Lectures 20 – 22 : Scaling limit of β-ensembles

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1 Edelman-Sutton Conjecture (cont.)

1.1 Existence of eigenvalue-eigenfunction pair for the smallest eigenvalue of cH_β

In the last lecture, we proved $\exists\{f_n\}$ and a nonnegative random variable K with $||f_n||_2^2 = 1$, $||f_n||_*^2 \langle K \; a.s. \; \forall n$, so that

$$\langle f_n, \mathcal{H}_\beta f_n\rangle \to \widetilde{\Lambda_0} = \inf_{\substack{||f||_2^2=1 \\ f\in L^*}} \langle f, \mathcal{H}_\beta f\rangle \; a.s.$$

Also, we proved the **lemma**: If $\{f_n\}$is bounded in L^*, then we can find a subsequence $\{f_{n_k}\}$ so that

1) $f_{n_k} \to \widetilde{f}$ in L_2
2) $f'_{n_k} \to \widetilde{f}'$ weak convergence in L^2
3) $f_{n_k} \to \widetilde{f}$ uniformly on compacts
4) $f_{n_k} \to \widetilde{f}$ weakly in L^*

Without loss of generality, we may assume $\{f_n\}$ itself has the above properties.
We will show that $\widetilde{\Lambda_0} = \langle \widetilde{f}, \mathcal{H}_\beta \widetilde{f}\rangle$ $a.s.$ and that $(\widetilde{\Lambda_0}, \widetilde{f})$ is an eigenvalue-eigenfunction pair with the smallest eigenvalue.

Proof. In the previous lecture, we proved $\exists$ a nonnegative random variable $C(B)$:

$$\sup_x \frac{\max(|\bar{B}'(x)|, |B(x) - \bar{B}(x)|)}{\sqrt{\log(2+x)}} < C(B) < \infty.$$

Hence, $\forall \epsilon > 0$, $\exists$ a random variable X, so that $|\bar{B}'(x)| \langle \epsilon(1+x)$ and $|B(x)-\bar{B}(x)| \langle \epsilon\sqrt{1+x}$, $\forall x > X$.
Then, $\forall f_n$,

$$\langle f_n, \mathcal{H}_\beta f_n\rangle = \int_0^\infty [(f'_n)^2 + xf_n^2]dx + \frac{2}{\sqrt{\beta}}\int_0^x f_n^2\bar{B}'dx + \frac{4}{\sqrt{\beta}}\int_0^x f'_n f_n(\bar{B} - B)dx + E \qquad (1)$$

where $|E| \le \epsilon C(\beta)||f_n||_*^2 \le \epsilon C(\beta) K$ $a.s.$, by Cauchy-Schwarz inequality, and $C(\beta)$ is a nonnegative finite constant.
Then, we want to show that

$$\widetilde{\Lambda_0} = \lim_{n=1} \langle f_n, \mathcal{H}_\beta f_n \rangle = \langle \widetilde{f}, \mathcal{H}_\beta \widetilde{f} \rangle \ a.s.$$

The second term in (1),

$$\frac{2}{\sqrt{\beta}} \int_0^x f_n^2 \bar{B}' dx \to \frac{2}{\sqrt{\beta}} \int_0^x \widetilde{f}^2 \bar{B}' dx \quad a.s.$$

because $f_n \to \widetilde{f}$ uniformly on compacts.
We **claim** that the third term in (1),

$$\frac{4}{\sqrt{\beta}} \int_0^x f_n' f_n (\bar{B} - B) dx \to \frac{4}{\sqrt{\beta}} \int_0^x \widetilde{f}' \widetilde{f} (\bar{B} - B) dx \quad a.s.$$

It's enough to show $\int_0^x |f_n f_n' - \widetilde{f} \widetilde{f}'| dx \to 0$ $a.s.$
Here we use triangle inequality. For large enough n, we have

$$\begin{aligned} & \int_0^x |f_n f_n' - \widetilde{f}\widetilde{f}'| dx \\ \le \ & \int_0^x |f_n f_n' - \widetilde{f}' f_n| dx + \int_0^x |\widetilde{f}' f_n - \widetilde{f}\widetilde{f}'| dx \\ \le \ & \int_0^x |f_n||f_n' - \widetilde{f}'| dx + ||\widetilde{f}'||_2 ||\widetilde{f} - f_n||_2 \\ \le \ & \int_0^x (|\widetilde{f}| + \epsilon)|f_n' - \widetilde{f}'| dx + ||\widetilde{f}'||_2 ||\widetilde{f} - f_n||_2 \\ \to \ & 0 \quad a.s. \end{aligned} \tag{2}$$

