Lectures 23 - 24 : Bulk Scaling of the General β-Ensemble

Notes prepared by: Nicos

1 Introduction

Recall the eigenvalue density of the $\beta-$ensemble

$$p_{n,\beta} = \frac{1}{Z_{n,\beta}} \prod |\lambda_i - \lambda_j|^{\beta} e^{-\frac{\beta}{4}\sum \lambda_i^2} \tag{1.1}$$

and the random vector on the real line

$$\Lambda_n = \{\lambda_1, \lambda_2, ..., \lambda_n\}. \tag{1.2}$$

The goal is to describe the limiting point process if we 'zoom in' around a point $c \in (-2, 2)$. (See Fig. 1) We are going to prove that

$$\sigma(c)\sqrt{n}\left(\Lambda_n - c\sqrt{n}\right) \Longrightarrow Sine_{\beta} \tag{1.3}$$

where $Sine_{\beta}$ is a point process that we will describe later and $\sigma(c) = \sqrt{4-c^2}\mathbf{1}\{|c| < 2\}$, the semicircle density. For the purposes of this limit we assume $c = 0$ and hence $\sigma(c) = 2$. Denote by $N(\lambda)$ the counting function of the process $Sine_{\beta}$.

1.1 Description of $N(\lambda)$

We have the following two equivalent descriptions for $N(\lambda)$.

First description. Let Z be a two dimensional Brownian Motion and consider the system of SDE (one equation for each λ)

$$d\alpha_{\lambda} = \lambda \frac{\beta}{4} e^{-\frac{\beta}{4}t}\, dt + Re\left(\left(e^{-i\alpha_{\lambda}} - 1\right) dZ\right) \tag{1.4}$$

with $\alpha_{\lambda}(0) = 0$. Note that the driving Brownian Motion is common for all λ. Then

$$N(\lambda) = \frac{1}{2\pi} \lim_{t\to+\infty} \alpha_{\lambda}(t). \tag{1.5}$$

At this point is not even clear that the limit in (1.5) exists, but we will show this in Section 2. Equivalence of descriptions 1 and 2 will be shown later.

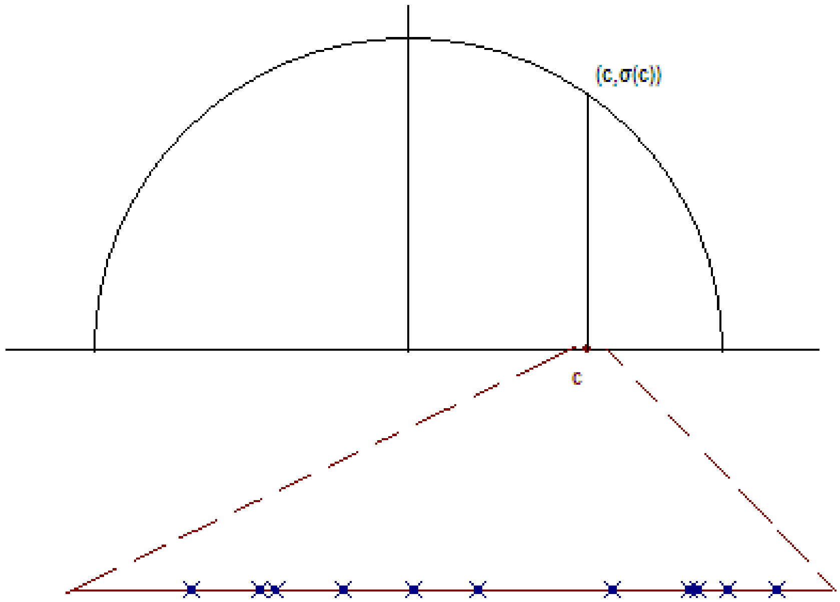

Figure 1: After multiplying $\{\Lambda_n - c\sqrt{n}\}$ by $\sigma(c)\sqrt{n}$ is like 'zooming in' around c.

Second description - The brownian Carousel. Consider the Poincare disc $\mathbb{D}$ and let $b(t)$ a hyperbolic Brownian Motion. In symbols, $b(t)$ is the solution to the following SDE:

$$dB = \frac{1 - |B|^2}{2} \, dZ, \tag{1.6}$$

where Z is a complex Brownian Motion. Given $b(t)$, for each λ define a process that gives the position of a particle on the unit circle (the boundary of $\mathbb{D}$) $Z_\lambda(t)$ described in the following way.

