

2 Grad, div, curl

Consider a scalar function of a vector variable: $f(\vec{r})$, for instance the pressure $p(\vec{r})$ as a function of position, or the temperature $T(\vec{r})$ at point $\vec{r}$, etc. One way to visualize such functions is to consider *isosurfaces* or *level sets*, these are the set of all $\vec{r}$'s for which $f(\vec{r}) = C_0$, for some constant C_0 . In cartesian coordinates $\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$ and the scalar function is a function of the three coordinates $f(\vec{r}) \equiv f(x, y, z)$, hence we can interpret an isosurface $f(x, y, z) = C_0$ as a single equation for the 3 unknowns x, y, z . In general, we are free to pick two of those variables, x and y say, then solve the equation $f(x, y, z) = C_0$ for z . For example, the isosurfaces of $f(\vec{r}) = x^2 + y^2 + z^2$ are determined by the equation $f(\vec{r}) = C_0$. This is the sphere of radius $\sqrt{C_0}$, if $C_0 \geq 0$.

2.1 Geometric definition of the Gradient

The value of $f(\vec{r})$ at a point $\vec{r}_0$ defines an isosurface $f(\vec{r}) = f(\vec{r}_0)$ through that point $\vec{r}_0$. The *gradient* of $f(\vec{r})$ at $\vec{r}_0$ can be defined geometrically as the *vector*, denoted $\vec{\nabla}f(\vec{r}_0)$, that

- (i) is perpendicular to the isosurface $f(\vec{r}) = f(\vec{r}_0)$ at the point $\vec{r}_0$ and points in the direction of increase of $f(\vec{r})$ and
- (ii) has a magnitude equal to the rate of change of $f(\vec{r})$ with distance from the isosurface.

From this geometric definition, we deduce that the plane tangent to the isosurface $f(\vec{r}) = f(\vec{r}_0)$ at $\vec{r}_0$ has the equation

$$(\vec{r} - \vec{r}_0) \cdot \vec{\nabla}f(\vec{r}_0) = 0 \quad (50)$$

Likewise, the plane $(\vec{r} - \vec{r}_0) \cdot \vec{\nabla}f(\vec{r}_0) = \epsilon$ is parallel to the tangent plane but further ‘up’ in the direction of the gradient if $\epsilon > 0$ or ‘down’ if $\epsilon < 0$. More generally, the function $f(\vec{r})$ can be locally (*i.e.* for $\vec{r}$ near $\vec{r}_0$) approximated by

$$f(\vec{r}) \approx f(\vec{r}_0) + (\vec{r} - \vec{r}_0) \cdot \vec{\nabla}f(\vec{r}_0). \quad (51)$$

Indeed since $\vec{\nabla}f$ points in the direction of fastest increase of $f(\vec{r})$ and has a magnitude equal to the rate of change of f with distance in that direction, the change in f as we move slightly away from $\vec{r}_0$ only depends on how much we have moved in the direction of the gradient. The actual change in f is the distance times the rate of change with distance, hence it is $(\vec{r} - \vec{r}_0) \cdot \vec{\nabla}f(\vec{r}_0)$. Equation (51) states that the isosurfaces of $f(\vec{r})$ look like planes perpendicular to $\vec{\nabla}f(\vec{r}_0)$ for $\vec{r}$ in the neighborhood for $\vec{r}_0$. Equation (51) is a *linear approximation* of $f(\vec{r})$ in the neighborhood of $\vec{r}_0$, just like $f(x) \approx f(x_0) + (x - x_0)f'(x_0)$ for functions of one variable. This is not exact, there is a small error that goes to zero faster than $|\vec{r} - \vec{r}_0|$ as $\vec{r} \rightarrow \vec{r}_0$. The limit as $\vec{r} \rightarrow \vec{r}_0$ can be written as the exact differential relation

$$\boxed{df(\vec{r}) = d\vec{r} \cdot \vec{\nabla}f(\vec{r})}, \quad (52)$$

where $df(\vec{r}) = f(\vec{r} + d\vec{r}) - f(\vec{r})$ is the differential change in $f(\vec{r})$ when $\vec{r}$ changes from $\vec{r}$ to $\vec{r} + d\vec{r}$. This differential relationship holds for any $\vec{r}$ and $d\vec{r}$. It is analogous to the differential relationship $df = f'(x)dx$ for functions of a single variable.

► It follows immediately from the geometric definition of the gradient that if $r = |\vec{r}|$ is the distance to the origin and $\hat{r} = \vec{r}/r$ is the unit radial vector, then $\vec{\nabla}r = \hat{r}$ and more generally $\vec{\nabla}f(r) = \hat{r}df/dr$. For instance, $\vec{\nabla}(1/r) = -\hat{r}/r^2$. ■

2.2 Directional derivative, gradient and the $\vec{\nabla}$ operator

The rate of change of $f(\vec{r})$ in the direction of the unit vector $\vec{n}$ at point $\vec{r}$, denoted $\partial f(\vec{r})/\partial n$ is defined as

$$\frac{\partial f(\vec{r})}{\partial n} = \lim_{h \rightarrow 0} \frac{f(\vec{r} + h\vec{n}) - f(\vec{r})}{h}. \quad (53)$$

From (51) and (52),

$$\frac{\partial f(\vec{r})}{\partial n} = \vec{n} \cdot \vec{\nabla} f(\vec{r}). \quad (54)$$

This is exact because the error of the linear approximation (51), $f(\vec{r} + h\vec{n}) \approx f(\vec{r}) + h\vec{n} \cdot \vec{\nabla} f(\vec{r})$, goes to zero faster than h as $h \rightarrow 0$.

In particular,

$$\frac{\partial f}{\partial x} = \vec{e}_x \cdot \vec{\nabla} f, \quad \frac{\partial f}{\partial y} = \vec{e}_y \cdot \vec{\nabla} f, \quad \frac{\partial f}{\partial z} = \vec{e}_z \cdot \vec{\nabla} f, \quad (55)$$

hence

$$\vec{\nabla} f = \vec{e}_x \frac{\partial f}{\partial x} + \vec{e}_y \frac{\partial f}{\partial y} + \vec{e}_z \frac{\partial f}{\partial z}, \quad (56)$$

for cartesian coordinates x, y, z . We can also deduce this important result from the *Chain rule* for functions of several variables. In cartesian coordinates, written (x_1, x_2, x_3) in place of (x, y, z) , the position vector reads $\vec{r} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$ and the scalar function $f(\vec{r}) \equiv f(x_1, x_2, x_3)$ is a function of the 3 coordinates. The directional derivative

$$\frac{\partial f(\vec{r})}{\partial n} = \lim_{h \rightarrow 0} \frac{f(\vec{r} + h\vec{n}) - f(\vec{r})}{h} = n_1 \frac{\partial f}{\partial x_1} + n_2 \frac{\partial f}{\partial x_2} + n_3 \frac{\partial f}{\partial x_3} = \vec{n} \cdot \vec{\nabla} f. \quad (57)$$

This follows from rewriting the difference $f(\vec{r} + h\vec{n}) - f(\vec{r})$ as the telescoping sum

$$\begin{aligned} & [f(x_1 + hn_1, x_2 + hn_2, x_3 + hn_3) - f(x_1, x_2 + hn_2, x_3 + hn_3)] \\ & + [f(x_1, x_2 + hn_2, x_3 + hn_3) - f(x_1, x_2, x_3 + hn_3)] \\ & + [f(x_1, x_2, x_3 + hn_3) - f(x_1, x_2, x_3)] \end{aligned} \quad (58)$$

then using continuity and the definition of the partial derivatives, *e.g.*

$$\lim_{h \rightarrow 0} \frac{f(\vec{r} + hn_3\vec{e}_3) - f(\vec{r})}{h} = n_3 \lim_{\delta \rightarrow 0} \frac{f(\vec{r} + \delta\vec{e}_3) - f(\vec{r})}{\delta} = n_3 \frac{\partial f}{\partial x_3}.$$

Either way, we establish that in cartesian coordinates

$$\boxed{\vec{\nabla} = \vec{e}_1 \frac{\partial}{\partial x_1} + \vec{e}_2 \frac{\partial}{\partial x_2} + \vec{e}_3 \frac{\partial}{\partial x_3} \equiv \vec{e}_i \partial_i} \quad (59)$$

where ∂_i is short for $\partial/\partial x_i$ and we have used the convention of summation over all values of the repeated index i .

