Lectures 6 – 7 : Marchenko-Pastur Law

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We will now turn our attention to rectangular matrices. Let

$$X = (\underline{\mathrm{X}}_1, \underline{\mathrm{X}}_2, \ldots, \underline{\mathrm{X}}_n) \in \mathbb{R}^{p\times n}$$

where X_{ij} are iid, $E(X_{ij}) = 0, E(X_{ij}^2) = 1$ and $p = p(n)$.
Define

$$S_n = \frac{1}{n}XX^T \ \in \mathbb{R}^{p\times p}$$

and let

$$\lambda_1 \le \lambda_2 \le \ldots \le \lambda_p$$

denote the eigenvalues of the matrix S_n.
Define the random spectral measure by

$$\mu_n = \frac{1}{p}\sum_{i=1}^{p}\delta_{\lambda_i}$$

We are now ready to state the Marchenko-Pastur law

Theorem 1. *Let S_n, μ_n be as above. Assume that $p/n \overset{n\to\infty}{\longrightarrow} y \in (0, 1]$. Then we have*

$$\mu_n(\cdot, \omega) \Rightarrow \mu \quad a.s$$

where μ is a deterministic measure whose density is given by

$$\frac{d\mu}{dx} = \frac{1}{2\pi xy}\sqrt{(b-x)(x-a)}1_{(a\le x\le b)} \tag{1}$$

Here a and b are functions of y given by

$$a(y) = (1-\sqrt{y})^2, \quad b(y) = (1+\sqrt{y})^2$$

Remark 2. If $y > 1$ then since $rank(S) = p \wedge n$ we will have roughly $n(y-1)$ zero eigenvalues. Since $\mu_n = \frac{1}{p}\sum_{i=1}^{p}\delta_{\lambda_i}$ we see that there will be a mass of $(1-y^{-1})$ at 0 in the limiting measure. Since the nonzero eigenvalues of XX^T and X^TX are same we can say that in this case the limiting distribution is

$$(1-y^{-1})\delta_0 + \mu$$

where μ satisfies (1)

Remark 3. Observe that if $y = 1$, then $a = 0, b = 4$, and thus

$$\frac{d\mu}{dx} = \frac{1}{2\pi x}\sqrt{(4-x)x}1_{(0\le x\le 4)}$$

In this case μ is the image of semicircle distribution under the mapping $x \to x^2$

Proof. : We now begin the proof of Marchenko Pastur Law. Since the support of μ is compact, μ is uniquely determined by its moments. So as in the Wigners case it is enough to show

$$\int x^k \, d\mu_n \to \int x^k \, d\mu$$

Again following Wigner's case, Borel Cantelli lemma says it is enough to show the following

1. $$E \int x^k \, d\mu_n \to \int x^k \, d\mu$$

2. $$\text{Var}(\int x^k \, d\mu_n) \leq \frac{C_k}{n^2}$$

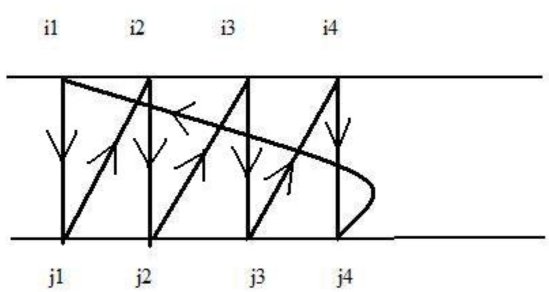
Computation of the second integral in 1 will show that

$$\int x^k \, d\mu = \sum_{r=0}^{k-1} \frac{y^r}{r+1} \binom{k}{r} \binom{k-1}{r}$$

Now notice that

$$\begin{aligned} E \int x^k \, d\mu_n &= \frac{1}{p} E(\sum_{i=1}^{p} \lambda_i^p) = \frac{1}{p} E[Tr(XX^T/n)^k] \\ &= \frac{1}{pn^k} E[\sum_{I,J} X_{i_1 j_1} X_{i_2 j_1} X_{i_2 j_2} X_{i_3 j_2} \ldots X_{i_k j_k} X_{i_1 j_k}] \equiv \frac{1}{pn^k} \sum_{I,J} E(I,J) \end{aligned}$$

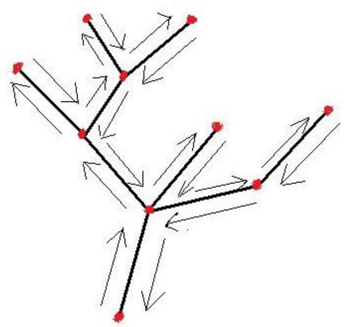
where $I \in [p]^k$ and $J \in [n]^k$.
Now this corresponds to a directed loop on a bipartite graph. For example if $k = 4$ then for typical $\{i_1, i_2, i_3, i_4\}$ and $\{j_1, j_2, j_3, j_4\}$ we have the following picture.



