COMBINATORIAL HODGE THEORY

1. Introduction

1.1. Overview of the course. Hodge theory has tremendous applications in algebraic geometry. From the classical theory developed by Hodge, to mixed Hodge structures by Deligne, variation of Hodge structures by Griffiths, Hodge modules by Saito, they all had fundamental influence in the development of algebraic geometry.

At the beginning of this century, examples of natural Hodge structures not defined from algebraic geometry start to appear. First, for irrational fans, there are no associated toric varieties. Nevertheless, their intersection cohomology groups can be defined combinatorially, and they satisfy the Kähler package as if they come from projective varieties. The second examples are the Soergel bimodules, which are defined for all Coxeter groups. When the Coxeter groups are not Weyl groups, there are no algebraic varieties from which the Soergel bimodules are defined. However, it is proved by Elias and Williamson that the Soergel bimodules also satisfy Kähler packages. As applications, they prove that all Kazhdan-Lusztig polynomials have nonnegative coefficients. More recently, various Chow rings and intersection cohomology groups are defined for arbitrary matroids. Again, for non-realizable matroids, we see Hodge structures not arising from algebraic varieties.

In this course, we plan to explain how Hodge structures are defined combinatorially from matroids. First, we will start with a short introduction on matroids and state the main results, which we will aim to prove. Next, we will give a crash course on algebraic geometry and Hodge theory. Then we move on to more detailed introduction of toric varieties, which will be essential to the remaining lectures. Finally, we will introduce the Chow ring of a matroid (as in [AHK]), prove its Kähler package (following the slightly simpler arguments in [BHM+]), and deduce the Heron-Rota-Welsh conjecture.

1.2. Motivating examples. As a motivation for the course, we present some results in combinatorics, whose proof uses ideas from algebraic geometry in an essential way.

First, we review the definition of chromatic polynomials of graphs. Given a finite graph $G$, a proper vertex coloring is a coloring of the graph such that the two vertices of any edge have different colors. Let $P(G,t)$ denote the number of proper vertex colorings with $t$ colors. Then $P(G,t)$ is a polynomial in $t$ with alternating coefficients (Exercise). For example,

- if $G$ is the complete graph with $n$-vertices, then $P_G(t) = t(t - 1) \cdots (t - n + 1)$;
- if $G$ is a tree with $n$-vertices, then $P_G(t) = t(t - 1)^{n-1}$;
- if $G$ contains a loop, then $P_G(t) = 0$.

The following theorem was first conjectured by Read in 1968 for the unimodular part, and later by Rota, Heron, and Welsh in a more general setting (using the language of matroids, which we will learn at the beginning of the semester).
Theorem 1.1 (Huh 2012). The absolute values of the coefficients of any chromatic polynomial is a log-concave sequence without internal zeros. More precisely, if
\[ P_G(t) = a_n t^n + \cdots + a_1 t, \]
and \( a_k \) is the last nonzero coefficient, then \( a_i \neq 0 \) for \( k \leq i \leq n \) and \( |a_i|^2 \geq |a_{i-1}a_{i+1}| \) for \( k+1 \leq i \leq n-1 \). In particular, the absolute values of the coefficients are unimodal.

Even though the statement does not involve any algebraic variety or any geometric objects, the original proof of Huh reduces the statement to the Hodge-Riemann relations of smooth complex projective varieties. The main goal of this course is to prove a more general version of this theorem, which is the Rota-Heron-Welsh conjecture and theorem of Adiprasito-Huh-Katz. Let us mention another result of similar nature.

Theorem 1.2 (Huh-Wang 2017). Given a \( d \)-dimensional vector space \( V \) and a finite generating set \( E \subset V \), let \( F \) be the collection of linear subspaces of \( V \) that is generated by a subset of \( E \). Denote the number of \( k \)-dimensional subspaces in \( F \) by \( W_k \), then \( W_k \leq W_{d-k} \) for \( k \leq d/2 \).

