Lectures 8 – 9 : Gaussian Ensembles

Notes prepared by: J. Hu

We've defined Gaussian Orthogonal, Unitary and Symplectic Ensembles last week. Recall that

Definition 1 (GOE)**.** Consider a real symmetric matrix where $X_{ij} = X_{ji} \sim N(0,1)$ and $X_{ii} \sim \sqrt{2}N(0,1)$. The resulting random matrix model is called Gaussian Orthogonal Ensemble.

Consider the space of $n \times n$ real symmetric matrices $\mathcal{H}_n$, we have $\binom{n}{2} + n$ free variables. The natural reference measure on this space is the Lebesgue measure on $\mathbb{R}^{\binom{n}{2}+n}$. The density for GOE with this reference measure is

$$\frac{1}{Z_{n,1}} e^{-\frac{1}{4} trH^2} \tag{1}$$

the measure is

$$\frac{1}{Z_{n,1}} e^{-\frac{1}{4} trH^2} dH \tag{2}$$

Here $H = [h_{ij}]_{i,j=1}^n$, $trH^2 = \sum_{i,j=1}^n h_{ij}^2 = \sum_{i=1}^n h_{ii}^2 + 2\sum_{i>j} h_{ij}^2$, then (2) is

$$\prod_{i=1}^n \frac{1}{\sqrt{4\pi}} e^{-\frac{h_{ii}^2}{4}} dh_{ii} \prod_{i>j} \frac{1}{\sqrt{2\pi}} e^{-\frac{h_{ij}^2}{2}} dh_{ij} \tag{3}$$

so $Z_{n,1} = (4\pi)^{\frac{n}{2}} (2\pi)^{\frac{1}{2}\binom{n}{2}}$.

In fact, Gaussian Ensembles can be unified by the following expression:

$$\frac{1}{Z_{n,\beta}} e^{-\frac{\beta}{4} trH^2} dH \tag{4}$$

where $\beta = 1, 2, 4$ denote GOE, GUE and GSE respectively.

Theorem 2. *GOE is invariant under conjugation with a orthogonal $n \times n$ matrix.*

Proof. Assume $C \in \mathbb{R}^{n \times n}$ is a orthogonal matrix ($C^T C = I$), and $H \sim GOE$. Define linear transformation:

$$H \to C^T HC = \tilde{H} \tag{5}$$

Then

$$tr(\tilde{H}^2) = tr(C^T HCC^T HC) = tr(C^T HHC) = tr(H^2) \tag{6}$$

so

$$\frac{1}{Z_{n,1}}e^{-\frac{1}{4}tr\tilde{H}^2} = \frac{1}{Z_{n,1}}e^{-\frac{1}{4}trH^2} \tag{7}$$

Now all we need to do is to show that the Jacobian of this transformation is 1. Rewrite H and $\tilde{H}$ in the following way:

$$\vec{H}_1 = (h_{11}, h_{22}, \cdots, h_{nn}, h_{12}, h_{13}, \cdots, h_{n-1,n})^T \tag{8}$$

$$\vec{H}_2 = (\tilde{h}_{11}, \tilde{h}_{22}, \cdots, \tilde{h}_{nn}, \tilde{h}_{12}, \tilde{h}_{13}, \cdots, \tilde{h}_{n-1,n})^T \tag{9}$$

then the linear transformation from $\vec{H}_1$ to $\vec{H}_2$ can be written as

$$\vec{H}_2 = A\vec{H}_1, \quad A \in \mathbb{R}^{[\binom{n}{2}+n]^2} \tag{10}$$

If we define a matrix $D \in \mathbb{R}^{[\binom{n}{2}+n]^2}$ as:

$$D = \text{diag}(1, 1, \cdots, 1, 2, \cdots, 2) \tag{11}$$

where the number of 1's is n, the number of 2's is $\binom{n}{2}$. Then it is easy to see that

$$\vec{H}_1^T D\vec{H}_1 = tr(H^2), \quad \vec{H}_2^T D\vec{H}_2 = tr(\tilde{H}^2) \tag{12}$$

Because of (6) and (10),

$$\vec{H}_1^T D\vec{H}_1 = H_2^T D\vec{H}_2 = \vec{H}_1^T A^T DA\vec{H}_1 \tag{13}$$

then

$$D = A^T DA \Rightarrow \det(D) = \det(D)\det(A)^2 \Rightarrow |\det(A)| = 1 \tag{14}$$

so $dH = d\tilde{H}$, this completes the proof. □

Besides the Gaussian Ensemble, there are lots of examples where the measure is invariant under the orthogonal (unitary, symplectic) conjugation, e.g. $\frac{1}{Z}e^{-trH^4}dH$. We can actually show the following:

Theorem 3. *H is a $n \times n$ symmetric matrix. Suppose that $P(H)dH$ is*

- *invariant under orthogonal conjugation*
- *the entries are independent*

then

$$P(H) = e^{-atrH^2 - btrH - c}, \tag{15}$$

where a, b, c are arbitrary constants.