1. $Z_\lambda(0) = 1$
2. $|Z_\lambda(t)| = 1$
3. $Z_\lambda(t)$ is rotated about $b(t)$ with speed $\lambda \frac{\beta}{4} e^{-\frac{\beta}{4}t}$.

For each λ we can compute the winding angle $\alpha_\lambda(t)$ at any time t. (This is the hyperbolic angle between 1, $b(t)$, $Z_\lambda(t)$.) Then,

$$N(\lambda) = \frac{1}{2\pi} \lim_{t \to +\infty} \alpha_\lambda(t). \tag{1.7}$$

$Z_\lambda(t)$ evolves according to

$$\partial_t Z_\lambda(t) = i\lambda\zeta \frac{|Z_\lambda - b|^2}{1 - |b|^2.} \tag{1.8}$$

2 The limit of $\alpha_\lambda(t)$

Consider (1.4) and look at the noise term

$$\begin{aligned} Re\left(\left(e^{-i\alpha_\lambda}-1\right)\,dZ\right) &= Re\left(\left(e^{-i\frac{\alpha_\lambda}{2}}-e^{i\frac{\alpha_\lambda}{2}}\right)e^{-\frac{i\alpha_\lambda}{2}}\,dZ\right) \\ &= Re\left(\sin(\frac{\alpha_\lambda}{2})(-2i)e^{-i\frac{\alpha_\lambda}{2}}\,dZ\right) \\ &= 2\sin(\frac{\alpha_\lambda}{2})Re\left(-ie^{-i\frac{\alpha_\lambda}{2}}\,dZ\right). \end{aligned} \tag{2.1}$$

Now assume we are only interested in a single value of λ. The expression in the real part of (2.1) is a rotation (depending only only on λ) of a two dimensional white noise and the real part projects it on the real line. We can write (2.1) as

$$Re\left(\left(e^{-i\alpha_\lambda}-1\right)\,dZ\right) = 2\sin(\frac{\alpha_\lambda}{2})\,dB \tag{2.2}$$

where B is a standard Brownian Motion. For a given λ, equation (1.4) becomes

$$d\alpha_\lambda = \lambda\frac{\beta}{4}e^{-\frac{\beta}{4}t} + 2\sin(\frac{\alpha_\lambda}{2})\,dB \tag{2.3}$$

Note that the noise term is 0 whenever $\alpha_\lambda = 2n\pi$. Define

$$X_\lambda(t) = \alpha_\lambda(t) - \lambda\int_0^1 \frac{\beta}{4}e^{-\frac{\beta}{4}s}\,ds. \tag{2.4}$$

We have that $X_\lambda(t)$ is a *local martingale*. Since $\alpha_\lambda(t)\geq 0$ and $0<\int_0^t \frac{\beta}{4}e^{-\frac{\beta}{4}s}\,ds$ we get $X_\lambda(t)\geq -\lambda$. Therefore X_λ converges a.s. From (2.4) we get that $\alpha_\lambda(t)$ converges almost surely. The convergence cannot happen while the noise term in (2.3) is in effect and we conclude that $\alpha_\lambda(t)$ must converge to an integer multiple of 2π. Hence

$$\lim_{t\to\infty}\frac{\alpha_\lambda(t)}{2\pi} = k, \qquad \text{for some } k\in\mathbb{Z}. \tag{2.5}$$

3 Discrete Sturm - Liuville theory

For the purposes of this section we are going to work with the triadiagonal matrix

$$M_n = \begin{bmatrix} a_n & b_{n-1} & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & \\ & b_{n-2} & \ddots & \ddots & \\ & & \ddots & \ddots & b_1 \\ & & & b_1 & a_1 \end{bmatrix}_{n\times n} \tag{3.1}$$