As in the Wigner's case we see that each edge must appear at least twice, otherwise $E(I, J) = 0$. Now we have $2k$ steps in the directed loop. Thus we see that we have at most k edges in the

skeleton, hence at most $k+1$ vertices in the skeleton.
Next assume that number of vertices $= m \le k$. Let $m = a + b$ where $a = \#$ of I vertices and $b = \#$ of J vertices. Then the total number of ways choosing a I vertices and b J vertices $\le Cp^a n^b$, where C is a constant independent of n. The contribution of these terms in the expectation $\le C' p^a n^b / pn^k \to 0$ as $n \to \infty$.
Thus we need to look at loops which have exactly $k+1$ vertices and k edges. These are exactly the double trees.

Reshuffle them to get the following structure.



Start with an I vertex. Vertices that can be reached in even steps are the I vertices, the rest are the J vertices.

Next we ask the question: How many double trees are there for a given shape? Here by the shape of a tree we mean the vertices numbered in order of appearence. For example

$$(2\ 3\ 4\ 5\ 4\ 6\ 7\ 6\ 8\ 6\ 4\ 3\ 9\ 3\ 10\ 11\ 10\ 3\ 2), \quad (12\ 13\ 14\ 15\ 16\ 17\ 16\ 18\ 16\ 14\ 13\ 19\ 13\ 6\ 7\ 6\ 13\ 12)$$

will give us the same shape, because after renumbering in order of appearance both will give us the following double tree

$$(1\ 2\ 3\ 4\ 3\ 5\ 6\ 5\ 7\ 5\ 3\ 2\ 8\ 2\ 9\ 10\ 9\ 2\ 1)$$

and all of them look like the figure above. Thats is we have to choose $r+1$ I vertices from $[p]$ and $k-r$ J vertices from $[n]$. This can be done in $P(p, r+1)P(n, k-r)$ where $P(n,k) = n(n-1)\dots(n-k+1)$ is permutation of k objects from n distinct objects. Notice that

$$P(p, r+1)P(n, k-r) = np^k y_n^r (1 + O(n^{-1})), \quad \text{where} \;\; y_n = p/n$$

Thus

$$E(\int x^k \, d\mu_n) = \frac{1}{pn^k} \sum_{I,J} E(I,J) = \sum_{r=0}^{k-1} y_n^r (1+O(n^{-1})) \times \#\{\text{double tree shapes with } r+1 \ I \text{ and } k-r \ J \text{ vertices}\}$$

Since $y_n \to y$, as $n \to \infty$, its clear that all we need now is to show that

$$\#\{\text{double tree shapes with } r+1 \;\; I \text{ and } k-r \;\; J \;\; \text{vertices}\} = \frac{1}{r+1}\binom{k}{r}\binom{k-1}{r}$$

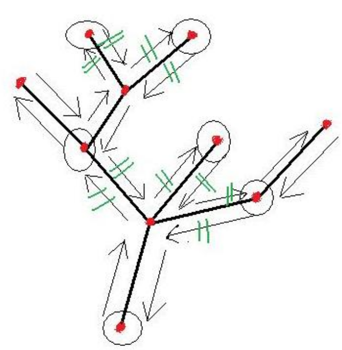
Towards this end we try to correspond each double tree shape with the following type of path/sequence of $2k$ steps.

1. If i is odd then $s_i \in \{-1, 0\}$
2. If i is even then $s_i \in \{0, 1\}$, $s_{2k} = 0$
3. For any $l = 1, 2, \ldots, 2k$, we have $\sum_{i=1}^{l} s_i \geq 0$. That is the path is never below 0.
4. $\#\{i : s_i = 1\} = \#\{i : s_i = -1\} = r$. Thats is there are exactly r upsteps and r downsteps
5. $\sum_{i=1}^{2k} s_i = 0$. That is we return to 0 at the end.