2.3 Div and Curl

We'll depart from our geometric point of view to first define divergence and curl computationally based on their cartesian representation. Here we consider *vector fields* $\vec{v}(\vec{r})$ which are vector functions of a vector variable, for example the velocity $\vec{v}(\vec{r})$ of a fluid at point $\vec{r}$, or the electric field $\vec{E}(\vec{r})$ at point $\vec{r}$, etc. For the geometric meaning of divergence and curl, see the sections on divergence and Stokes' theorems.

The *divergence* of a vector field $\vec{v}(\vec{r})$ is defined as the dot product $\vec{\nabla} \cdot \vec{v}$. Now since the unit vectors $\vec{e}_i$ are constant for cartesian coordinates, $\vec{\nabla} \cdot \vec{v} = (\vec{e}_i \partial_i) \cdot (\vec{e}_j v_j) = (\vec{e}_i \cdot \vec{e}_j) \partial_i v_j = \delta_{ij} \partial_i v_j$ hence

$$\boxed{\vec{\nabla} \cdot \vec{v} = \partial_i v_i = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3.} \quad (60)$$

Likewise, the *curl* of a vector field $\vec{v}(\vec{r})$ is the cross product $\vec{\nabla} \times \vec{v}$. In cartesian coordinates, $\vec{\nabla} \times \vec{v} = (\vec{e}_j \partial_j) \times (\vec{e}_k v_k) = (\vec{e}_j \times \vec{e}_k) \partial_j v_k$. Recall that $\epsilon_{ijk} = \vec{e}_i \cdot (\vec{e}_j \times \vec{e}_k)$, or in other words ϵ_{ijk} is the i component of the vector $\vec{e}_j \times \vec{e}_k$, thus $\vec{e}_j \times \vec{e}_k = \vec{e}_i \epsilon_{ijk}$ and

$$\boxed{\vec{\nabla} \times \vec{v} = \vec{e}_i \epsilon_{ijk} \partial_j v_k.} \quad (61)$$

(Recall that $\vec{a} \times \vec{b} = \vec{e}_i \epsilon_{ijk} a_j b_k$.) The right hand side is a *triple sum* over all values of the repeated indices i, j and k ! But that triple sum is not too bad since $\epsilon_{ijk} = \pm 1$ depending on whether (i, j, k) is a cyclic (=even) permutation or an acyclic (=odd) permutation of $(1, 2, 3)$ and vanishes in all other instances. Thus (61) expands to

$$\vec{\nabla} \times \vec{v} = \vec{e}_1 (\partial_2 v_3 - \partial_3 v_2) + \vec{e}_2 (\partial_3 v_1 - \partial_1 v_3) + \vec{e}_3 (\partial_1 v_2 - \partial_2 v_1). \quad (62)$$

We can also write that the i -th cartesian component of the curl is

$$\boxed{\vec{e}_i \cdot (\vec{\nabla} \times \vec{v}) = (\vec{\nabla} \times \vec{v})_i = \epsilon_{ijk} \partial_j v_k.} \quad (63)$$

Note that the divergence is a scalar but the curl is a vector.

2.4 Vector identities

In the following $f = f(\vec{r})$ is an arbitrary scalar field while $\vec{v}(\vec{r})$ and $\vec{w}(\vec{r})$ are vector fields. Two crucial relationships that follow easily from $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$ and $\vec{a} \times (\alpha \vec{a}) = 0$ are

$$\boxed{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0,} \quad (64)$$

and

$$\boxed{\vec{\nabla} \times (\vec{\nabla} f) = 0.} \quad (65)$$

The divergence of a curl and the curl of a gradient vanish identically (assuming all those derivatives exist). These can be verified using indicial notation. Other useful identities are

$$\vec{\nabla} \cdot (f \vec{v}) = (\vec{\nabla} f) \cdot \vec{v} + f(\vec{\nabla} \cdot \vec{v}), \quad (66)$$

$$\vec{\nabla} \times (f \vec{v}) = (\vec{\nabla} f) \times \vec{v} + f(\vec{\nabla} \times \vec{v}), \quad (67)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \nabla^2 \vec{v}, \quad (68)$$

where $\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the *Laplacian* operator.

The identity (66) is verified easily using indicial notation

$$\vec{\nabla} \cdot (f \vec{v}) = \partial_i (f v_i) = (\partial_i f) v_i + f(\partial_i v_i) = (\vec{\nabla} f) \cdot \vec{v} + f(\vec{\nabla} \cdot \vec{v}).$$

Likewise the second identity (67) follows from $\epsilon_{ijk} \partial_j (f v_k) = \epsilon_{ijk} (\partial_j f) v_k + f \epsilon_{ijk} (\partial_j v_k)$. The identity (68) is easily remembered from the double cross product formula $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ but note that the $\vec{\nabla}$ in the first term *must* appear first since $\vec{\nabla}(\vec{\nabla} \cdot \vec{v}) \neq (\vec{\nabla} \cdot \vec{v}) \vec{\nabla}$. That identity

can be verified using indicial notation if one knows the double cross product identity in terms of the permutation tensor (see earlier notes on ϵ_{ijk} and index notation) namely

$$\epsilon_{ijk}\epsilon_{klm} = \epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}. \quad (69)$$

A slightly trickier identity is

$$\vec{\nabla} \times (\vec{v} \times \vec{w}) = (\vec{w} \cdot \vec{\nabla})\vec{v} - (\vec{\nabla} \cdot \vec{v})\vec{w} + (\vec{\nabla} \cdot \vec{w})\vec{v} - (\vec{v} \cdot \vec{\nabla})\vec{w}, \quad (70)$$

where $(\vec{w} \cdot \vec{\nabla})\vec{v} = (w_j \partial_j)v_i$ in indicial notation. This can be verified using (69) and can be remembered from the double cross-product identity with the additional input that $\vec{\nabla}$ is a vector *operator*, not just a regular vector, hence $\vec{\nabla} \times (\vec{v} \times \vec{w})$ represents derivatives of a product and this doubles the number of terms of the resulting expression. The first two terms are the double cross product $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ for derivatives of $\vec{v}$ while the last two terms are the double cross products for derivatives of $\vec{w}$.