The last inequality is because of the uniformly convergence of $\{f_n\}$ in L^2.
Since $(|\widetilde{f}| + \epsilon)$ is in L^2 and $f_n' \to \widetilde{f}'$ weak convergence in L^2, the first term in (2) $\to 0$ $a.s.$.
Since $||\widetilde{f}'||_2$ is finite and $f_n \to \widetilde{f}$ in L^2, the second term in (2) $\to 0$ $a.s.$.
Then the **claim** follows.
Finally, we deal with the first term in (1).
By Fatou's lemma,

$$\liminf \int_0^\infty x f_n^2 dx \ge \int_0^\infty x \widetilde{f}^2 dx \ a.s..$$

And

$$||\widetilde{f}'||_2^2 = \int_0^\infty \widetilde{f}' \widetilde{f}' dx = \int_0^\infty f_n' \widetilde{f}' dx \le \liminf ||f_n'||_2 ||\widetilde{f}'||_2 \ a.s.,$$

which implies

$$\liminf \int_0^\infty (f_n')^2 dx = \liminf ||f_n'||_2^2 \ge ||\widetilde{f}'||_2^2 = \int_0^\infty (\widetilde{f}')^2 dx \ a.s..$$

Hence,

$$
\begin{aligned}
\widetilde{\Lambda_0} &= \lim_{n=1} \langle f_n, \mathcal{H}_\beta f_n \rangle \\
&= \lim_{n=1} \left\{ \int_0^\infty [(f_n')^2 + x f_n^2] dx + \frac{2}{\sqrt{\beta}} \int_0^x f_n^2 \bar{B}' dx + \frac{4}{\sqrt{\beta}} \int_0^x f_n' f_n (\bar{B} - B) dx + E \right\} \\
&\geq \int_0^\infty [(\widetilde{f}')^2 + x \widetilde{f}^2] dx + \frac{2}{\sqrt{\beta}} \int_0^x \widetilde{f}^2 \bar{B}' dx + \frac{4}{\sqrt{\beta}} \int_0^x \widetilde{f}' \widetilde{f} (\bar{B} - B) dx - \epsilon C(\beta) K \\
&\geq \langle \widetilde{f}, \mathcal{H}_\beta \widetilde{f} \rangle - 2\epsilon C(\beta) K \ a.s.
\end{aligned}
$$

Since ϵ is arbitrary,

$$\widetilde{\Lambda_0} \geq \langle \widetilde{f}, \mathcal{H}_\beta \widetilde{f} \rangle \ a.s..$$

According to the definition of $\widetilde{\Lambda_0}$,

$$\widetilde{\Lambda_0} = \langle \widetilde{f}, \mathcal{H}_\beta \widetilde{f} \rangle \ a.s..$$

Next, we prove the second part of the statement.
First, let's go back to the matrices example. Assume that

$$\lambda_0 = \inf_{||x||=1} \langle x, Ax \rangle = \min_{||x||=1} = \langle x_0, Ax_0 \rangle$$

Then, for any $\epsilon > 0$

$$
\begin{aligned}
& \frac{1}{||x_0 + \epsilon y||^2} \langle (x_0 + \epsilon y), A(x_0 + \epsilon y) \rangle - \langle x_0, Ax_0 \rangle \\
&= (1 - 2\epsilon \langle x_0, y \rangle + O(\epsilon^2))(\langle x_0, Ax_0 \rangle + 2\epsilon \langle y, Ax_0 \rangle + O(\epsilon^2)) - \langle x_0, Ax_0 \rangle \\
&= 2\epsilon(\langle y, Ax_0 \rangle - \langle x_0, y \rangle \langle x_0, Ax_0 \rangle) + O(\epsilon^2) \\
&= 2\epsilon(\langle y, Ax_0 - \lambda_0 x_0 \rangle) + O(\epsilon^2)
\end{aligned}
$$