The above theorem is the realizable case of the so-called “top-heavy” conjecture posted by Dowling-Wilson in 1975. Even though the statement only involves vector configurations, the proof somehow reduces to the Hodge theory of some singular projective algebraic variety. For the non-realizable case, the same statement is proved more recently by Braden-Huh-Matherne-Proudfoot-Wang. Because the proof is much more technical, it will not be covered in this course. For interested students, we refer to the survey paper [Oko22].

2. Introduction to matroids

2.1. Definitions. Let \( E \) be a finite set. A matroid structure on \( E \) can be defined equivalently in many different ways. We will study the ones using rank functions, independent sets, bases, and flats. We refer to [Oxl11] for more detailed introductions.

Definition 2.1. A matroid rank function is a function \( r : 2^E \to \mathbb{Z}_{\geq 0} \) satisfying the following properties:
\begin{itemize}
  \item \( r(S) \leq r(T) \) for \( S \subset T \subset E \);
  \item \( r(S) \leq |S| \);
  \item \( r(S) + r(T) \geq r(S \cup T) + r(S \cap T) \), that is, \( r \) is submodular.
\end{itemize}

Definition 2.2. A collection \( \mathcal{I} \) of subsets of \( E \) is called independent sets, if they satisfy the following properties:
\begin{itemize}
  \item \( \emptyset \in \mathcal{I} \);
  \item if \( A \subset B \), and \( B \in \mathcal{I} \), then \( A \in \mathcal{I} \);
  \item if \( A, B \in \mathcal{I} \) and \( A \) has more elements than \( B \), then there exists \( x \in A \) such that \( B \cup \{x\} \in \mathcal{I} \).
\end{itemize}
The last condition is called the independent set exchange property.

Definition 2.3. A collection \( \mathcal{B} \) of subsets of \( E \) is called bases, if they satisfy the following properties:
• $\mathcal{B}$ is nonempty;
• if $A, B \in \mathcal{B}$ and $x \in A \setminus B$, then there exists $y \in B \setminus A$ such that $(A \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

The second property is called the basis exchange property, and it implies that all bases have the same cardinality.

**Definition 2.4.** A collection $\mathcal{F}$ of subsets of $E$ are called flats, if they satisfy the following properties:
• $E \in \mathcal{F}$;
• if $F, G \in \mathcal{F}$, then $F \cap G \in \mathcal{F}$;
• for any flat $F \in \mathcal{F}$, the minimal $G \in \mathcal{F}$ strictly containing $F$ induce a partition of $E \setminus F$.

**Theorem/Definition 2.5.** All the above structures on $E$ are equivalent, and they define a matroid structure on $E$.

**Proof.** Exercise! □

At the end of this subsection, we introduce a few more terminologies.

**Definition 2.6.** Let $M$ be a matroid defined over a set $E$ using one of the above equivalent definitions. Then $E$ is called the ground set of $M$. The rank of $M$ is defined to be $r(E)$, and also denoted by $r(M)$. An element $i \in E$ is called a loop if $r(i) = 0$,

\[1\]
which is equivalent to that $i$ is not contained in any independent set, and is further equivalent to that $i$ is contained in every flat. Two elements $i, j \in E$ are called parallel, if $r(i) = r(j) = r(\{i, j\}) = 1$. If a matroid has no loops, then we say it is loopless. If a matroid has no loops or parallel elements, then we say it is simple.

**Remark 2.7.** For the moment, we skip some important concepts in matroid theory: circuits, dual matroids.

### 2.2. Examples.

A uniform matroid on $E = \{1, \ldots, n\}$ with rank $r$, denoted by $U_{r,n}$, is a matroid whose bases are all $r$-element subsets of $E$. When $r = n$, the matroid $U_{r,n}$ is called a Boolean matroid.

The bases $\mathcal{B} = \{\{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$ define a rank 3 matroid on $E = \{1, 2, 3, 4\}$.

Given a connected graph $G$ with vertex set $V$ and edge set $E$ (here we allow loops and multiple edges), we let $\mathcal{B}$ be the set of spanning trees. Then $\mathcal{B}$ form the bases of a matroid. More generally, any graph $G$ defines a matroid whose independent set $\mathcal{I}$ is the set of forests in $G$.