Another similar identity is

$$\vec{v} \times (\vec{\nabla} \times \vec{w}) = (\vec{\nabla} \vec{w}) \cdot \vec{v} - (\vec{v} \cdot \vec{\nabla})\vec{w}. \quad (71)$$

In indicial notation this reads

$$\epsilon_{ijk}v_j(\epsilon_{klm}\partial_l w_m) = (\partial_i w_j)v_j - (v_j \partial_j)w_i. \quad (72)$$

Note that this last identity involves the gradient of a vector field $\vec{\nabla} \vec{w}$. This makes sense and is a *tensor*, i.e. a geometric object whose components with respect to a basis form a *matrix*. In indicial notation, the components of $\vec{\nabla} \vec{w}$ are $\partial_i w_j$ and there are 9 of them. This is very different from $\vec{\nabla} \cdot \vec{w} = \partial_i w_i$ which is a scalar.

The bottom line is that these identities can be reconstructed relatively easily from our knowledge of regular vector identities for dot, cross and double cross products, however $\vec{\nabla}$ is a vector of derivatives and one needs to watch out more carefully for the order and the product rules. If in doubt, jump to indicial notation.

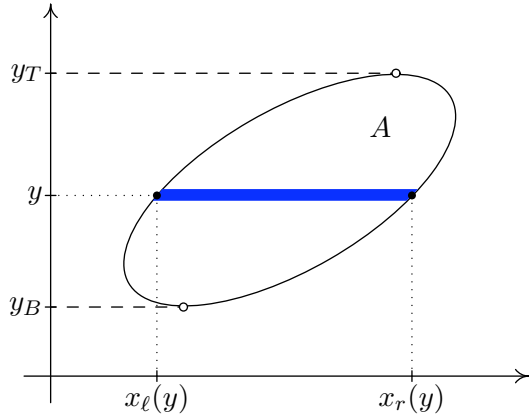
Exercises

1. Verify (64) and (65).
2. Digest and verify the identity (69) *ab initio*.
3. Verify the double cross product identities $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ and $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{a}(\vec{b} \cdot \vec{c})$ in indicial notation using (69).
4. Verify (70) and (71) using indicial notation and (69).
5. Use (69) to derive vector identities for $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$ and $(\vec{\nabla} \times \vec{v}) \cdot (\vec{\nabla} \times \vec{w})$.
6. Show that $\vec{\nabla} \cdot (\vec{v} \times \vec{w}) = \vec{w} \cdot (\vec{\nabla} \times \vec{v}) - \vec{v} \cdot (\vec{\nabla} \times \vec{w})$ and explain how to reconstruct this from the rules for the mixed (or box) product $\vec{a} \cdot (\vec{b} \times \vec{c})$ of three regular vectors.
7. Find the fastest way to show that $\vec{\nabla} \cdot \vec{r} = 3$ and $\vec{\nabla} \times \vec{r} = 0$.
8. Find the fastest way to show that $\vec{\nabla} \cdot (\hat{r}/r^2) = 0$ and $\vec{\nabla} \times (\hat{r}/r^2) = 0$ for all $\vec{r} \neq 0$.
9. Quickly calculate $\vec{\nabla} \cdot \vec{B}$ and $\vec{\nabla} \times \vec{B}$ when $\vec{B} = (\vec{e}_z \times \vec{r})/|\vec{e}_z \times \vec{r}|^2$ (cf. Biot-Savart law for a line current) [Hint: use both vector identities and cartesian coordinates where convenient].

3 Fundamental theorems of vector calculus

3.1 Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$

The integral of a function of two variables $f(x, y)$ over a domain A of $\mathbb{R}^2$ denoted $\int_A f(x, y) dA$ can be defined as the limit of a $\sum_n f(x_n, y_n) \Delta A_n$ where the A_n 's, $n = 1, \dots, N$ provide an (approximate) partition of A that breaks up A into a set of small area elements, squares or triangles for instance. ΔA_n is the area of those element n and (x_n, y_n) is a point inside that element, for instance the center of area of the triangle. The integral would be the limit of such sums when the area of the triangles goes to zero and their number N must then go to infinity. This limit should be such that the aspect ratios of the triangles remain bounded away from 0 so we get a finer and finer sampling of A . This definition also provides a way to approximate the integral by such a finite sum.



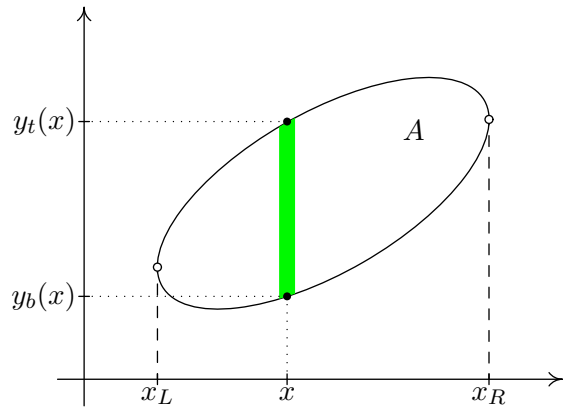
If we imagine breaking up A into small squares aligned with the x and y axes then the sum over all squares inside A can be performed row by row. Each row-sum then tends to an integral in the x -direction, this leads to the conclusion that the integral can be calculated as *iterated* integrals

$$\int_A f(x, y) dA = \int_{y_B}^{y_T} dy \int_{x_l(y)}^{x_r(y)} f(x, y) dx \quad (73)$$

We can also imagine summing up column by column instead and each column-sum then tends to an integral in the y -direction, this leads to the *iterated* integrals

$$\int_A f(x, y) dA = \int_{x_L}^{x_R} dx \int_{y_b(x)}^{y_t(x)} f(x, y) dy. \quad (74)$$

Note of course that the limits of integrations differ from those of the previous iterated integrals.



This iterated integral approach readily extends to integrals over three-dimensional domains in $\mathbb{R}^3$ and more generally to integrals in $\mathbb{R}^n$.

3.2 Fundamental theorem of Calculus

The fundamental theorem of calculus can be written

$$\boxed{\int_a^b \frac{dF}{dx} dx = F(b) - F(a).} \quad (75)$$

Once again we can interpret this in terms of limits of finite differences. The derivative is defined as

$$\frac{dF}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta F}{\Delta x} \quad (76)$$

where $\Delta F = F(x + \Delta x) - F(x)$, while the integral

$$\int_a^b f(x)dx = \lim_{\Delta x_n \rightarrow 0} \sum_{n=1}^N f(\tilde{x}_n) \Delta x_n \quad (77)$$

where $\Delta x_n = x_n - x_{n-1}$ and $x_{n-1} \leq \tilde{x}_n \leq x_n$, with $n = 1, \dots, N$ and $x_0 = a$, $x_N = b$, so the set of x_n 's provides a partition of the interval $[a, b]$. The best choice for $\tilde{x}_n$ is the midpoint $\tilde{x}_n = (x_n + x_{n-1})/2$. This is the *midpoint* scheme in numerical integration methods. Putting these two limits together and setting $\Delta F_n = F(x_n) - F(x_{n-1})$ we can write

$$\int_a^b \frac{dF}{dx} dx = \lim_{\Delta x_n \rightarrow 0} \sum_{n=1}^N \frac{\Delta F_n}{\Delta x_n} \Delta x_n = \lim_{\Delta x_n \rightarrow 0} \sum_{n=1}^N \Delta F_n = F(b) - F(a). \quad (78)$$