Proof. $P(H)$ can be written as

$$P(H) = \prod_{i=1}^{n} f_i(h_{ii}) \prod_{i>j} g_{ij}(h_{ij}) \tag{16}$$

we can assume that $f_i = f$, $g_{ij} = g$, then

$$P(H) = \prod_{i=1}^{n} f(h_{ii}) \prod_{i>j} g(h_{ij}) \tag{17}$$

Let matrix C be

$$\begin{bmatrix} \cos\alpha & \sin\alpha & & & \\ -\sin\alpha & \cos\alpha & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}_{n\times n}$$

where all the other elements are zero and $|\alpha| \ll 1$. Then C can be approximated by

$$\begin{bmatrix} 1 & \alpha & & & \\ -\alpha & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}_{n\times n}$$

up to order $O(\alpha)$. $\tilde{H} = C^T H C$ is approximated by

$$\begin{bmatrix} h_{11} - 2\alpha h_{12} & h_{12} + \alpha(h_{11} - h_{22}) & \cdots & h_{1j} - \alpha h_{2j} & \cdots \\ & h_{22} + 2\alpha h_{12} & \cdots & h_{2j} + \alpha h_{1j} & \cdots \\ & & & h_{ij} & \end{bmatrix}_{n\times n}$$

up to order $O(\alpha)$. Since $P(H) = P(\tilde{H})$, one has

$$\prod_{i=1}^{n} f(h_{ii}) \prod_{i>j} g(h_{ij}) = \prod_{i=1}^{n} f(\tilde{h}_{ii}) \prod_{i>j} g(\tilde{h}_{ij}) \tag{18}$$

Take log on both sides, and let $\alpha \to 0$, we get

$$\frac{(h_{11} - h_{22})g'(h_{12})}{g(h_{12})} - 2\frac{h_{12}f'(h_{11})}{f(h_{11})} + 2\frac{h_{12}f'(h_{22})}{f(h_{22})} - \sum_{j=3}^{n}[\frac{h_{2j}g'(h_{1j})}{g(h_{1j})} - \frac{h_{1j}g'(h_{2j})}{g(h_{2j})}] = 0 \tag{19}$$

this means the last term in (19) must be zero,

$$\frac{h_{2j}g'(h_{1j})}{g(h_{1j})} = \frac{h_{1j}g'(h_{2j})}{g(h_{2j})} \tag{20}$$

Let $x = h_{2j}$, $y = h_{1j}$, this is just

$$\frac{xg'(y)}{g(y)} = \frac{yg'(x)}{g(x)} \tag{21}$$

i.e.

$$\frac{g'(y)}{yg(y)} = \frac{g'(x)}{xg(x)} \tag{22}$$

this holds for any x and y, so there is a constant b such that

$$\frac{g'(x)}{xg(x)} = -b \tag{23}$$

solution of this ODE is

$$g(x) = C_1 e^{-\frac{bx^2}{2}} \tag{24}$$

The sum of the first three terms in (19) is also zero,

$$h_{12}[-b(h_{11} - h_{22}) - 2\frac{f'}{f}(h_{11}) + 2\frac{f'}{f}(h_{22})] = 0 \tag{25}$$

Let $x = h_{11}$, $y = h_{22}$, this is just

$$-b(x-y) - 2\frac{f'(x)}{f(x)} + 2\frac{f'(y)}{f(y)} = 0 \tag{26}$$

i.e.

$$-bx - 2\frac{f'(x)}{f(x)} = -2\frac{f'(y)}{f(y)} - by \tag{27}$$

this holds for any x and y, so there is a constant c such that

$$bx + \frac{2f(x)}{f(x)} = -c \tag{28}$$

solve this ODE gives

$$f(x) = C_2 e^{-\frac{bx^2}{4} - \frac{cx}{2}} \tag{29}$$

so

$$P(H) = \prod_{i=1}^{n} f(h_{ii}) \prod_{i>j} g(h_{ij}) = Ce^{-\frac{b}{4}trH^2 - \frac{c}{2}trH} \tag{30}$$

therefore $P(H)$ is in the form of (15). □

We know that the density of the Gaussian Ensembles can be represented by

$$\frac{1}{Z_{n,\beta}} e^{-\frac{\beta}{4}trH^2} dH \tag{31}$$

where $\beta = 1, 2, 4$ denote GOE, GUE and GSE respectively. Now we are going to show:

Theorem 4. *The joint eigenvalue density is given by*

$$P(\lambda_1, \cdots, \lambda_n) = \frac{1}{\tilde{Z}_{n,\beta}} \prod_{i>j} |\lambda_i - \lambda_j|^{\beta} e^{-\frac{\beta}{4}\sum_{i=1}^{n} \lambda_i^2} \tag{32}$$

where $\lambda_1 < \lambda_2 < \cdots < \lambda_n$.