**Definition 2.8.** A matroid defined by a graph as above is called a graphic matroid.

**Exercise 2.9.** Prove the above rank 3 matroid is defined by a graph. Prove that the uniform matroid of rank 2 on 4 elements is not defined by any graph.

**Definition 2.10.** Let $V$ be a finite-dimensional vector space and let $E \subset V$ be a finite subset (usually we assume $E$ to be spanning). Let $\mathcal{I}$ be all the subsets of $E$ that are independent in $V$. Then $\mathcal{I}$ form independent sets and hence define a matroid. We call such $E$ a vector configuration, and we call a matroid realizable if it is defined by a vector

\[\text{Instead of } r(\{i\}), \text{ we will simply write } r(i).\]
configuration. More specifically, given a field $\mathbb{K}$, we say a matroid $M$ is realizable over $\mathbb{K}$, it it can be realized by such a pair $E \subset V$ with $V$ a $\mathbb{K}$-vector space.

**Proposition 2.11.** A graphic matroid is realizable over any field.

*Proof.* Given a finite graph $G$, we label its vertices by $1, 2, \ldots, n$. Fixing any field $\mathbb{K}$, we let $V$ be the $\mathbb{K}$-vector space with basis $v_1, \ldots, v_n$. Let the set of edges be $E = \{e_1, \ldots, e_m\}$, and assume that the two ends of each $e_i$ be $e_1^i, e_2^i \in \{1, \ldots, n\}$. Now, we choose the subset $E_V \subset V$ to be $\{e_1^i - e_2^i | i = 1, \ldots, m\}$. Then it is easy to check that a subset of $E$ form a forest if and only if the corresponding set of vectors in $V$ are independent. $\Box$

**Example 2.12.** The fano matroid, defined as in the following picture, is realizable over any field of characteristic 2, and not over any other characteristics.

The non-Pappus matroid, defined as in the following, is not realizable over any field.

2.3. Hyperplane arrangements. We have seen that a vector configuration $E \subset V$ defines a matroid. An equivalent way to see this construction is through hyperplane arrangement. Fixing a $d$-dimensional vector space $W$, a *central hyperplane arrangement* is a collection $A = \{H_1, \ldots, H_n\}$ of $(d-1)$-dimensional linear subspaces (also called hyperplanes). Here we allow two hyperplanes to be the same. The hyperplane arrangement $A$ is called *essential*, if the intersection of all the hyperplanes $H_i$ is equal to zero.

**Proposition 2.13.** Given such a central hyperplane arrangement $A$, it defines a matroid $M$ on $E = \{1, \ldots, n\}$ whose rank function is given by

$$r(S) = d - \dim \bigcap_{i \in S} H_i.$$  

*Proof.* Let $V = W^\vee$ be the dual vector space of $W$. Each hyperplane $H_i$ determines a vector $v_i \in V$, unique up to scalar, by the formula

$$H_i = \{w \in W \mid v_i(w) = 0\}.$$
If we identify $E$ with $\{v_1, \ldots, v_n\}$, then the above rank function is exactly equal to the rank function defined by the set $\{v_1, \ldots, v_n\} \subset V$, which we know is the rank function of a matroid.

**Remark 2.14.** The above proposition shows that for loopless matroids, being realizable by a vector configuration is equivalent to being realizable by a hyperplane arrangement.

**Remark 2.15.** If the hyperplane arrangement $\mathcal{A}$ is not essential, we can let $W_0 = H_1 \cap \cdots \cap H_n$ and replace $\mathcal{A}$ by the hyperplane arrangement $\{H_1/W_0, \ldots, H_n/W_0\}$ in $W/W_0$ without changing the associated matroid.

The advantage of realizing a matroid by a hyperplane arrangement instead of a vector configuration is that the hyperplane arrangement has richer geometry that can be used to study the combinatorics of the associated matroid, as we will see later in this section.

### 2.4. Deletion, contraction, lower and upper intervals.

Given a matroid $M$ and an element $e$ in the ground set $E$, we can form two new matroids $M \setminus e$ and $M/e$, the deletion and contraction of $e$.

First, we explain these operations in terms of graphs, vector configurations, and hyperplane arrangements.