Since x_0 is the minimizer, the first term in the above equation must be 0 for all y. That is, $\langle y, Ax_0 - \lambda_0 x_0 \rangle = 0$ for all y. Hence, (λ_0, x_0) is an eigenvalue-eigenvector pair.
Then, we use the same idea for $\widetilde{f}$ and $\mathcal{H}_\beta$. Define that

$$f^{\epsilon,\varphi} = \frac{\widetilde{f} + \epsilon \varphi}{||\widetilde{f} + \epsilon \varphi||_2}$$

Then,

$$
\begin{aligned}
& \langle f^{\epsilon,\varphi} - \mathcal{H}_\beta \widetilde{f^{\epsilon,\varphi}} \rangle - \langle \widetilde{f}, \mathcal{H}_\beta \widetilde{f} \rangle \\
&= 2\epsilon \left[-\langle \widetilde{f}, \mathcal{H}_\beta \widetilde{f} \rangle \int_0^\infty \widetilde{f} \varphi dx + \int_0^\infty (\widetilde{f}' \varphi' + x \widetilde{f} \varphi) dx - \frac{2}{\sqrt{\beta}} \int_0^\infty \bar{B}' \widetilde{f} \varphi dx \right. \\
& \left. - \frac{2}{\sqrt{\beta}} \int_0^\infty (\bar{B} - B)(\widetilde{f}\varphi)' dx) \right] + O(\epsilon^2) \\
&= 2\epsilon [\langle \varphi, \mathcal{H}_\beta \widetilde{f} - \widetilde{\Lambda_0} \widetilde{f} \rangle] + O(\epsilon^2)
\end{aligned}
$$

Use the same argument that we used in matrices example, $(\widetilde{\Lambda_0}, \widetilde{f})$ is an eigenvalue-eigenfunction pair for $\mathcal{H}_\beta$. And we also proved the $\widetilde{\Lambda_0}$ is the smallest eigenvalue. □

Define $\mathcal{H}_0$ as the function space of all the eigenfunction corresponding to $\widetilde{\Lambda_0}$.
Claim $\mathcal{H}_0$ is finite dimensional.

Proof. Suppose $\exists$ countable infinite $f_1, f_2, \cdots$, which form an orthonormal(L_2) basis in $\mathcal{H}_0$. Then we can find a random variable K so that $||f_i||_*^2 \langle K$ $a.s., \forall i \in \mathbf{N}$. By the lemma in the previous lecture, $\exists$ a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that $f_{n_k} \stackrel{L_2}{\to} f$ and $f \in \mathcal{H}_0$. Then $\forall n$, we can find k such that $n_k \geq n$. Since $f_n \perp \{f_{n_k}, f_{n_{(k+1)}}, \cdots\}$, we get $f_n \perp f$, which is a contradiction to the fact that $\{f_1, f_2, \cdots\}$ is an orthonormal basis. Hence, $\mathcal{H}_0$ is finite dimensional. $\square$

We can define (find) the other eigenvalues inductively:$\widetilde{\Lambda_0} \leq \widetilde{\Lambda_1} \leq \widetilde{\Lambda_2} \leq \cdots$.

1.2 Counting function of eigenvalues

First let's look at a much simpler problem:

$$\begin{cases} -\partial_{XX} f = \lambda f \\ f(0) = 0, f(L) = 0 \end{cases}$$

We have

$$f_{k-1}(x) = \sin(k\frac{\pi}{L}x), \quad k = 1, 2 \cdots$$

$$\lambda_{(k-1)} = (k\frac{\pi}{L})^2$$

Then, k^{th} eigenfunction has k roots in $(0, L)$.
For any λ:

$$\begin{cases} f_\lambda : -\partial_{XX} f = \lambda f \\ f(0) = 0 \end{cases}$$

λ is an eigenvalue $\Leftrightarrow f_\lambda(L) = 0$.
Then, f_λ has k roots in $(0, L) \Leftrightarrow \lambda_k \geq \lambda \geq \lambda_{k-1}$.
This is true in general on $[0, L]$:

$$\begin{cases} -\partial_{XX} f + Vf = \lambda f \\ f(0) = 0 \end{cases}$$

And for such kind problem, we can use the Ricalti transformation:

$$p = \frac{f'}{f} \Rightarrow p' = -(\frac{f'}{f})^2 + \frac{f''}{f} = p^2 + \frac{f''}{f}$$