Proof. First, we want to perform the Householder transformation (orthogonal transformation) to change the $n \times n$ symmetric matrix M_n to a tridiagonal symmetric matrix which looks like

$$\begin{bmatrix} a_n & b_{n-1} & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & \\ & b_{n-2} & \ddots & \ddots & \\ & & \ddots & \ddots & b_1 \\ & & & b_1 & a_1 \end{bmatrix}_{n\times n}$$

Important Observation: If we apply the Householder transformation to G O/U/S E, then a_k's and b_k's will be independent with a computable distribution.

Assume $M_n = [X_{ij}]_{i,j=1}^n \sim GOE$, $X_{ii} = \sqrt{2}\xi_{ii} \sim \sqrt{2}N(0,1)$, $X_{ij} = \xi_{ij} \sim N(0,1)$. Write M_n as:

$$M_n = \begin{bmatrix} \sqrt{2}\xi_n & Z_{n-1}^T \\ Z_{n-1} & M_{n-1} \end{bmatrix}_{n\times n}$$

where $Z_{n-1}^T = [X_{12}, X_{13}, \cdots, X_{1n}]$, $M_{n-1} \in \mathbb{R}^{(n-1)\times(n-1)} \sim GOE$. Define

$$U_n = \begin{bmatrix} 1 & \\ & \tilde{U}_n \end{bmatrix}_{n\times n}$$

here $\tilde{U}_n \in \mathbb{R}^{(n-1)\times(n-1)}$ can be chosen (we will show this later) orthogonal such that

$$\tilde{U}_n Z_{n-1} = (||Z_{n-1}||_2, 0, \cdots, 0)^T \tag{33}$$

$\tilde{U}_n$ will only depend on $(X_{12}, \cdots, X_{1n})$. Then

$$U_n M_n U_n^T = \begin{bmatrix} \sqrt{2}\xi_n & ||Z_{n-1}||_2 & 0 & \cdots & 0 \\ ||Z_{n-1}||_2 & & & & \\ 0 & & & & \\ \vdots & & \tilde{U}_n M_{n-1} \tilde{U}_n^T & & \\ 0 & & & & \end{bmatrix}_{n\times n}$$

$\tilde{M}_{n-1} = \tilde{U}_n M_{n-1} \tilde{U}_n^T$ has the same distribution as M_{n-1} and it is independent of ξ_n, $||Z_{n-1}||_2$.
Now we repeat the above procedure. Write $\tilde{M}_{n-1}$ as:

$$\tilde{M}_{n-1} = \begin{bmatrix} \sqrt{2}\xi_{n-1} & Z_{n-2}^T \\ Z_{n-2} & M_{n-2} \end{bmatrix}_{(n-1)\times(n-1)}$$

Define

$$U_{n-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \tilde{U}_{n-1} \end{bmatrix}_{n\times n}$$

here $\tilde{U}_{n-1} \in \mathbb{R}^{(n-2)\times(n-2)}$ can be chosen orthogonal such that

$$\tilde{U}_{n-1} Z_{n-2} = (||Z_{n-2}||_2, 0, \cdots, 0)^T \tag{34}$$

Then

$$U_{n-1}U_nM_nU_n^TU_{n-1}^T = \begin{bmatrix} \sqrt{2}\xi_n & ||Z_{n-1}||_2 & 0 & \cdots & 0 \\ ||Z_{n-1}||_2 & \sqrt{2}\xi_{n-1} & ||Z_{n-2}||_2 & & \\ 0 & ||Z_{n-2}||_2 & & & \\ \vdots & & \tilde{U}_{n-1}\tilde{M}_{n-1}\tilde{U}_{n-1}^T & & \\ 0 & & & & \end{bmatrix}_{n\times n}$$