Let $G$ be a graph, and let $e$ be an edge of $G$ with endpoints $v_1, v_2$. Then $G \setminus e$ is the new graph whose vertices are the same as $G$, and its edges are all the edges of $G$ except $e$. For the graph $G/e$, we keep all the vertices of $G$, except merging $v_1$ and $v_2$ together, and we delete the edge $e$, keeping all the other remaining edges. In the case $v_1 = v_2$, the two graphs $G \setminus e$ and $G/e$ are the same.

Given a vector configuration $E \subset V$ and an element $e \in E$, we will define two new vector configurations, the deletion and the contraction. The deletion is simply defined by $E \setminus \{e\}$ as a subset of $V$. The contraction is defined to be the image of $E \setminus \{e\}$ in $V/\langle e \rangle$. In the case $e = 0$, the deletion is equal to the contraction.

Given a hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ in $W$, the deletion of $H_i$ is defined to be the hyperplane arrangement $\{H_1, \ldots, \hat{H}_i, \ldots, H_n\}$ in $W$. The contraction of $H_i$ is defined to be the hyperplane arrangement $\{H_1 \cap H_i, \ldots, H_{i-1} \cap H_i, H_{i+1} \cap H_i, \ldots, H_n \cap H_i\}$ in $H_i$.

Now we give the formal definition of deletion and contraction of a matroid using rank functions.

**Definition 2.16.** Let $M$ be a matroid with ground set $E$. For any $e \in E$, we define the **deletion matroid** $M \setminus e$ to be the matroid on $E \setminus \{e\}$ with rank functions $r_{M \setminus e}(S) = r_M(S)$ for any $S \subset E$. We define the **contraction matroid** $M/e$ to be the matroid on $E \setminus \{e\}$ with rank function $r_{M/e}(S) = r(S \cup \{e\}) - r(e)$.

**Remark 2.17.** Equivalently, the deletion and contraction can be defined using independent sets:

$$\mathcal{I}_{M \setminus e} = 2^{E \setminus \{e\}} \cap \mathcal{I}_M$$

and

$$\mathcal{I}_{M/e} = \{I \subset E \setminus e \mid I \cup \{e\} \in \mathcal{I}_M\}.$$
**Remark 2.18.** If $e$ is a loop, then $M \setminus e = M/e$, whose flats are naturally bijective to the flats of $M$. If $e$ is not a loop, then $r(M/e) = r(M) - 1$. On the other hand, if $e$ appears in every basis (in this case, $e$ is called a coloop), then $r(M \setminus e) = r(M) - 1$; otherwise, $r(M \setminus e) = r(M)$.

**Definition 2.19.** Given a flat $F$ of a matroid $M$, we define the matroid $M^F$ to be the matroid deleting every element of $E \setminus F$ from $M$. We also define the matroid $M^F$ to be the matroid contracting every element of $F$ from $M$.

**Proposition 2.20.** The poset of flats of $M^F$ is naturally isomorphic to the poset of the flats $\{G \in \mathcal{F} \mid G \leq F\}$. Similarly, the poset of flats of $M^F$ is naturally isomorphic to the poset of the flats $\{G \in \mathcal{F} \mid G \geq F\}$.

**Proof.** Exercise! □

**Definition 2.21.** Another way to construct new matroid is taking direct sum. Let $M_1$ and $M_2$ be matroids with ground sets $E_1$ and $E_2$ respectively. Then their direct sum $M_1 \oplus M_2$ is defined to be the matroid with ground set $E_1 \sqcup E_2$ and