Then,

$$\begin{cases} -\partial_{XX} f + Vf = \lambda f \\ f(0) = 0 \end{cases} \Leftrightarrow \begin{cases} p' = -\lambda + V - p^2 \\ p(0) = \infty \end{cases}$$

# of explosions= # of eigenvalues below λ.
This is same on $[0,\infty)$:

$$\left\{\begin{array}{c}\mathcal{H}_\beta f=\lambda f\\ f(0)=0\end{array}\right.\Leftrightarrow\left\{\begin{array}{c}-\partial_{XX}f+Vf=\lambda f\\ f(0)=0\end{array}\right.$$

where $V=x+\frac{2}{\sqrt{\beta}}B'(x)$.
Let $p=f'/f$, then

$$\begin{aligned}
&p'(x)=\frac{2}{\sqrt{\beta}}B'(x)+x-\lambda-p^2(x)\\
\Leftrightarrow\quad &dp=\frac{2}{\sqrt{\beta}}dB+(x-\lambda-p^2)dx\\
\Leftrightarrow\quad &p(x_1)-p(x_0)=\frac{2}{\sqrt{\beta}}(B(x_1)-B(x_0))+\int_{x_0}^{x_1}[x-\lambda-p^2(x)]dx
\end{aligned}$$

The last equation is a diffusion model which has a general form

$$dX_t=\sigma(X_t,t)dB_t+f(X_t,t)dt.$$

The discrete analogue of the diffusion model is

$$X_{t+\epsilon}=X_t+\eta_t$$

where η_t is asymptotically normal with expectation $f(X_t,t)\cdot\epsilon$ and variance $\sigma(X_t,t)^2\cdot\epsilon$.

If σ and f are appropriate, it's solvable. In our case, for the equation

$$\mathcal{H}_\beta f=\lambda f,\quad f(0)=0,$$

if the solution f decays fast enough, it can be proved that (λ,f) is an ev-ef pair.

1.3 Limit of the matrix form

Recall that

$$M_n^\beta=\begin{bmatrix}a_n & b_{n-1} & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & \\ & b_{n-2} & \ddots & \ddots & \\ & & \ddots & \ddots & b_1\\ & & & b_1 & a_1\end{bmatrix},\qquad H_n^\beta=n^{\frac{1}{6}}(2\sqrt{n}I_n-M_n)$$

where $a_k \sim \frac{1}{\sqrt{\beta}} N(0,2)$ and $b_k \sim \frac{1}{\sqrt{\beta}} \chi_{\beta k}$, and

$$H_n^\beta = n^{\frac{2}{3}} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} + n^{\frac{1}{6}} \begin{bmatrix} 0 & \gamma_1 & & & \\ \gamma_1 & 0 & \gamma_2 & & \\ & \gamma_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \gamma_{n-1} \\ & & & \gamma_{n-1} & 0 \end{bmatrix}$$
$$+ \frac{n^{\frac{1}{6}}}{\sqrt{2\beta}} \begin{bmatrix} 2\rho_0 & \zeta_1 & & & \\ \zeta_1 & 2\rho_1 & \zeta_2 & & \\ & \zeta_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \zeta_{n-1} \\ & & & \zeta_{n-1} & 2\rho_{n-1} \end{bmatrix}$$

where $\gamma_k = \sqrt{n} - E\chi_{\beta(n-k)} \approx \frac{k}{2\sqrt{n}}$.

We know that

$$w_{n,1}(x) = n^{-\frac{1}{6}} \sum_{i=1}^{\lfloor x \cdot n^{1/3} \rfloor} \rho_i \Rightarrow B_1(x)$$
$$w_{n,2}(x) = n^{-\frac{1}{6}} \sum_{i=1}^{\lfloor x \cdot n^{1/3} \rfloor} \zeta_i \Rightarrow B_2(x)$$
$$2n^{-\frac{1}{6}} \sum_{i=1}^{\lfloor x \cdot n^{1/3} \rfloor} \gamma_i \to \frac{x^2}{2}$$