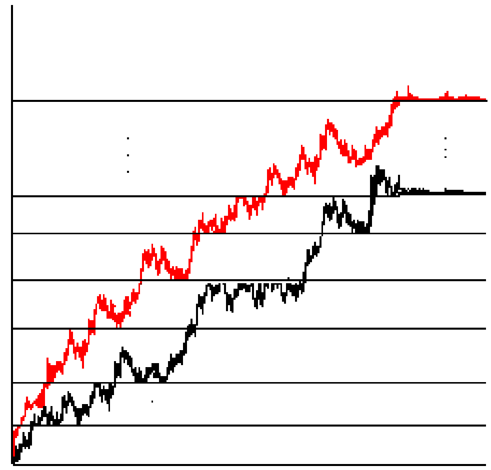

Figure 2: Two possible evolutions for the angle $\alpha_\lambda(t)$ for two different positive values $0 < \lambda_1$ (black) $< \lambda_2$ (red) before they stabilize. Every time the two diffusions get close to each other, the difference in the drifts pushes the red one higher. Notice that the two diffusions never go below the horizontal level lines $y = 2\pi n$ and eventually they stabilize.

where $b_k > 0$.

Let $\vec{u} = [u_1, ...u_n]^t$ and consider the eigenvalue equation

$$M_n \vec{u} = \lambda \vec{u}. \tag{3.2}$$

Coordinate-wise we we get the difference equation

$$b_{n-k} u_{k-1} + a_{n-k} u_k + b_{n-k-1} u_{k+1} = \lambda u_k, \tag{3.3}$$

with initial condition $u_0 = 0$ and terminal condition $u_{n+1} = 0$. Recall the following three properties of (3.3).

1. A number λ is an eigenvalue of M_n with eigenvector $\vec{u}$ if and only if verifies (3.3) with the terminal condition.
2. If $\vec{u}$ is an eigenvector, then it cannot have two consecutive 0 entries.
3. The eigenvalues of M_n are simple (since $b_k > 0$).

Define $r_k = \dfrac{u_{k+1}}{u_k}$, $r_0 = +\infty$. For the remaining part we allow $\frac{1}{0} = \infty$, and property (2) implies that we will never have to worry about the expression $\frac{0}{0}$.

Divide (3.3) by u_k and massage the expression a bit. We get the one term recursion now

$$r_k = -\frac{b_{n-k}}{b_{n-k-1}}\frac{1}{r_{k-1}} + \frac{\lambda - a_{n-k}}{b_{n-k-1}}. \tag{3.4}$$

Equivalently, set $\frac{b_{n-k}}{b_{n-k-1}} = c_k > 0$, $\frac{1}{b_{n-k-1}} = e_k > 0$ and $\frac{-a_{n-k}}{b_{n-k-1}} = d_k$ to get

$$r_k = \frac{-c_k}{r_{k-1}} + d_k + \lambda e_k := R_k(r_{k-1}) \tag{3.5}$$

Property (1) now becomes

$$\lambda \text{ is an eigenvalue } \Leftrightarrow r_n = 0 \tag{3.6}$$

Since r_k is a function of λ as well, we are going to write $r_{k,\lambda}$.

Define X to be the space we get if we connect countable copies of $\mathbb{R}$ in the following manner:

$$X = \ldots \overset{\mathbb{R}}{\overline{\qquad\quad}} \infty \overset{\mathbb{R}}{\overline{\qquad\quad}} \infty \overset{\mathbb{R}}{\overline{\qquad\quad}} \ldots$$

Each copy of $\mathbb{R}$ is indexed by an integer and we say the n-th copy is indexed by the indeger n.

Lemma 1. *For each k, the map $R_k : \mathbb{R} \to X$ that takes $r_{k-1,\lambda} \longmapsto r_{k,\lambda}$ is monotonically increasing in λ (for $\lambda > 0$), in the sense that if we get the value infinity for some k, λ, we move on to the next copy of $\mathbb{R}$. We assume that for $k = 1, \lambda > 0$ we have the value infinity at the beginning of the copy of $\mathbb{R}$ indexed by the integer 1.*

Proof. Let $0 < \lambda_1 < \lambda_2$. Assume that $r_{k-1,\lambda_1} \leq r_{k-1,\lambda_2}$. The map R_k can break into a composition of the following four functions:

$$r_{k-1,\lambda} \longmapsto \frac{-1}{r_{k-1,\lambda}} \longmapsto c_k\frac{-1}{r_{k-1,\lambda}} \longmapsto \frac{-c_k}{r_{k-1,\lambda}} + d_k \longmapsto \frac{-c_k}{r_{k-1,\lambda}} + d_k + \lambda e_k = r_{k,\lambda}. \tag{3.7}$$

Now observe that the first map in increasing (so is order preserving), given that we move to a different copy of R if we ever get the value ∞. The other maps are either multiplication by a positive number or shifting by a constant. Throughout we are assuming $\infty + c = \infty$.