$$I_{M_1 \oplus M_2} = \{I_1 \cup I_2 \mid I_1 \in I_{M_1}, I_2 \in I_{M_2}\}.$$ 

**Remark 2.22.** Equivalently, the direct sum matroid can be defined using basis

$$B_{M_1 \oplus M_2} = \{B_1 \cup B_2 \mid B_1 \in B_{M_1}, B_2 \in B_{M_2}\},$$

or rank function

$$r_{M_1 \oplus M_2}(S_1 \cup S_2) = r_{M_1}(S_1) + r_{M_2}(S_2),$$

or flats

$$F_{M_1 \oplus M_2} = \{F_1 \cup F_2 \mid F_1 \in F_{M_1}, F_2 \in F_{M_2}\}.$$ 

2.5. **Characteristic polynomials.**

**Definition 2.23.** Let $M$ be a matroid with ground set $E$. Its characteristic polynomial is defined as

$$\chi_M(t) = \sum_{S \subseteq E} (-1)^{|S|} t^{r(M) - r(S)}.$$ 

**Proposition 2.24.** The characteristic polynomial $\chi_M(t)$ satisfies the following properties:

1. (loop property) if $M$ has a loop, then $\chi_M(t) = 0$;
2. (normalization) the characteristic polynomial of the uniform matroid $U_{1,1}$ satisfies

$$\chi_{U_{1,1}}(t) = t - 1;$$
3. (direct sum) if $M = M_1 \oplus M_2$, then

$$\chi_{M_1 \oplus M_2}(t) = \chi_{M_1}(t) \cdot \chi_{M_2}(t);$$
4. (deletion/contraction) if $e$ is not a coloop of $M$, then

$$\chi_M(t) = \chi_{M \setminus e}(t) - \chi_{M/e}(t).$$

Moreover, the characteristic polynomial is the unique way to associate each matroid a polynomial such that all the above properties are satisfied.
Proof. Properties (2), (3), (4), can be checked directly from the definition of the characteristic polynomial. Property (1) follows from (4).

The uniqueness can be proved inductively on the cardinality of the ground set. When the ground set consists of one element, the uniqueness follows from (1) and (2). If a matroid has an element that is not a coloop, then the uniqueness follows from induction hypothesis and (4). If all elements are coloops, then the matroid is Boolean, and it is a direct sum of $U_{1,1}$. So the uniqueness follows from (2) and (3).

□

It is a generalization of the chromatic polynomial in the following sense.

**Proposition 2.25.** Let $G$ be a graph, and let $M$ be the associated matroid. Then,

\[(1) \quad P_G(t) = l \cdot \chi_M(t)\]

where $l$ is the number of connected components of $G$.

Proof. The chromatic polynomial also has a deletion/contraction formula

$$P_G(t) = P_{G\setminus e}(t) - P_{G/e}(t)$$

for any edge $e$. The equality (1) can be proved using induction by showing $t^l \cdot \chi_M(t)$ also satisfies the above identity.

□

The following two propositions demonstrate that the characteristic polynomial of a matroid is closely related to the geometry of the complement of a hyperplane arrangement.

**Proposition 2.26.** Suppose that $A = \{H_1, \ldots, H_n\}$ is an essential hyperplane arrangement in $W = (\mathbb{F}_q)^d$, and denote the associated matroid by $M$. Let

$$U = W \setminus \left( \bigcup_{1 \leq i \leq n} H_i \right)$$

be the complement of the hyperplane arrangement. Then the number of $\mathbb{F}_q$ points in $U$ is equal to $\chi_M(q)$.

Proof. We can count the number of $\mathbb{F}_q$ points using inclusion-exclusion principle, and this will lead to the defining formula of $\chi_M(t)$.

□

**Proposition 2.27.** Suppose that $A = \{H_1, \ldots, H_n\}$ is an essential hyperplane arrangement in $W = \mathbb{C}^d$, and denote the associated matroid by $M$ (with ground set $E = \{1, \ldots, n\}$). Let $U$ be the complement of the hyperplane arrangement. Then

\[(2) \quad \chi_M(t) = \sum_{0 \leq k \leq d} (-1)^k \dim_{\mathbb{Q}} H^k(U, \mathbb{Q}) t^{d-k} \]

\[(3) \quad = \sum_{0 \leq k \leq d} (-1)^{d-k} \dim_{\mathbb{Q}} H^{d+k}_c(U, \mathbb{Q}) t^k.\]