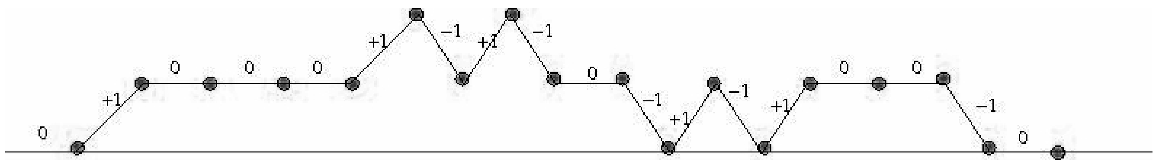
Given any such sequence $\{s_i\}_{i=1}^{2k}$, clearly we can construct a tree as following:

- Suppose i is odd. If $s_i = -1$ then we go down the double tree, if $s_i = 0$ then we go up from an I vertex but we will return
- Suppose now i is even. If $s_i = 1$ then we go one step up in the double tree. If $s_i = 0$ then we go one step down

Next given a double tree shape we construct such a sequence $\{s_i\}_{i=1}^{2k}$ First for each I vertex we mark the first edge leading to it and the last edge leaving it. After marking the previous double will look like the following. The circled vertices are the I vertices.



Now put $s_i = 1$ if the i-th edge is marked and its going up, $s_i = -1$ if i-th edge is marked and going down, $s_i = 0$ otherwise. For example the above double tree will give the following path.



We have to verify this allocation of $-1, 0, 1$ would still make $\{s_i\}_{i=1}^{2k}$ satisfy condition (3). Suppose if possible we have a first l such that

$$\sum_{i=1}^{2l-1} s_i = -1, \quad \text{hence} \sum_{i=1}^{2l} s_i = 0, \quad \text{and } s_{2l-1} = -1$$

Then the other part tells us that we can construct a double tree with vertices $\{1, 2 \ldots, 2l\}$, and since $s_{2l-1} = -1$, the second bullet from the other part says that we are not moving up to a new vertex,but going down to an old vertex in $\{1, 2 \ldots, 2l\}$. But this destroys the double tree shape giving us a contradiction. Hence we see that if we allocate $-1, 0, 1$ by the above rule then we indeed get a sequence $\{s_i\}_{i=1}^{2k}$ satisfying required conditions.

Thus the set of the double tree shapes is in bijection with the set of sequence $\{s_i\}_{i=1}^{2k}$, so all we need to do now is count such sequences $\{s_i\}_{i=1}^{2k}$.
Since $s_{2k} \neq +1$, not considering condition (3) for the moment we see that out of $k-1$ positions for $+1$ and k positions for -1 we have to choose r each. Therefore the number of such sequences is $\binom{k-1}{r}\binom{k}{r}$.
Lets now count the number of sequences which fail condition (3). Since those paths hit -1 there exists a first l, such that $\sum_{i=1}^{2l-1} s_i = -1$ (By construction of the sequence s_k can be -1 only when k is odd.). We now construct a new sequence $\{s_i'\}_{i=1}^{2k}$ by 'reflection'. Put

$$s_i' = s_i, \quad \text{for } i = 1, \ldots, 2l-1, \; s_{2k}' = s_{2k} = 0$$

For $l \leq i \leq k-1$, put

$$\begin{aligned} (s_{2i}', s_{2i+1}') &= (1, -1) \quad \text{if } (s_{2i}, s_{2i+1}) = (1, -1) \\ &= (0, 0) \quad \text{if } (s_{2i}, s_{2i+1}) = (0, 0) \\ &= (1, 0) \quad \text{if } (s_{2i}, s_{2i+1}) = (0, -1) \\ &= (0, -1) \quad \text{if } (s_{2i}, s_{2i+1}) = (1, -0) \end{aligned}$$

Clearly the set of all sequences $\{s_i\}_{i=1}^{2k}$, which fail condition (3) is in bijection with the set of sequences $\{s_i'\}_{i=1}^{2k}$ But to count the number of such sequences $\{s_i'\}_{i=1}^{2k}$ we just have to count the number of ways we can choose $r-1$ '$+1$' from $k-1$ of them, and $r+1$ '-1' from k of them. This can be done in $\binom{k-1}{r-1}\binom{k}{r+1}$. So the total number of sequences $\{s_i\}_{i=1}^{2k}$ which satisfies condition (1 - 4) is given by

$$\binom{k-1}{r}\binom{k}{r} - \binom{k-1}{r-1}\binom{k}{r+1} = \frac{1}{r+1}\binom{k-1}{r}\binom{k}{r}$$

This proves the fact about expectation and the proof of the variance bound is similar to that of Wigner Matrix. □

We now move to some particular type of random matrices, namely the **Gaussian Ensembles**.