We can also write this in the integral form

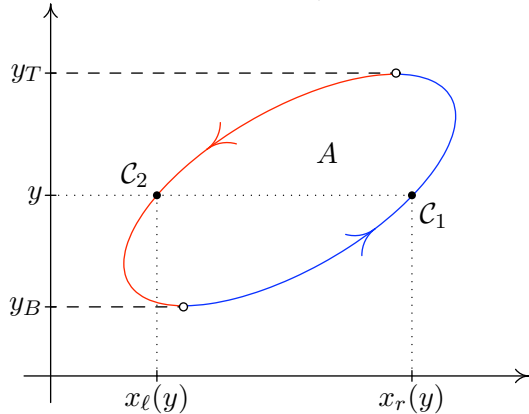
$$\int_a^b \frac{dF}{dx} dx = \int_{F(a)}^{F(b)} dF = F(b) - F(a). \quad (79)$$

3.3 Fundamental theorem in $\mathbb{R}^2$

From the fundamental theorem of calculus and the reduction of integrals on a domain A of $\mathbb{R}^2$ to iterated integrals on intervals in $\mathbb{R}$ we obtain for a function $F(x, y)$

$$\int_A \frac{\partial F}{\partial x} dA = \int_{y_B}^{y_T} dy \int_{x_\ell(y)}^{x_r(y)} \frac{\partial F}{\partial x} dx = \int_{y_B}^{y_T} [F(x_r(y), y) - F(x_\ell(y), y)] dy. \quad (80)$$

This looks nice enough but we can rewrite the integral on the right-hand side as a line integral over the boundary of A . The boundary of A is a closed curve $\mathcal{C}$ often denoted ∂A (not to be confused with a partial derivative). The boundary $\mathcal{C}$ has two parts $\mathcal{C}_1$ and $\mathcal{C}_2$.



The curve $\mathcal{C}_1$ can be parametrized in terms of y as $\vec{r}(y) = \vec{e}_x x_r(y) + \vec{e}_y y$ with $y = y_B \rightarrow y_T$, hence

$$\int_{\mathcal{C}_1} F(x, y) \vec{e}_y \cdot d\vec{r} = \int_{y_B}^{y_T} F(x_r(y), y) dy.$$

Likewise, the curve $\mathcal{C}_2$ can be parametrized using y as $\vec{r}(y) = \vec{e}_x x_\ell(y) + \vec{e}_y y$ with $y = y_T \rightarrow y_B$, hence

$$\int_{\mathcal{C}_2} F(x, y) \vec{e}_y \cdot d\vec{r} = \int_{y_T}^{y_B} F(x_\ell(y), y) dy.$$

Putting these two results together the right hand side of (80) becomes

$$\int_{y_B}^{y_T} [F(x_r(y), y) - F(x_\ell(y), y)] dy = \int_{\mathcal{C}_1} F \vec{e}_y \cdot d\vec{r} + \int_{\mathcal{C}_2} F \vec{e}_y \cdot d\vec{r} = \oint_{\mathcal{C}} F \vec{e}_y \cdot d\vec{r},$$

where $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$ is the *closed* curve bounding A . Then (80) becomes

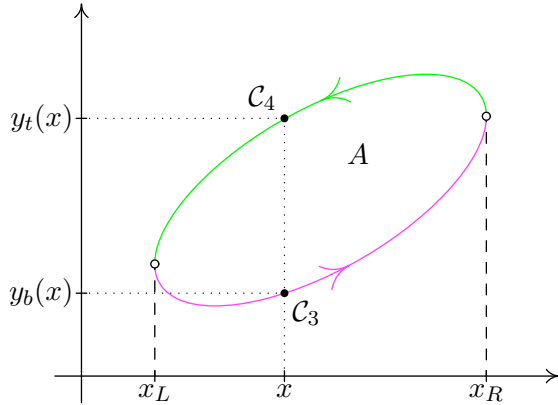
$$\int_A \frac{\partial F}{\partial x} dA = \oint_{\mathcal{C}} F \vec{e}_y \cdot d\vec{r}. \quad (81)$$

The symbol $\oint$ is used to emphasize that the integral is over a closed curve. Note that the curve $\mathcal{C}$ has been oriented *counter-clockwise* such that the interior is to the left of the curve.

Similarly the fundamental theorem of calculus and iterated integrals lead to the result that

$$\int_A \frac{\partial F}{\partial y} dA = \int_{x_L}^{x_R} dx \int_{y_b(x)}^{y_t(x)} \frac{\partial F}{\partial y} dy = \int_{x_L}^{x_R} [F(x, y_t(x)) - F(x, y_b(x))] dx, \quad (82)$$

and the integral on the right hand side can be rewritten as a line integral around the boundary curve $\mathcal{C} = \mathcal{C}_3 + \mathcal{C}_4$.



The curve $\mathcal{C}_3$ can be parametrized in terms of x as $\vec{r}(x) = \vec{e}_x x + \vec{e}_y y_b(x)$ with $x = x_L \rightarrow x_R$, hence

$$\int_{\mathcal{C}_3} F(x, y) \vec{e}_x \cdot d\vec{r} = \int_{x_L}^{x_R} F(x, y_b(x)) dx.$$

Likewise, the curve $\mathcal{C}_4$ can be parametrized using x as $\vec{r}(x) = \vec{e}_x x + \vec{e}_y y_t(x)$ with $x = x_R \rightarrow x_L$, hence

$$\int_{\mathcal{C}_4} F(x, y) \vec{e}_x \cdot d\vec{r} = \int_{x_R}^{x_L} F(x, y_t(x)) dx.$$

The right hand side of (82) becomes

$$\int_{x_L}^{x_R} [F(x, y_t(x)) - F(x, y_b(x))] dx = - \int_{\mathcal{C}_4} F \vec{e}_x \cdot d\vec{r} - \int_{\mathcal{C}_3} F \vec{e}_y \cdot d\vec{r} = - \oint_{\mathcal{C}} F \vec{e}_z \cdot d\vec{r},$$

where $\mathcal{C} = \mathcal{C}_3 + \mathcal{C}_4$ is the *closed* curve bounding A oriented *counter-clockwise* as before. Then (82) becomes

$$\int_A \frac{\partial F}{\partial y} dA = - \oint_{\mathcal{C}} F \vec{e}_x \cdot d\vec{r}. \quad (83)$$

3.4 Green and Stokes' theorems

The two results (81) and (83) can be combined into a single important formula. Write (81) for a function $G(x, y)$ in place of $F(x, y)$ and subtract from it (83) to deduce the *curl form of Green's theorem*

$$\boxed{\int_A \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dA = \oint_{\mathcal{C}} (F \vec{e}_x + G \vec{e}_y) \cdot d\vec{r} = \oint_{\mathcal{C}} (F dx + G dy).} \quad (84)$$

Note that dx and dy in the last integral are *not* independent quantities, they are the projection of the line element $d\vec{r}$ onto the basis vectors $\vec{e}_x$ and $\vec{e}_y$ as written in the middle line integral. If $(x(t), y(t))$ is a parametrization for the curve then $dx = (dx/dt)dt$ and $dy = (dy/dt)dt$ and the t -bounds of integration should be picked to correspond to counter-clockwise orientation. Note also that Green's theorem (84) is the formula to remember since it includes both (81) when $F = 0$ and (83) when $G = 0$.