Proof. The second equality follows from Poincare duality. For the first equality, we follow arguments of [Dim17, Proposition 3.3]. The idea is to prove that the right-hand side of (2) also satisfies the deletion-contraction relations. Let

$$U' = V \setminus \left( \bigcup_{1 \leq i \leq n-1} H_i \right) \quad \text{and} \quad U'' = U' \cap H_n.$$
Then $U''$ is a closed complex submanifold of $U'$ and $U$ is the complement. So we have a relative cohomology long exact sequence (all cohomology groups are with $\mathbb{Q}$-coefficients):

\[(4) \quad \cdots \to H^k(U', U) \to H^k(U') \to H^k(U) \to H^{k+1}(U', U) \to \cdots.\]

Let $T$ be a tubular neighborhood of $U''$ in $U'$ (see [BT, page 65-66] for the definition and existence of tubular neighborhoods). Then by excision, we have

\[H^k(U', U) = H^k(U', U \setminus U'') \cong H^k(T, T \setminus U'').\]

Since the tubular neighborhood $T$ is diffeomorphic to a rank one $\mathbb{C}$-vector bundle of $U''$, it follows from Thom Isomorphism (see [BT, Theorem 6.17]) that

\[H^k(T, T \setminus U'') \cong H^{k-2}(U'').\]

In fact, we can see the above isomorphism more directly. Clearly, the normal bundle of $H_n$ in $W = \mathbb{C}^n$ is trivial. Thus, the normal bundle of $U''$ in $U'$ is also trivial. Since a tubular neighborhood is diffeomorphic to the normal bundle, we have the following diffeomorphism of a pair

\[(T, T \setminus U'') \cong (U'' \times \mathbb{C}, U'' \times \mathbb{C}^*)\]

where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. By Künneth formula,

\[H^k(U'' \times \mathbb{C}, U'' \times \mathbb{C}^*) \cong \bigoplus_{0 \leq i \leq k} H^{k-i}(U'') \otimes H^i(\mathbb{C}, \mathbb{C}^*) \cong H^{k-2}(U'') \otimes H^2(\mathbb{C}, \mathbb{C}^*) \cong H^{k-2}(U'').\]

Thus, the long exact sequence (4) can be rewritten as

\[\cdots \to H^{k-2}(U'') \to H^k(U') \to H^k(U) \to H^{k-1}(U'') \to \cdots\]

We claim that the maps $H^{k-2}(U'') \to H^k(U')$ in the long exact sequence vanish and the long exact sequence splits into short exact sequences

\[(5) \quad 0 \to H^k(U') \to H^k(U) \to H^{k-1}(U'') \to 0.\]

The vanishing of maps $H^{k-2}(U'') \to H^k(U')$ can be proved using the fact that it is a morphism of mixed Hodge structures, together with the fact that $H^k(U')$ has weight $2k$ and $H^{k-2}(U'') = H^{k-2}(U'', \mathbb{Q}_U(1))$ has weight $2(k-2) + 2 = 2k - 2$, where (1) denotes the Tate twist.

The vanishing of $H^{k-2}(U'') \to H^k(U')$ is also an intermediate step in the proof of the theorem that the cohomology ring of $H^*(U)$ is isomorphic to the Orlik-Solomon algebra, and it can be proved using induction (see [Dim17, Theorem 3.5]). We will discuss this in next subsection.

Now, we prove the equation (2) using (5) and induction on $n$. When $n = 1$, $U = \mathbb{C}^*$ and equation (2) obviously holds. For larger $n$, we first assume that $H_n$ does not correspond to a coloop in the matroid $M$, that is, $n \in E = \{1, \ldots, n\}$ is not a coloop of $M$. In this case, the deletion $\mathcal{A}' \setminus \{H_n\}$ is also an essential hyperplane arrangement. Moreover,
the complement of the deletion hyperplane arrangement is $U'$ and the complement of the contraction hyperplane arrangement is $U''$. Thus, by Proposition 2.24 (4),

$$\chi_M(t) = \chi_{M\setminus n}(t) - \chi_{M/n}(t),$$

and by induction hypothesis, we know that

$$\chi_{M\setminus n}(t) = \sum_{0 \leq k \leq d} (-1)^k \dim H^k(U') t^{d-k}$$

and

$$\chi_{M/n}(t) = \sum_{0 \leq k \leq d-1} (-1)^k \dim H^k(U'') t^{d-1-k}.$$

Combining the above three equations and using the short exact sequence (5), we have