Define $||v||_{2,n}^2 = n^{-\frac{1}{3}} \sum v_k^2$ and

$$\langle v, H_n^\beta v \rangle_{2,n} = n^{\frac{1}{3}} \sum (v_{k+1} - v_k)^2 + 2n^{-\frac{1}{6}} \sum \gamma_k v_k v_{k+1} + n^{-\frac{1}{6}} \sqrt{\frac{2}{\beta}} \sum \rho_k v_k^2$$
$$+ n^{-\frac{1}{6}} \sqrt{\frac{2}{\beta}} \sum \zeta_k v_k v_{k+1}$$

which are scaled version of L_2 norm and inner product of vectors, and

$$||v||_*^2 = n^{\frac{1}{3}} \sum (v_{k+1} - v_k)^2 + n^{-\frac{2}{3}} \sum k v_k^2 + n^{-\frac{1}{3}} \sum v_k^2$$
$$= n^{-\frac{1}{3}} \sum \left(\frac{v_{k+1} - v_k}{n^{-1/3}} \right)^2 + n^{-\frac{1}{3}} \sum (k \cdot n^{-\frac{1}{3}}) v_k^2 + n^{-\frac{1}{3}} \sum v_k^2$$

which is a discrete version of L^* norm if we view v_k as the mesh sampled by size of $n^{-\frac{1}{3}}$.

Lemma 1. *There exists a sequence of random constants $C_1(n), C_2(n), C_3(n)$ such that*

$$C_1 ||v||_*^2 - C_2 ||v||_2^2 \le \langle v, H_n^\beta v \rangle \le C_3 ||v||_*^2.$$

Also $C_1(n), C_2(n), C_3(n)$ are tight.

Define $L_{2,n} = \{f \in L_2 : \text{step functions with mesh } n^{-\frac{1}{3}}, f(0) = 0, f(x) = 0, x \geq n^{\frac{2}{3}}\}$. For each $f \in L_{2,n}$, it can be treated as a vector of length n. Let $P_n : L_2 \to L_{2,n}$ be the projection from L_2 to $L_{2,n}$, then we can define

$$\langle f, \hat{H}_n^\beta f \rangle_{L_2} = \langle P_n f, H_n^\beta P_n f \rangle_{2,n}.$$

For each n, let λ_n denote the minimum eigenvalue of $\hat{H}_n^\beta$ and f_n denote the corresponding eigen function satisfying $||f_n|| = 1$. We have that $\lambda_n \to \lambda$ (maybe subsequence) and $f_n \xrightarrow{L_2} f$.

Lemma 2.

1. *If $f_n \in L_{2,n}$ with $f_n \to f$ in L_2 weakly, $n^{\frac{1}{3}}(f_n(x + n^{-\frac{1}{3}}) - f_n(x)) \to f'$ in L_2 weakly, then for each $\varphi \in C_0^\infty$,*

$$\langle \varphi, \hat{H}_n^\beta f_n \rangle \to \langle \varphi, \mathcal{H}_\beta f \rangle$$

   Notice that the left term is a partial summation.

2. *If $f_n \in L_{2,n}$ is bounded in $L_{*,n}$, then there exists a subsequence $f_{n_k} \to f$ in L_2 such that*

$$\langle \varphi, \hat{H}_n^\beta f_n \rangle \to \langle \varphi, \mathcal{H}_\beta f \rangle$$

Therefore, for all $\varphi \in C_0^\infty$,

$$\begin{aligned} \lambda \langle \varphi, f \rangle &= \lim \langle \varphi, \lambda_n f_n \rangle \\ &= \lim \langle \varphi, \hat{H}_n^\beta f_n \rangle \\ &= \langle \varphi, \mathcal{H}_\beta f \rangle \end{aligned}$$

which implies (λ, f) is an ev-ef pair of $\mathcal{H}_\beta$.