Gaussian Orthogonal Ensemble (GOE): Here we look at matrices M_n of the form $M_n = [X_{i,j}]_{i,j=1}^n$ where

$$X_{i,j} = X_{j,i}, \quad X_{i,j} \overset{iid}{\sim} N(0,1),\ i < j, \quad \text{and } X_{i,i} \sim \sqrt{2}N(0,1)$$

and they are all independent.
We can construct them in following way. Take a matrix $A = [Y_{i,j}]_{i,j=1}^n$, where $Y_{i,j} \overset{iid}{\sim} N(0,1)$. Then

$$M_n = (A + A^T)/\sqrt{2}$$

is a GOE.

Gaussian Unitary Ensemble (GUE): These are very similar to GOE. Here we look at matrices M_n of the form $M_n = [X_{i,j}]_{i,j=1}^n$ where

$$X_{i,j} = \bar{X}_{j,i}, \quad X_{i,j} \sim N(0,1/2) + iN(0,1/2),\ i < j, \quad \text{and } X_{i,i} \sim N(0,1)$$

and they are all independent. We can construct them in them in the following way. Take a matrix $A = [Y_{i,j}]_{i,j=1}^n$, where $Y_{i,j} \overset{iid}{\sim} N(0,1/2) + iN(0,1/2)$] Then

$$M_n = (A + A^*)/\sqrt{2}$$

is GUE

Gaussian Symplectic Ensemble (GSE) Define Z as the following block diagonal matrix $Z_{2n\times 2n} = diag(A, A \ldots, A)$, where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Call a matrix $M \in \mathbb{C}^{2n\times 2n}$ symplectic if

$$Z = MZM^T$$

We next define the space of quaternions. Define the following 2×2 matrices

$$\mathbf{e_1} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \mathbf{e_2} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \mathbf{e_3} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that

$$\mathbf{e_1} \cdot \mathbf{e_2} = -\mathbf{e_2} \cdot \mathbf{e_1} = \mathbf{e_3}, \quad \mathbf{e_1}^2 = \mathbf{e_2}^2 = \mathbf{e_3}^2 = -\mathbf{1}$$

The conjugation rule is as follows

$$\bar{\mathbf{1}} = \mathbf{1}, \bar{\mathbf{e}}_\mathbf{2} = -\mathbf{e_2}, \bar{\mathbf{e}}_\mathbf{3} = -\mathbf{e_3}, \bar{\mathbf{e}}_\mathbf{4} = -\mathbf{e_4}$$

The vector space generated by $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{1}\}$ over $\mathbb{C}$ is called the space of quaternions.
A quaternion $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is real if $q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)}$ are real where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = q^{(1)} \cdot \mathbf{1} + q^{(2)} \cdot \mathbf{e_2} + q^{(3)} \cdot \mathbf{e_3} + q^{(4)} \cdot \mathbf{e_4}$$

A random quaternion $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called real standard quaternion if its real and if

$$q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)} \overset{iid}{\sim} N(0, 1/4)$$

Let $Q_{i,j} = q_{i,j}^{(1)} \cdot \mathbf{1} + q_{i,j}^{(2)} \cdot \mathbf{e_2} + q_{i,j}^{(3)} \cdot \mathbf{e_3} + q_{i,j}^{(4)} \cdot \mathbf{e_4}$

A GSE is defined by $M_n = [Q_{i,j}]_{i,j=1}^n$ where for $i < j$, $Q_{i,j}$ are iid standard quaternions, $Q_{i,j} = \bar{Q}_{j,i}$, and on the diagonal $i = j$ we have $q_{i,i}^{(0)} \sim N(0, 1/2)$ We can construct such a matrix as follows. Let $A = [Y_{i,j}]_{i,j=1}^n$ where $Y_{i,j}$ are iid real standard quaternions. Then $M_n = (A + A^*)/\sqrt{2}$ is GSE.

Let dM be the reference lebesgue measure , based on the determining entries. Define the density function w.r.t dM as

$$\frac{1}{Z_{n,\beta}} \exp(-\frac{\beta}{4} Tr(M^2))$$

Then this defines the density of GOE, GUE and GSE for $\beta = 1, 2, 4$ respectively.