Green's theorem can be written in several equivalent forms. Define the vector field $\vec{v} = F(x, y)\vec{e}_x + G(x, y)\vec{e}_y$. A simple calculation verifies that its curl is purely in the $\vec{e}_z$ direction indeed (62) gives $\vec{\nabla} \times \vec{v} = \vec{e}_z (\partial G/\partial x - \partial F/\partial y)$ thus (84) can be rewritten in the form

$$\int_A (\vec{\nabla} \times \vec{v}) \cdot \vec{e}_z dA = \oint_{\mathcal{C}} \vec{v} \cdot d\vec{r}, \quad (85)$$

This result also applies to any 3D vector field $\vec{v}(x, y, z) = F(x, y, z)\vec{e}_x + G(x, y, z)\vec{e}_y + H(x, y, z)\vec{e}_z$ and any planar surface A perpendicular to $\vec{e}_z$ since $\vec{e}_z \cdot (\vec{\nabla} \times \vec{v})$ still equals $\partial G/\partial x - \partial F/\partial y$ for such 3D vector fields and the line element $d\vec{r}$ of the boundary curve $\mathcal{C}$ of such planar area is perpendicular to $\vec{e}_z$ so $\vec{v} \cdot d\vec{r}$ is still equal to $F(x, y, z)dx + G(x, y, z)dy$. The extra z coordinate is a mere parameter for the integrals and (85) applies equally well to 3D vector field $\vec{v}(x, y, z)$ provided A is a planar area perpendicular to $\vec{e}_z$.

In fact that last restriction on A itself can be removed. This is *Stokes' theorem* which reads

$$\boxed{\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S} = \oint_{\mathcal{C}} \vec{v} \cdot d\vec{r}}, \quad (86)$$

where S is a bounded orientable surface in 3D space, not necessarily planar, and $\mathcal{C}$ is its closed curve boundary. The orientation of the surface as determined by the direction of its normal $\vec{n}$, where $d\vec{S} = \vec{n}dS$, and the orientation of the boundary curve $\mathcal{C}$ must obey the right-hand rule. Thus a corkscrew turning in the direction of $\mathcal{C}$ would go through S in the direction of its normal $\vec{n}$. That restriction is a direct consequence of the fact that the right hand rule enters the definition of the curl as the cross product $\vec{\nabla} \times \vec{v}$.

Stokes' theorem (86) provides a geometric interpretation for the curl. At any point $\vec{r}$, consider a small disk of area A perpendicular to an arbitrary unit vector $\vec{n}$, then Stokes' theorem states that

$$(\vec{\nabla} \times \vec{v}) \cdot \vec{n} = \lim_{A \rightarrow 0} \oint_{\mathcal{C}} \vec{v} \cdot d\vec{r} \quad (87)$$

where $\mathcal{C}$ is the circle bounding the disk A oriented with $\vec{n}$. The line integral $\oint \vec{v} \cdot d\vec{r}$ is called the *circulation* of the vector field $\vec{v}$ around the closed curve $\mathcal{C}$.

Partial proof of Stokes' Theorem

► We'll sketch a proof of Stokes' theorem and do a non-trivial exercise in vector calculus in the process. Imagine a bounded surface S in 3D space that can be parametrized in terms of cartesian coordinates (x, y) , that is $S : z = h(x, y)$ with x, y in some domain A in the (x, y) -plane then $\vec{r}(x, y) = \vec{e}_x x + \vec{e}_y y + \vec{e}_z h(x, y)$ and

$$\frac{\partial \vec{r}}{\partial x} = \vec{e}_x + \vec{e}_z \frac{\partial h}{\partial x}, \quad \frac{\partial \vec{r}}{\partial y} = \vec{e}_y + \vec{e}_z \frac{\partial h}{\partial y}, \quad (88)$$

so

$$d\vec{S} = \left(\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right) dx dy = \left(\vec{e}_z - \vec{e}_x \frac{\partial h}{\partial x} - \vec{e}_y \frac{\partial h}{\partial y} \right) dx dy. \quad (89)$$

Now if we write $\vec{v} = \vec{e}_x u + \vec{e}_y v + \vec{e}_z w$ then from (62)

$$\vec{\nabla} \times \vec{v} = \vec{e}_x \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + \vec{e}_y \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) + \vec{e}_z \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right),$$

and from (89)

$$\begin{aligned} (\vec{\nabla} \times \vec{v}) \cdot \left(\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right) &= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) - \frac{\partial h}{\partial x} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) - \frac{\partial h}{\partial y} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \\ &= \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \frac{\partial h}{\partial x} \right) - \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial h}{\partial y} \right) + \left(\frac{\partial h}{\partial y} \frac{\partial w}{\partial x} - \frac{\partial h}{\partial x} \frac{\partial w}{\partial y} \right). \end{aligned} \quad (90)$$

Now each of these expressions are partial derivatives of functions of three variables, thus $\partial v/\partial x$ for instance means derivative with respect to x with y and z fixed. However in the surface integral (86), all these expressions must be evaluated on the surface S , meaning at $z = h(x, y)$ with x, y in A . The *chain rule* then says that

$$\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \frac{\partial h}{\partial x}\right)_S = \frac{\partial \bar{v}}{\partial x} \quad \text{and} \quad \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial h}{\partial y}\right)_S = \frac{\partial \bar{u}}{\partial y} \quad (91)$$

where

$$\bar{u}(x, y) = u(x, y, h(x, y)), \quad \bar{v}(x, y) = v(x, y, h(x, y)). \quad (92)$$

and the subscript S indicates that the expression must be evaluated on S meaning at $z = h(x, y)$ with x, y in A . We can massage the last term in parentheses on the right hand side of (90)

$$\begin{aligned} \frac{\partial h}{\partial y} \frac{\partial w}{\partial x} - \frac{\partial h}{\partial x} \frac{\partial w}{\partial y} &= \frac{\partial h}{\partial y} \left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial z} \frac{\partial h}{\partial x} \right) - \frac{\partial h}{\partial x} \left(\frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} \frac{\partial h}{\partial y} \right) \\ &= \frac{\partial h}{\partial y} \frac{\partial \bar{w}}{\partial x} - \frac{\partial h}{\partial x} \frac{\partial \bar{w}}{\partial y} \\ &= \frac{\partial}{\partial x} \left(\bar{w} \frac{\partial h}{\partial y} \right) - \frac{\partial}{\partial y} \left(\bar{w} \frac{\partial h}{\partial x} \right). \end{aligned} \quad (93)$$

where $\bar{w}(x, y) = w(x, y, h(x, y))$ as in (92) and the last step follows from the cancellation of the mixed partials $\partial^2 h/\partial y \partial x = \partial^2 h/\partial x \partial y$. Substituting (91) and (93) into (90) yields

$$\begin{aligned} (\vec{\nabla} \times \vec{v}) \cdot \left(\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right) &= \frac{\partial}{\partial x} \left(\bar{v} + \bar{w} \frac{\partial h}{\partial y} \right) - \frac{\partial}{\partial y} \left(\bar{u} + \bar{w} \frac{\partial h}{\partial x} \right) \\ &\equiv \frac{\partial G(x, y)}{\partial x} - \frac{\partial F(x, y)}{\partial y}. \end{aligned} \quad (94)$$

Hurrah! we have managed to write

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S} = \int_A \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dA. \quad (95)$$