Since $e_k > 0$, by the last mapping we get that for $\lambda_1 < \lambda_2$ we have $r_{k,\lambda_1} \leq r_{k,\lambda_2}$ as required. A straightforward induction argument finishes the proof for all k. The base case of the induction is covered by the assumption $r_{0,\lambda} = \infty$ for all λ. □

We want to connect $r_{k,\lambda}$ with a certain angle. This is not really far fetched if you notice that the space X defined above is a covering space for a circle (and isomorphic to $\mathbb{R}$.) It is going to be relevant to define the *index function I*

$$I : X \longrightarrow \mathbb{Z}, \quad I(r_{k,l}) = m \tag{3.8}$$

where m is the index of the copy of $\mathbb{R}$ that $r_{k,\lambda}$ lies in. We set $I(r_{0,\lambda}) = 0$ for all λ. This will correspond to to the number of times we winded around the circle.

Conformal Mappings

Consider the map from the upper-half plane to the unit disk $\mathbb{D}$,

$$U : z \longmapsto \frac{i-z}{i+z}. \tag{3.9}$$

U takes the real line to the boundary of the disk (the unit circle). In paticular $U(0) = 1, U(\infty)(= U(-\infty)) = -1$. Therefore, $U(r_{k,l}) = Z_{k,\lambda}$ where $|Z_{k,\lambda}| = 1$.

To get the recursion for $Z_{k,\lambda}$ use the inverse map U^{-1} and R_k (in (3.5)). Then

$$Z_{k,\lambda} = UR_kU^{-1}(Z_{k-1,\lambda}). \tag{3.10}$$

Note that the map UR_kU^{-1} is a conformal mapping form the disk to itself, so it has the form $UR_kU^{-1} = \zeta\frac{z-a}{1-\bar{a}z}$ for some a and $\zeta, |\zeta| = 1$. It maps the boundary circle to itself, and it preserves the order.

Any complex number w on the unit circle can be written as $w = e^{i\theta_w}$ for some value of θ_w. Therefore, there exist angles $\Theta_{k,\lambda}$ such that

$$Z_{k,\lambda} = e^{i\Theta_{k,\lambda}}. \tag{3.11}$$

Statement (3.6) now becomes

$$\lambda \text{ is an eigenvalue } \Leftrightarrow r_{n,\lambda} = 0 \Leftrightarrow Z_{n,\lambda} = 1 \Leftrightarrow \Theta_{n,\lambda} = 0 \mod 2\pi. \tag{3.12}$$

Using the function (3.8), we can define the function

$$\phi_n(\lambda) = 2I(r_{n,\lambda})\pi + \Theta_{n,\lambda}. \tag{3.13}$$

Lemma 2 (Counting eigenvalues)**.** *Let $\lambda_1 < \lambda_2$. Then*

$$\#\{ \textit{ eigenvalues } \in [\lambda_1, \lambda_2]\} = \#\{n : \phi_n(\lambda_1) \le 2n\pi \le \phi_n(\lambda_2)\} \tag{3.14}$$

Proof. By Lemma 1 and from the fact that UR_kU^{-1} preserves the order of points on the boundary of the circle, we conclude that $\phi_n(\lambda)$ is strictly increasing in λ. For this reason, the number of eigenvalues in $[\lambda_1, \lambda_2]$ is the difference of the indices $I(r_{n,\lambda_2}) - I(r_{n,\lambda_1})$, since for $r_{n,\lambda}$ to be 0, we must have $r_{n-1,\lambda} = \infty$. But every time we hit $+\infty$, we move to the next copy of $\mathbb{R}$ in X. To finish the proof, use (3.12). □

4 Derivation of equation (1.4) using Sturm-Liouville Theory

Recall that

$$M_n = \frac{1}{\sqrt{\beta}} \begin{bmatrix} a_n & b_{n-1} & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & \\ & b_{n-2} & \ddots & \ddots & \\ & & \ddots & \ddots & b_1 \\ & & & b_1 & a_1 \end{bmatrix}_{n\times n}, \tag{4.1}$$

with $a_k \sim N(0,2)$, $b_k \sim \chi_{k\beta} > 0$ a.s. Recall that this matrix is symmetric, so the entries are not independent. To analyze this matrix (and by extension the corresponding equation (3.4)) we need to move the randomness on and above the diagonal, in way that we have independent entries but without changing the eigenvalues. We need a basic fact from Linear algebra.