$$\chi_M(t) = \sum_{0 \leq k \leq d} (-1)^k \dim H^k(U') t^{d-k} - \sum_{0 \leq k \leq d-1} (-1)^k \dim H^k(U'') t^{d-1-k}$$

$$= \sum_{0 \leq k \leq d} (-1)^k \dim H^k(U') + \sum_{0 \leq k \leq d-1} (-1)^k \dim H^{k-1}(U'') t^{d-k}$$

$$= \sum_{0 \leq k \leq d} (-1)^k \dim H^k(U') t^{d-k}.$$

Hence, equation (2) follows.

The remaining case is when $n$ is a coloop of $M$. In this case, there exists an isomorphism $U' \cong U'' \times \mathbb{C}$ and under this isomorphism the subset $U'' \subset U'$ is identified with $U'' \times \{0\}$. Hence, $U \cong U'' \times \mathbb{C}^*$. By Künneth formula,

$$H^k(U) \cong H^k(U'') \oplus H^{k-1}(U'').$$

On the other hand, since $n$ is a coloop of $M$, $M \cong (M \setminus n) \oplus U_{1,1}$. Thus, by Proposition 2.24 (2) and (3),

$$\chi_M(t) = (t - 1)\chi_{M\setminus n}(t).$$

By induction hypothesis,

$$\chi_{M\setminus n}(t) = \sum_{0 \leq k \leq d-1} (-1)^k \dim H^k(U'') t^{d-1-k}.$$

Hence,

$$\chi_M(t) = (t - 1)\chi_{M\setminus n}(t) = \sum_{0 \leq k \leq d} (-1)^k \dim H^k(U') + \dim H^{k-1}(U'') t^{d-k}$$

and by (6),

$$\chi_M(t) = \sum_{0 \leq k \leq d} (-1)^k \dim H^k(U') t^{d-k}.$$

We have completed the proof. \qed
2.6. Orlik-Solomon algebra and the geometry of hyperplane arrangement complements. Let $M$ be a matroid with ground set $E = \{1, \ldots, n\}$. Let $V_E$ be an $n$-dimensional $\mathbb{Q}$-vector space with basis $e_1, \ldots, e_n$. Let $\bigwedge^\bullet V_E$ be the exterior algebra of $V_E$. We consider $\bigwedge^\bullet V_E$ as a graded algebra by setting $\deg e_i = 1$ for all $i$. Define a degree $-1$ linear map $\partial_E : \bigwedge^\bullet V_E \to \bigwedge^\bullet V_E$ by setting $\partial_E e_1 = 0$, $\partial_E e_i = 1$, and $
abla \partial_E (e_S) = \sum_{1 \leq k \leq l} (-1)^k e_{S \setminus \{i_k\}}$ for any $S \subset E$.

Here, for any subset $S \subset E$, we denote $e_{i_1} \cdots e_{i_l}$ by $e_S$. The Orlik-Solomon algebra of $M$, denoted by $\text{OS}^\bullet(M)$, is the quotient of $\bigwedge^\bullet V_E$ by the ideal generated by $\partial_E e_S$ for all dependent sets $S$ of $M$. It is a graded commutative algebra.

Exercise 2.28. The differential $\partial_E$ descents to a differential $\partial$ on $\text{OS}^\bullet(M)$, and the complex $(\text{OS}^\bullet(M), \partial)$ is acyclic when the rank of $M$ is positive.

Now, let $A = \{H_1, \ldots, H_n\}$ be a hyperplane arrangement in $W = \mathbb{C}^d$. Let $f_i : W \to \mathbb{C}$ be the homogeneous linear function whose zero locus is equal to $H_i$. Then, we have a linear map

$$F = (f_1, \ldots, f_n) : W \to \mathbb{C}^n.$$ Let $U$ be the complement of the hyperplane arrangement. Then $F$ induces a holomorphic map

$$F_U : U \to (\mathbb{C}^*)^n.$$