Now for any point $\vec{r}(x, y)$ on the surface S , we have from (88)

$$d\vec{r} = \frac{\partial \vec{r}}{\partial x} dx + \frac{\partial \vec{r}}{\partial y} dy = \vec{e}_x dx + \vec{e}_y dy + \vec{e}_z \left(\frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy \right)$$

where dx and dy would be independent for any point on S except on its boundary $\mathcal{C}$ where dx and dy must be such that $d\vec{r}$ is tangent to $\mathcal{C}$. In any case, at any point on the surface S

$$\vec{v} \cdot d\vec{r} = \left(\bar{u} + \bar{w} \frac{\partial h}{\partial x} \right) dx + \left(\bar{v} + \bar{w} \frac{\partial h}{\partial y} \right) dy = F(x, y) dx + G(x, y) dy.$$

Hurrah again! finally if $\vec{r}$ is on the boundary $\mathcal{C}$ of S then (x, y) is on the boundary $\mathcal{C}_A$ of A and we have

$$\oint_{\mathcal{C}} \vec{v} \cdot d\vec{r} = \oint_{\mathcal{C}_A} F dx + G dy. \quad (96)$$

The results (95) and (96) reduce Stokes' theorem (86) to Green's theorem (84), thereby proving it in this particular case where S can be parametrized by x and y . ■

More general proof of Stokes' theorem

Indicial notation enables a fairly straightforward proof of Stokes' theorem for the more general case of a surface S in 3D space that can be parametrized by a 'good' function $\vec{r}(s, t)$ (differentiable and integrability as needed). Such a surface S can fold and twist (it could even intersect itself!) and is therefore of a more general kind than those that can be parametrized by the cartesian coordinates x and y . The main restriction on S is that it must be 'orientable'. This means that it must have an 'up' and a 'down' as defined by the direction of the normal $\partial\vec{r}/\partial s \times \partial\vec{r}/\partial t$. The famous Mobius strip only has one side and is the classical example of a non-orientable surface. The boundary of the Mobius strip forms a knot.

► Let $x_i(s, t)$ represent the i component of the position vector $\vec{r}(s, t)$ in the surface S with $i = 1, 2, 3$. Then

$$\begin{aligned}
 (\vec{\nabla} \times \vec{v}) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) &= \epsilon_{ijk} \frac{\partial v_k}{\partial x_j} \epsilon_{ilm} \frac{\partial x_l}{\partial s} \frac{\partial x_m}{\partial t} = \epsilon_{ijk} \epsilon_{ilm} \frac{\partial v_k}{\partial x_j} \frac{\partial x_l}{\partial s} \frac{\partial x_m}{\partial t} \\
 &= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) \frac{\partial v_k}{\partial x_j} \frac{\partial x_l}{\partial s} \frac{\partial x_m}{\partial t} \\
 &= \frac{\partial v_k}{\partial x_j} \frac{\partial x_j}{\partial s} \frac{\partial x_k}{\partial t} - \frac{\partial v_k}{\partial x_j} \frac{\partial x_k}{\partial s} \frac{\partial x_j}{\partial t} \\
 &= \frac{\partial \bar{v}_k}{\partial s} \frac{\partial x_k}{\partial t} - \frac{\partial \bar{v}_k}{\partial t} \frac{\partial x_k}{\partial s} \\
 &= \frac{\partial}{\partial s} \left(\bar{v}_k \frac{\partial x_k}{\partial t} \right) - \frac{\partial}{\partial t} \left(\bar{v}_k \frac{\partial x_k}{\partial s} \right) \\
 &= \frac{\partial G(s, t)}{\partial s} - \frac{\partial F(s, t)}{\partial t},
 \end{aligned} \tag{97}$$

where we have used (69), then the chain rule

$$\frac{\partial v_k}{\partial x_j} \frac{\partial x_j}{\partial s} = \frac{\partial v_k}{\partial x_1} \frac{\partial x_1}{\partial s} + \frac{\partial v_k}{\partial x_2} \frac{\partial x_2}{\partial s} + \frac{\partial v_k}{\partial x_3} \frac{\partial x_3}{\partial s} = \frac{\partial \bar{v}_k}{\partial s},$$

and equality of mixed partials $\partial^2 x_k / \partial s \partial t = \partial^2 x_k / \partial t \partial s$ with $\bar{v}_k(s, t) = v_k(x_1(s, t), x_2(s, t), x_3(s, t))$. The result (97) is equivalent to (94) with s, t in place of x, y . It is not only more general but also easier to obtain using indicial notation. That's because the cartesian parametrization actually breaks the symmetry of the problem. This proof demonstrates the power of the indicial notation and the summation convention. The first line of (97) is a *quintuple* sum over all values of the indices i, j, k, l and m ! This would be unmanageable without the compact notation.

Now for any point on the surface S with position vector $\vec{r}(s, t)$

$$\vec{v} \cdot d\vec{r} = \bar{v}_i \left(\frac{\partial x_i}{\partial s} ds + \frac{\partial x_i}{\partial t} dt \right) = \left(\bar{v}_i \frac{\partial x_i}{\partial s} \right) ds + \left(\bar{v}_i \frac{\partial x_i}{\partial t} \right) dt = F(s, t) ds + G(s, t) dt, \tag{98}$$

and we have again reduced Stokes' theorem to Green's theorem (84) but expressed in terms of s and t instead of x and y . In details we have shown that

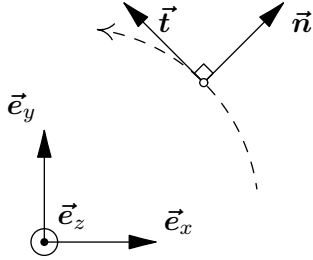
$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S} = \int_A (\vec{\nabla} \times \vec{v}) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) ds dt = \int_A \left(\frac{\partial G(s, t)}{\partial s} - \frac{\partial F(s, t)}{\partial t} \right) ds dt, \tag{99}$$

$$\oint_C \vec{v} \cdot d\vec{r} = \oint_{C_A} F(s, t) ds + G(s, t) dt. \tag{100}$$

The right hand sides of (99) and (100) are equal by Green's theorem (84). ■

3.5 Divergence form of Green's theorem

The fundamental theorems (81) and (83) in $\mathbb{R}^2$ can be rewritten in a more palatable form.