Fact: Let D be an invertible matrix and M a matrix. Then $\widetilde{M} = DMD^{-1}$ and M have the same eigenvalues.

In the special case where D is an invertible diagonal matrix

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}_{n\times n}, \tag{4.2}$$

we get

$$\widetilde{M}_n = \frac{1}{\sqrt{\beta}} \begin{bmatrix} a_n & b_{n-1}\frac{d_1}{d_2} & & & \\ b_{n-1}\frac{d_2}{d_1} & a_{n-1} & b_{n-2}\frac{d_2}{d_3} & & \\ & b_{n-2}\frac{d_2}{d_3} & \ddots & \ddots & \\ & & \ddots & \ddots & b_1\frac{d_{n-1}}{d_n} \\ & & & b_1\frac{d_n}{d_{n-1}} & a_1 \end{bmatrix}_{n\times n}. \tag{4.3}$$

Check to see that this type of conjugation with D keeps the product of symmetric off-diagonal elements the same. With an appropriate choice of d_i we can transform M_n to the following matrix,

$$M_n \sim \frac{1}{\sqrt{\beta}} \begin{bmatrix} N(0,2) & \frac{\chi^2_{(n-1)\beta}}{c_1} & & & \\ c_1 & N(0,2) & \frac{\chi^2_{(n-2)\beta}}{c_2} & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{\chi^2_{\beta}}{c_{n-1}} \\ & & & c_{n-1} & N(0,2) \end{bmatrix}_{n\times n}. \tag{4.4}$$

Finally, define $X_k \sim \frac{1}{\sqrt{\beta}}N(0,2)$, $s_j = \sqrt{n-j-\frac{1}{2}}$, $Y_j = \frac{\chi^2_{(n-j-1)\beta}}{\beta s_{j+1}} - s_j$ and do one final conjugation to get the matrix

$$\widetilde{M}_n = \begin{bmatrix} X_0 & -(s_0+Y_0) & & & \\ -s_1 & X_1 & s_1+Y_1 & & \\ & s_2 & X_2 & -(s_2+Y_2) & \\ & & \ddots & \ddots & (-1)^n(s_{n-2}+Y_{n-2}) \\ & & & (-1)^n s_{n-1} & X_{n-1} \end{bmatrix}_{n\times n}. \tag{4.5}$$

We have the following moment estimates for Y_j.

1. $|\mathbb{E}Y_j| = \mathcal{O}\left((n-j)^{-3/2}\right)$
2. $|\mathbb{E}Y_j^2| = \frac{2}{\beta} + \mathcal{O}\left((n-j)^{-1}\right)$
3. $|\mathbb{E}Y_j^4| = \mathcal{O}\left(1\right)$

For this matrix, we have the equivalent recursion for $r_{k,\tilde{\lambda}}$, $k \geq 0$, $\tilde{\lambda} \geq 0$

$$r_{k+1,\tilde{\lambda}} = -\frac{s_k+Y_k}{s_{k+1}}\frac{1}{r_{k,\tilde{\lambda}}} + \frac{\tilde{\lambda}-X_k}{(-1)^k s_{k+1}}. \tag{4.6}$$

Let $\mathbf{u} = [u_1, u_2, ..., u_n] = [F_1, G_1, F_2, G_2, ...]$, and consider the eigenvalue equation

$$\mathbf{u}\widetilde{M}_n = \tilde{\lambda}\mathbf{u}.$$

Then we get the pair of equations

$$\begin{aligned} 0 &= F_{i+1}s_{2i} - G_i(X_{2i-1}-\tilde{\lambda}) - F_i(s_{2i-2}+Y_{2i-2}) \\ 0 &= G_{i+1}s_{2i+1} + F_i(X_{2i}-\tilde{\lambda}) - G_i(s_{2i-1}+Y_{2i-1}) \end{aligned} \tag{4.7}$$