Repeat this $(n-3)$ times, M_n can be transformed to the tridiagonal form:

$$\begin{bmatrix} \sqrt{2}\xi_n & ||Z_{n-1}||_2 & & & \\ ||Z_{n-1}||_2 & \sqrt{2}\xi_{n-1} & ||Z_{n-2}||_2 & & \\ & ||Z_{n-2}||_2 & \ddots & \ddots & \\ & & \ddots & \ddots & ||Z_1||_2 \\ & & & ||Z_1||_2 & \sqrt{2}\xi_1 \end{bmatrix} = \begin{bmatrix} a_n & b_{n-1} & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & \\ & b_{n-2} & \ddots & \ddots & \\ & & \ddots & \ddots & b_1 \\ & & & b_1 & a_1 \end{bmatrix}$$

where $a_n, a_{n-1}, \cdots, a_1, b_{n-1}, \cdots, b_1$ are independent. $a_k \sim \sqrt{2}N(0,1)$, $b_k \sim \chi_k$, which is defined by:

$$\chi_n^2 \sim Y_n = \sum_{i=1}^{n} X_i^2, \quad X_1, X_2, \cdots, X_n \ i.i.d. \sim N(0,1) \tag{35}$$

$$\sqrt{Y_n} \sim \chi_n \sim \Gamma(\tfrac{n}{2}, \tfrac{1}{2}) \tag{36}$$

and the density of χ_n^2 is:

$$\frac{1}{\Gamma(\frac{n}{2})}(\frac{1}{2})^{\frac{n}{2}}\chi^{\frac{n}{2}-1}e^{-\frac{x}{2}} \tag{37}$$

Now the question remained is how to choose $\tilde{U}_n$? We claim that $\tilde{U}_n$ defined in the following way suffices.

$$\tilde{U}_n = I_{n-1} - 2\frac{v_{n-1}v_{n-1}^T}{|v_{n-1}|^2} \tag{38}$$

$$v_{n-1} = Z_{n-1} - ||Z_{n-1}||_2 \cdot e_1 \tag{39}$$

$$e_1 = (1, 0, \cdots, 0)^T \tag{40}$$

It's easy to check that $\tilde{U}_n\tilde{U}_n^T = I$ and $\tilde{U}_nZ_{n-1} = (||Z_{n-1}||_2, 0, \cdots, 0)$.

Generally speaking, Gaussian Ensembles have the tridiagonal representation:

$$\frac{1}{\sqrt{\beta}}\begin{bmatrix} N(0,\sqrt{2}) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,\sqrt{2}) & \chi_{(n-2)\beta} & & \\ & \chi_{(n-2)\beta} & \ddots & \ddots & \\ & & \ddots & \ddots & \chi_\beta \\ & & & \chi_\beta & N(0,\sqrt{2}) \end{bmatrix}_{n\times n}$$

This matrix will have the same joint eigenvalue density as the original ensemble.

For any symmetric tridiagonal matrix, we can conjugate it with a diagonal matrix $D = diag(d_1, \cdots, d_n)$ to get $b_i \geq 0$ for all $1 \leq i \leq n-1$. Since

$$D^T \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix} D = \begin{bmatrix} a_1 & b_1\frac{d_2}{d_1} & & & \\ b_1\frac{d_1}{d_2} & a_2 & b_2\frac{d_3}{d_2} & & \\ & b_2\frac{d_2}{d_3} & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1}\frac{d_n}{d_{n-1}} \\ & & & b_{n-1}\frac{d_{n-1}}{d_n} & a_n \end{bmatrix}$$

choose $d_1, d_2, \cdots, d_n = \pm 1$ in a way that $b_k \frac{d_k}{d_{k-1}} \geq 0$. So the final tridiagonal form can be assumed as:

$$A = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix}_{n\times n}$$

with $b_k > 0$. Suppose the eigenvalues are $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, $v_1, v_2, \cdots, v_n$ are the normalized eigenvectors ($< v_1, v_2 >= \delta_{ij}$). We show the following facts:

- The eigenvalues are simple: $\lambda_1 < \lambda_2 < \cdots < \lambda_n$.

  The eigenvalue equations $Au = \lambda u$ are:

$$(a_1 - \lambda)u_1 + b_1 u_2 = 0 \tag{41}$$

$$b_{k-1}u_{k-1} + (a_k - \lambda)u_k + b_k u_{k+1} = 0, \quad k = 2, \cdots, (n-1) \tag{42}$$

$$b_{n-1}u_{n-1} + (a_n - \lambda)u_n = 0 \tag{43}$$

  If $u_1 = 0$, then each $u_k = 0$. So we can assume that $u_1 \neq 0$, in particular we can assume $u_0 = 1$. From u_1, we can get $u_2, u_3, \cdots, u_{n+1}$ successively. If $u_{n+1} = 0$, then λ is an eigenvalue.

$$\begin{bmatrix} \frac{\lambda - a_k}{b_k} & -\frac{b_{k-1}}{b_k} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_k \\ u_{k-1} \end{bmatrix} = \begin{bmatrix} u_{k+1} \\ u_k \end{bmatrix}$$

  Since $b_k > 0$, the rank of this matrix is 2.

□