The line element $d\vec{r}$ at a point on the curve $\mathcal{C}$ is in the direction of the unit tangent $\vec{t}$ at that point, so $d\vec{r} = \vec{t}dr$, where $\vec{t}$ points in the counterclockwise direction of the curve. Then $\vec{t} \times \vec{e}_z = \vec{n}$ is the unit *outward* normal $\vec{n}$ to the curve at that point and $\vec{e}_z \times \vec{n} = \vec{t}$ so

$$\vec{e}_x \cdot \vec{t} = \vec{e}_x \cdot (\vec{e}_z \times \vec{n}) = (\vec{e}_x \times \vec{e}_z) \cdot \vec{n} = -\vec{e}_y \cdot \vec{n} \quad (101)$$

$$\vec{e}_y \cdot \vec{t} = \vec{e}_y \cdot (\vec{e}_z \times \vec{n}) = (\vec{e}_y \times \vec{e}_z) \cdot \vec{n} = \vec{e}_x \cdot \vec{n}. \quad (102)$$

Hence since $d\vec{r} = \vec{t}dr$, the fundamental theorems (81) and (83) can be rewritten

$$\int_A \frac{\partial F}{\partial x} dA = \oint_{\mathcal{C}} F \vec{e}_y \cdot d\vec{r} = \oint_{\mathcal{C}} F \vec{e}_x \cdot \vec{n} dr, \quad (103)$$

$$\int_A \frac{\partial F}{\partial y} dA = - \oint_{\mathcal{C}} F \vec{e}_x \cdot d\vec{r} = \oint_{\mathcal{C}} F \vec{e}_y \cdot \vec{n} dr. \quad (104)$$

The right hand side of these equations is easier to remember since they have $\vec{e}$ going with $\partial/\partial x$ and $\vec{e}_y$ with $\partial/\partial y$ and both equations have positive signs. But there are hidden subtleties. The arclength element $dr = |d\vec{r}|$ is positive by definition and $\vec{n}$ must be the unit *outward* normal to the boundary, so if an explicit parametrization of the boundary curve is known, the bounds of integration should be picked so that $\oint_{\mathcal{C}} dr$ would be the length of the curve with a positive sign. For the $d\vec{r}$ line integrals, the bounds of integration must correspond to counter-clockwise orientation of $\mathcal{C}$.

Formulas (103) and (104) can be combined in more useful forms. First, if s is the signed distance in a direction of the unit vector $\hat{s}$, then the derivative in the direction $\hat{s}$ is $\partial F/\partial s = \hat{s} \cdot \vec{\nabla} F = s_x \partial F/\partial x + s_y \partial F/\partial y$, where $\hat{s} = s_x \vec{e}_x + s_y \vec{e}_y$, therefore combining (103) and (104) accordingly we obtain

$$\boxed{\int_A \frac{\partial F}{\partial s} dA = \oint_{\mathcal{C}} F \hat{s} \cdot \vec{n} dr.} \quad (105)$$

This result is written in a coordinate-free form. It applies to *any* direction $\hat{s}$ in the x,y plane. Another useful combination is to add (103) to (104) written for a function $G(x,y)$ in place of $F(x,y)$ yielding the *divergence-form of Green's theorem*

$$\boxed{\int_A \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right) dA = \oint_{\mathcal{C}} (F \vec{e}_x + G \vec{e}_y) \cdot \vec{n} dr.} \quad (106)$$

The left hand side integrand is easily recognized as the divergence $\vec{\nabla} \cdot \vec{v}$ of the vector field $\vec{v}(x,y) = F \vec{e}_x + G \vec{e}_y$.

3.6 Gauss' theorem

Gauss' theorem is the 3D version of the divergence form of Green's theorem. It is proved by first extending the fundamental theorem of calculus to 3D.

If V is a bounded volume in 3D space and $F(x,y,z)$ is a scalar function of the cartesian coordinates (x,y,z) , then we have

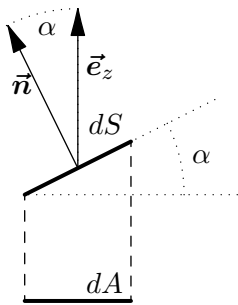
$$\int_V \frac{\partial F}{\partial z} dV = \oint_S F \vec{e}_z \cdot \vec{n} dS, \quad (107)$$

where S is the *closed* surface enclosing V and $\vec{n}$ is the unit *outward* normal to S .

The proof of this result is similar to that for (81). Assume that the surface can be parametrized using x and y in two pieces: an upper ‘hemisphere’ at $z = z_u(x, y)$ and a lower ‘hemisphere’ at $z = z_l(x, y)$ with x, y in a domain A , the projection of S onto the x, y plane, that is the *same domain* for both the upper and lower surfaces. The closed surface S is not a sphere in general but we used the word ‘hemisphere’ to help visualize the problem. For a sphere, A is the equatorial disk, $z = z_u(x, y)$ is the northern hemisphere and $z = z_l(x, y)$ is the southern hemisphere.

Iterated integrals with $dV = dA dz$ and the fundamental theorem of calculus give

$$\int_V \frac{\partial F}{\partial z} dV = \int_A dA \int_{z_l(x,y)}^{z_u(x,y)} \frac{\partial F}{\partial z} dz = \int_A [F(x, y, z_u(x, y)) - F(x, y, z_l(x, y))] dA. \quad (108)$$



We can interpret that integral over A as an integral over the entire closed surface S that bounds V . All we need for that is a bit of geometry. If dA is the projection of the surface element $d\vec{S} = \vec{n}dS$ onto the x, y plane then we have $dA = \cos \alpha dS = \pm \vec{e}_z \cdot \vec{n} dS$. The $+$ sign applies to the upper surface for which $\vec{n}$ is pointing up (*i.e.* in the direction of $\vec{e}_z$) and the $-$ sign for the bottom surface where $\vec{n}$ points down (and would be opposite to the $\vec{n}$ on the side figure). Thus we obtain

$$\int_V \frac{\partial F}{\partial z} dV = \oint_S F \vec{e}_z \cdot \vec{n} dS, \quad (109)$$

where $\vec{n}$ is the unit *outward* normal to S .

We can obtain similar results for the volume integrals of $\partial F/\partial x$ and $\partial F/\partial y$ and combine those to obtain

$$\boxed{\int_V \frac{\partial F}{\partial s} dV = \oint_S F \hat{s} \cdot \vec{n} dS,} \quad (110)$$

for arbitrary but fixed direction $\hat{s}$. This is the 3D version of (105) and of the fundamental theorem of calculus.

We can combine this theorem into many useful forms. Writing it for $F(x, y, z)$ in the $\vec{e}$ direction, with the $\vec{e}_y$ version for a function $G(x, y, z)$ and the $\vec{e}_z$ version for a function $H(x, y, z)$ we obtain *Gauss's theorem*

$$\boxed{\int_V \vec{\nabla} \cdot \vec{v} dV = \oint_S \vec{v} \cdot \vec{n} dS,} \quad (111)$$

where $\vec{v} = F\vec{e}_x + G\vec{e}_y + H\vec{e}_z$. This is the 3D version of (106). Note that both (110) and (111) are expressed in coordinate-free forms. These are general results. The integral $\oint_S \vec{v} \cdot \vec{n} dS$ is the *flux* of $\vec{v}$ through the surface S . If $\vec{v}(\vec{r})$ is the velocity of a fluid at point $\vec{r}$ then that integral represents the time-rate at which volume of fluid flows through the surface S .

Gauss' theorem provides a coordinate-free interpretation for the divergence. Consider a small sphere of volume V and surface S centered at a point $\vec{r}$, then Gauss' theorem states that

$$\vec{\nabla} \cdot \vec{v} = \lim_{V \rightarrow 0} \oint_S \vec{v} \cdot \vec{n} dS. \quad (112)$$

Note that (110) and (111) are equivalent. We deduced (111) from (110), but we can also deduce (110) from (111) by considering the special $\vec{v}(\vec{r}) = F(\vec{r})\hat{s}$ where $\hat{s}$ is a unit vector independent of $\vec{r}$. Then from our vector identities $\vec{\nabla} \cdot (F\hat{s}) = \hat{s} \cdot \vec{\nabla} F = \partial F/\partial s$.