Next, we want to show that λ is the smallest eigenvalue of $\mathcal{H}_\beta$. Let Λ_0 be the smallest eigenvalue, so there exists minimizing function f_0 for Λ_0. By definition the discretized version

$$\lambda_n \leq \frac{\lim \langle f_{0,n}, \hat{H}_n^\beta f_{0,n} \rangle}{\lim \langle f_{0,n}, f_{0,n} \rangle} \to \Lambda_0$$

where the left hand side $\lambda_n \to \lambda$. So it implies that $\lambda \leq \Lambda_0$, and together with the fact that $\lambda \geq \Lambda_0$,

$$\Rightarrow \lambda = \Lambda_0.$$

We can use the induction to deal with other eigenvalues, then

$$(\lambda_0^{(n)}, \lambda_1^{(n)}, \lambda_2^{(n)}, \cdots) \Rightarrow (\Lambda_0, \Lambda_1, \Lambda_2, \cdots)$$

Remark: we can also obtain the edge scaling of the Laguerre ensemble $M = AA^T \overset{d}{=} \frac{1}{\beta} BB^T$, where B is a bidiagonal matrix in the form

$$B = \begin{bmatrix} \chi_{\beta m} & & \\ \chi_{\beta(n-1)} & \chi_{\beta(m-1)} & \\ & \ddots & \ddots \end{bmatrix}$$

2 Description of the limiting process

Recall that the density of eigenvalues are given by

$$P_{\beta,n}(\lambda_1, \cdots, \lambda_n) = \frac{1}{Z_{n,\beta}} \prod_{i>j} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta}{4}\sum \lambda_i^2}, \quad \lambda_1 < \lambda_2 < \cdots < \lambda_n.$$

2.1 Edge scaling limit

By Tracy-Widom, in the cases $\beta = 1, 2, 4$, $n^{\frac{1}{6}}(\lambda_n - 2\sqrt{n}) \to TW_\beta = -\Lambda_0$.

For any $a > 0$,

$$P(n^{\frac{1}{6}}(\lambda_n - 2\sqrt{n}) \geq a) = \exp(-\frac{2}{3}\beta a^{\frac{3}{2}}(1 + o(1))) \tag{3}$$

$$P(n^{\frac{1}{6}}(\lambda_n - 2\sqrt{n}) \leq -a) = \exp(-\frac{1}{24}\beta a^3(1 + o(1))). \tag{4}$$

For $\beta = 1, 2, 4$, one can prove this using the explicit formulas for the limits. By Ramirez-Rode-Virag, these asymptotic will hold for all $\beta > 0$.

$$P(TW_\beta > a) = P(\Lambda_0 < -a)$$

where

$$\begin{aligned}\Lambda_0 &= \inf_{\substack{f \in L^* \\ ||f||_2^2 = 1}} \left\{ \int_0^\infty (f'^2 + xf^2)dx + \frac{2}{\sqrt{\beta}} \int_0^\infty f^2 dB \right\} \\ &\leq \frac{1}{||g||_2^2} \left\{ \int_0^\infty (g'^2 + xg^2)dx + \frac{2}{\sqrt{\beta}} \int_0^\infty g^2 dB \right\} \qquad \forall g \in L^*\end{aligned}$$

in which $\frac{2}{\sqrt{\beta}} \int_0^\infty g^2 dB = N(0,1) \times \frac{2}{\sqrt{\beta}} \left(\int_0^\infty g^4 dx\right)^{\frac{1}{2}}$. Therefore,

$$\begin{aligned}P(\Lambda_0 < -a) &\geq P\left(\frac{2}{\sqrt{\beta}} \left(\int_0^\infty g^4 dx \right)^{\frac{1}{2}} \times N(0,1) < -a \int g^2 dx - \int (g'^2 + xg^2)dx \right) \\ &= \Phi\left(\frac{-a \int g^2 dx - \int (g'^2 + xg^2)dx}{\frac{2}{\sqrt{\beta}} \left(\int_0^\infty g^4 dx\right)^{\frac{1}{2}}} \right)\end{aligned}$$

Since it holds for any $g \in L^*$, we can choose a particular function $g = \text{sech}(\sqrt(a)(x-1))$ to obtain the lower bound for up tail in (3). Similarly

$$P(TW_\beta < -a) \leq P(\langle g, \mathcal{H}_\beta g \rangle > a||g||_2^2)$$

With a suitable test function, it gives you the upper bound in (4). And the other two inequality can be obtained by diffusion description.

2.2 Bulk scaling limit of the β-ensemble

We have showed that, for $|c| < 2$,

$$2\pi\sqrt{n}\sigma(c)(\Lambda_n - 2\sqrt{n}) \Rightarrow \text{some point process limit}, \qquad \sigma(c) = \frac{1}{2\pi}\sqrt{4-c^2}$$

By V-Viray, for each $\beta > 0$, there is a limiting point process(Sine_β) which does not depend on c.

