exactly the quartic defining such curve under its canonical embedding. [Den24] proves for a specific familfy of trigonal genus 4 curves, the Green-Griffith invariant $\overline{\delta_C}(\nu)$ relates to its canonical embedding. One might wonder

Question 6. *For non-hyperelliptic curve C of genus 4, how does its Green-Griffith invariant $\overline{\delta_C}(\nu)$ relate to the quadric and cubic defining the curve under its canonical embedding.*

It would not be entirely unreasonable to ask the following

Question 7. *For a curve C of any genus $g\geq 3$, how does its Green-Griffith invariant $\overline{\delta_C}(\nu)$ relate to the canonial syzygies of the curve C?*

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