3.7 Other useful forms of the fundamental theorem in 3D

A useful form of (110) is to write it in indicial form as

$$\boxed{\int_V \frac{\partial F}{\partial x_j} dV = \oint_S n_j F dS.} \quad (113)$$

Then with $f(\vec{r})$ in place of $F(\vec{r})$ we deduce that

$$\int_V \vec{e}_j \frac{\partial f}{\partial x_j} dV = \oint_S \vec{e}_j n_j f dS \quad (114)$$

since the cartesian unit vectors $\vec{e}_j$ are independent of position. This result can be written in coordinate-free form as

$$\int_V \vec{\nabla} f dV = \oint_S f \vec{n} dS. \quad (115)$$

One application of this form of the fundamental theorem is to prove *Archimedes' principle*.

Next, writing (113) for v_k in place of F yields

$$\int_V \frac{\partial v_k}{\partial x_j} dV = \oint_S n_j v_k dS, \quad (116)$$

which can be multiplied by the position-independent ϵ_{ijk} and summed over all j and k to give

$$\int_V \epsilon_{ijk} \frac{\partial v_k}{\partial x_j} dV = \oint_S \epsilon_{ijk} n_j v_k dS. \quad (117)$$

The coordinate-free form of this is

$$\int_V \vec{\nabla} \times \vec{v} dV = \oint_S \vec{n} \times \vec{v} dS. \quad (118)$$

The integral theorems (115) and (118) provide yet other geometric interpretations for the gradient

$$\vec{\nabla} f = \lim_{V \rightarrow 0} \oint_S f \vec{n} dS, \quad (119)$$

and the curl

$$\vec{\nabla} \times \vec{v} = \lim_{V \rightarrow 0} \oint_S \vec{n} \times \vec{v} dS. \quad (120)$$

These are similar to the result (112) for the divergence. However, the geometric definition of the gradient as the vector pointing in the direction of greatest rate of change (sect. 2.1) and of the $\vec{n}$ component of the curl as the limit of the local circulation per unit area as given by Stokes' theorem (87) are perhaps more fundamental.

In applications we use the fundamental theorem as we do in 1D, namely to reduce a 3D integral to a 2D integral, for instance. However we also use them the other way, to evaluate a complicated surface integral as a simpler volume integral for instance.

Exercises

1. If $\mathcal{C}$ is any closed curve in 3D space (i) calculate $\oint_{\mathcal{C}} \vec{r} \cdot d\vec{r}$ in two ways, (ii) calculate $\oint_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{r}$ in two ways, where $f(\vec{r})$ is a scalar function. [Hint: by direct calculation and by Stokes theorem].

2. If $\mathcal{C}$ is any closed curve in 3D space not passing through the origin calculate $\oint_{\mathcal{C}} r^{-3} \vec{r} \cdot d\vec{r}$ in two ways.
3. Calculate the circulation of the vector field $\vec{B} = (\vec{e}_z \times \vec{r})/|\vec{e}_z \times \vec{r}|^2$ (i) about a circle of radius R centered at the origin in a plane perpendicular to $\vec{e}_z$; (ii) about any closed curve $\mathcal{C}$ in 3D that does not go around the z -axis; (iii) about any closed curve $\mathcal{C}_0$ that does go around the $\vec{e}_z$ axis. What's wrong with the z -axis anyway?
4. Consider $\vec{v} = \vec{\omega} \times \vec{r}$ where $\vec{\omega}$ is a constant vector, independent of $\vec{r}$. (i) Evaluate the circulation of $\vec{v}$ about the circle of radius R centered at the origin in the plane perpendicular to the direction $\vec{n}$ by direct calculation of the line integral; (ii) Calculate the curl of $\vec{v}$ using vector identities; (iii) calculate the circulation of $\vec{v}$ about a circle of radius R centered at $\vec{r}_0$ in the plane perpendicular to $\vec{n}$.
5. If $\mathcal{C}$ is any closed curve in 2D, calculate $\oint_{\mathcal{C}} \vec{n} dr$ where $\vec{n}(\vec{r})$ is the unit outside normal to $\mathcal{C}$ at the point $\vec{r}$ of $\mathcal{C}$.
6. If S is any closed surface in 3D, calculate $\oint \vec{n} dS$ where $\vec{n}(\vec{r})$ is the unit outside normal to S at a point $\vec{r}$ on S .
7. If S is any closed surface in 3D, calculate $\oint p(\vec{r}) \vec{n} dS$ where $\vec{n}$ is the unit outside normal to S and $p(\vec{r}) = (p_0 - \rho g \vec{e}_z \cdot \vec{r})$ where p_0 , ρ and g are constants (This is *Archimedes' principle* with ρ as fluid density and g as the acceleration of gravity.)
8. Calculate the flux of $\vec{r}$ through (i) the surface of a sphere of radius R centered at the origin in two ways; (ii) through the surface of the sphere of radius R centered at $\vec{r}_0$; (iii) through the surface of a cube of side L with one corner at the origin in two ways.
9. (i) Calculate the flux of $\vec{v} = \vec{r}/r^3$ through the surface of a sphere of radius ϵ centered at the origin. (ii) Calculate the flux of that vector field through a closed surface that does not enclose the origin [Hint: use the divergence theorem] (iii) Calculate the flux through an arbitrary closed surface that encloses the origin [Hint: use divergence theorem and (i) to isolate the origin. What's wrong with the origin anyway?]
10. Calculate $\vec{\nabla} |\vec{r} - \vec{r}_0|^{-1}$ and $\vec{\nabla} \cdot \vec{v}$ with $\vec{v} = (\vec{r} - \vec{r}_0)/|\vec{r} - \vec{r}_0|^3$ where $\vec{r}_0$ is a constant vector.
11. Calculate the flux of $\vec{v}$ through the surface of a sphere of radius R centered at the origin when $\vec{v} = \alpha_1 (\vec{r} - \vec{r}_1)/|\vec{r} - \vec{r}_1|^3 + \alpha_2 (\vec{r} - \vec{r}_2)/|\vec{r} - \vec{r}_2|^3$ where α_1 and α_2 are scalar constants and (i) $|\vec{r}_1|$ and $|\vec{r}_2|$ are both less than R ; (ii) $|\vec{r}_1| < R < |\vec{r}_2|$. Generalize to $\vec{v} = \sum_{i=1}^N \alpha_i (\vec{r} - \vec{r}_i)/|\vec{r} - \vec{r}_i|^3$.
12. Calculate $\vec{F}(\vec{r}) = \int_{V_0} \vec{f} dV_0$ where V_0 is the inside of a sphere of radius R centered at O , $\vec{f} = (\vec{r} - \vec{r}_0)/|\vec{r} - \vec{r}_0|^3$ with $|\vec{r}| > R$ and the integral is over $\vec{r}_0$. Warning: this is essentially the gravity field or force at $\vec{r}$ due to a sphere of uniform mass density. Supposedly, it took Newton 20 years to figure it out... of course he was never taught calculus, he had to invent it, and vector calculus came much after him and he never knew about Gauss' theorem. The integral can be cranked out if you're good at analytic integration. But the smart solution is to realize that the integral over $\vec{r}_0$ is essentially a sum over $\vec{r}_i$ as in the previous exercise so we can figure out the flux of $\vec{F}$ through any closed surface enclosing all of V_0 . Now by symmetry $\vec{F}(\vec{r}) = F(r)\hat{r}$, so knowing the flux is enough to figure out $F(r)$. Newton would be impressed!