How can we describe the process Sine_β?
We will describe the counting function $N(\lambda)$ of Sine_β.

$$N(\lambda) = \begin{cases} 0 & \lambda = 0 \\ \#\{\text{points in } [0,\lambda)\} & \lambda > 0 \\ \#\{\text{points in } [-\lambda, 0)\} & \lambda < 0 \end{cases}$$

For any $a < b$, # of points in $[a,b) = N(b) - N(a) \overset{d}{=} N(b-a)$ which means it is translation invariant.

Next we will introduce two equivalent description of the process.

2.2.1 First description

Let $Z(t)$ be complex BM. $Z(t) = B_1(t) + iB_2(t)$ where B_1, B_2 are independent standard BM.
Consider the following family of stochastic differential equations.

$$\alpha_\lambda(t), \quad \lambda \in \mathbb{R}, \quad t \in [0,\infty)$$

$$d\alpha_\lambda(t) = \lambda\frac{\beta}{4}e^{-\frac{\beta}{4}t}dt + Re\left(\left(e^{-i\alpha_\lambda} - 1\right)dZ\right), \quad \alpha_\lambda(0) = 0$$

where $Re\left(\left(e^{-i\alpha_\lambda} - 1\right)dZ\right) = (\cos\alpha_\lambda - 1)dB_1 + \sin\alpha_\lambda dB_2$.

$$\alpha_\lambda(t) = \int_0^t \lambda\frac{\beta}{4}e^{-\frac{\beta}{4}s}ds + \int_0^t Re\left(\left(e^{-i\alpha_\lambda(s)} - 1\right)dZ\right)$$

Claim:

1. This will have a solution.
2. $\frac{1}{2\pi}\lim\limits_{t\to\infty}\alpha_\lambda(t)$ exists a.s. and it's an integer. Define

$$N(\lambda) = \frac{1}{2\pi}\lim_{t\to\infty}\alpha_\lambda(t).$$

3. $N(\lambda)$ is increasing in λ. $N(\lambda)$ is an integer-valued step function.
4. $N(\lambda)$ is the counting function of Sine_β.

2.2.2 Second description

In this section, we need some knowledge about 2D hyperbolic space. Specifically, we will use the Poincaré disk model on $\mathbb{U} = \{|z| < 1\}$.

lines:	1) lines passing 0
	2) circles $\perp \{|z| = 1\}$
angles:	same as in the euclidean plane
distance:	$4\frac{1}{1-|z|^2}$

It's confirmal invariant. For $|\alpha| < 1$, the transformation $T(z) = e^{iv}\frac{z-\alpha}{1-\bar{\alpha}z}$ keep distance, angle, $\cdots$

Rotation around a point.

- rotation around 0: $z \to e^{iv} \cdot z$
- rotation around $\alpha(|\alpha| < 1)$: first shift $T(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$, then rotation and transform back

$$z \to T^{-1}(e^{iv} \cdot T(z))$$

BM in the hyperbolic plane - scaling limit of a simple random walk in the hyperbolic plane.

$$dB = \frac{1-|B|^2}{2}dZ$$

where $B(t)$ is hyperbolic BM and $Z(t)$ is complex BM.

For each λ, we will follow a point $Z_\lambda(t) \in \mathbb{U}$ on the boundary $z_\lambda(0) = 1$. We rotate (in hyp) $Z_\lambda(t)$ about the point B(t) with speed $\lambda\frac{\beta}{4}e^{-\frac{\beta}{4}t}$, and

$$\frac{\partial Z_\lambda(t)}{\partial t} = i\lambda\frac{\beta}{4}e^{-\frac{\beta}{4}t}\frac{|z_\lambda - B|^2}{1-|B|^2}$$

For each λ, we also follow the winding angle of z_λ about $B(t)$.

$$\alpha_\lambda(t) = \langle 1, B(t), Z(t) \rangle$$

Define $N(\lambda)$ as the total winding number of $Z_\lambda(t)$.

Claim:

1. $N(\lambda)$ is increasing.
2. $N(\lambda)$ is the counting function of Sine_β.