Let $s(t) = \sqrt{1-t}$. This implies that $s_j \approx s(\frac{j}{n})$. Also let F, G be nice enough functions so that $F_i = F\left(\frac{2i-1}{n}\right)$ and $G_i = G\left(\frac{2i}{n}\right)$. Now replace $\tilde{\lambda} = \frac{\lambda}{2\sqrt{n}}$ and divide everything in (4.7) by $\sqrt{n}$. The first equation in (4.7) becomes

$$F\left(\frac{2i+1}{n}\right)s\left(\frac{2i}{n}\right) - F\left(\frac{2i-1}{n}\right)s\left(\frac{2i-2}{n}\right) = \frac{\lambda}{2n}G\left(\frac{2i}{n}\right) + \frac{X_{2i-1}}{\sqrt{n}}G\left(\frac{2i}{n}\right) + \frac{Y_{2i-2}}{\sqrt{n}}F\left(\frac{2i-1}{n}\right) \tag{4.8}$$

This (and the corresponding one for the second equation in (4.7)) is a discretization of the diffusions

$$\begin{aligned} d(Fs) &= -\frac{\lambda}{4}G\,dt + \beta^{-1/2}G\,dW_1 + \beta^{-1/2}F\,dW_2 \\ d(Gs) &= \frac{\lambda}{4}F\,dt - \beta^{-1/2}G\,dW_3 + \beta^{-1/2}G\,dW_4, \end{aligned} \tag{4.9}$$

with boundary conditions $F(0) = 1, G(0) = 0$ and with W_1, W_2, W_3, W_4 independent standard Brownian Motions. λ is an eigenvalue for (4.9) if and only if $F(1) = 0$. Setting $w = F + iG$ and $u = sw$ we combine the two equations in (4.9) into

$$s(t)\,du = i\frac{\lambda}{4}u\,dt + (2\beta)^{-1/2}u\,dZ_1 + (2\beta)^{-1/2}\overline{u}\,dZ_2, \quad t \in [0,1] \tag{4.10}$$

where Z_1, Z_2 are independent complex Brownian Motion. The time change $t \longmapsto 1 - e^{-\beta t/4}$ makes time to run in $[0, +\infty)$ and (4.10) is transformed into

$$du = i\frac{\lambda}{2}u\frac{\beta}{4}e^{-\beta t/4}\,dt + \frac{1}{2}\overline{u}\,dZ_2 + \frac{1}{2}u\,dZ_1, \quad t \in [0, \infty). \tag{4.11}$$

Define $\varphi_\lambda = \arg(F+iG) = \arg(u) = Im\{\log u\}$. This is the same as the index function (3.8)multiplied by π (used for example in (3.13)) since the discrete Sturm - Liouville theory we explained does not depend on the fact that the matrix M_n is symmetric.

The chain rule for stochastic calculus gives

$$d(\log u) = i\frac{\lambda}{2}\frac{\beta}{4}e^{-\beta t/4}\,dt + \frac{1}{2}\frac{\overline{u}}{u}\,dZ_2 + \frac{1}{2}\,dZ_1, \quad t \in [0, \infty). \tag{4.12}$$

Take imaginary parts on (4.12) to get the SDE for ϕ_λ and recall that $\frac{\overline{u}}{u} = e^{-2i\varphi_\lambda}$. Then

$$d\varphi_\lambda = \frac{\lambda}{2}\frac{\beta}{4}e^{-\beta t/4}\,dt + Im\left\{\frac{1}{2}e^{-2i\varphi_\lambda}\,dZ_2\right\} + \frac{1}{2}\,dB, \quad t \in [0, \infty). \tag{4.13}$$

Finally, set $a_\lambda = 2(\varphi_\lambda - \varphi_0)$ and from the fact that $i\,dZ =_{\mathcal{D}} dZ$ if Z is a complex brownian motion, we get (1.4)

$$d\alpha_\lambda = \lambda\frac{\beta}{4}e^{-\frac{\beta}{4}t}\,dt + Re\left(\left(e^{-i\alpha_\lambda} - 1\right)\,dZ\right). \tag{4.14}$$