Lectures 18 - 19 : Edge Scaling of the General β-Ensemble

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1 A Review of Several Concepts

1.1 Brownian Motion

Definition 1 (Brownian Motion)**.** The process $x \mapsto B(x)$ with $B(0) = 0$ is a Brownian Motion if for $0 < t_1 < t < 2 < ... < t_n$ we have that $B(t_1), B(t_2) - B(t_1), ..., B(t_n) - B(t_{n-1})$ are independent, mean zero normal random variables with variance $t_1, t_2 - t_1, ..., t_n - t_{n-1}$ respectively.

Theorem 2. *Brownian Motion exists.*

Theorem 3. *Brownian Motion is almost surely nowhere differentiable.*

1.2 Distributions (generalized functions)

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a 'nice' function, then we can define an associated linear functional. For φ a smooth compactly supported function we have that $\varphi \mapsto \int_{\mathbb{R}} f(x)\varphi(x)dx$. Other examples of linear functionals include $\delta_0 : \varphi \to \varphi(0)$.

Suppose $T : \varphi \to T\varphi$ is a linear functional, then we can define the derivative of T by $T' : \varphi \to -T\varphi'$. In the case of a linear functional associated to f we have $\int f'(x)\varphi(x) = -\int f(x)\varphi'(x)dx$, the function $g(x)$ that satisfies $\int g(x)\varphi(x)dx = -\int f(x)\varphi'(x)dx$ for all smooth compactly supported φ is called the weak derivative of f.

1.3 White Noise and Scaling Limits

Definition 4. Formally White Noise is defined as the derivative of Brownian Motion $B'(x)$. This is defined in the weak sense since Brownian Motion is nowhere differentiable.

$B'(x)$ is a random distribution (generalized function)

$$\varphi \mapsto \int \varphi(x)B'(x)dx = \int \varphi(x)dB(x) \sim N(0, \textstyle\int \varphi^2(x)dx)$$

Suppose that $\tau_1, \tau_2, ...$ are i.i.d. random variables with $P(\tau = \pm 1) = 1/2$ then the Central Limit theorem tells us that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tau_i \Rightarrow N(0,1).$$

Now take $S_\epsilon(t) = \sqrt{\epsilon} \sum_{\epsilon i \leq 1} \tau_i$ then if we let $\epsilon \to 0$ we have that $S_\epsilon(t) \Rightarrow B(t)$ where $B(t)$ is Brownian Motion. In other word with the appropriate scaling if we look at a random walk from far away we can see Brownian Motion.

To see White Noise in the limit let φ be a smooth compactly supported function then

$$\sqrt{\epsilon} \sum \varphi(k\epsilon)\tau_k \Rightarrow N(0, \textstyle\int \varphi^2).$$

Another possible description of White Noise is as follows: Let $\{\varphi_1, \varphi_2, \varphi_3, ...\}$ be and orthonormal basis of $L^2(\mathbb{R})$ and choose and i.i.d. sequence of standard normals $N(0,1)$, $\xi_1, \xi_2, \xi_3 ...$, then we define $\int \varphi_i dB = \xi_i$. For $f \in L^2(\mathbb{R})$ there exist constants $c_1, c_2, c_3, ...$ such that

$$f(x) = \sum_{i=1}^{\infty} c_i \varphi_i \qquad \text{and} \qquad \sum_{i=1}^{\infty} c_i^2 < \infty.$$

This gives us that $\int f(x) dB(x) = \sum_{i=1}^{\infty} c_i \xi_i$. To get Brownian Motion from White Noise take $B(x) = \int 1[0,x] dB$.

2 Edelman-Sutton Conjecture (cont.)

Recall that for the general β-ensemble the joint intensity of the eigenvalues $\Lambda_n = \{\lambda_1, \ldots \lambda_n\}$ is the same as for the matrix

$$M_n = \begin{bmatrix} a_n & b_{n-1} & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & \\ & b_{n-2} & \ddots & \ddots & \\ & & \ddots & \ddots & b_1 \\ & & & b_1 & a_1 \end{bmatrix}_{n \times n} \tag{1}$$

where $a_k \sim \frac{\sqrt{2}}{\beta} N(0,1)$ and $b_k \sim \frac{1}{\sqrt{\beta}} \mathcal{X}_{\beta k}$. The Edelman-Sutton conjecture states that the limiting point process of $n^{1/6}(2\sqrt{n} - \Lambda_n)$ is given by

$$-\partial_{xx} + x + \frac{2}{\sqrt{\beta}} B'(x).$$

Recall that the idea behind this conjecture is the following: Consider the tridiagonal matrix $n^{\frac{1}{6}}(2\sqrt{n} I_n - M_n)$:
Diagonal entries: $n^{\frac{1}{6}}(2\sqrt{n} - \sqrt{\frac{2}{\beta}} \rho_i)$

Off-diagonal entries: $-n^{\frac{1}{6}}\sqrt{n}+\frac{k}{2\sqrt{n}}n^{\frac{1}{6}}+n^{\frac{1}{6}}\frac{1}{\sqrt{2\beta}}\zeta_k$
These terms can be grouped to yield the equation

$$n^{\frac{1}{6}}(2\sqrt{n}I_n - M_n) = n^{\frac{2}{3}}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} + \frac{n^{\frac{-1}{3}}}{2}\begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 2 & & \\ & 2 & \ddots & \ddots & \\ & & \ddots & \ddots & n-1 \\ & & & n-1 & 0 \end{bmatrix}$$
$$+\frac{n^{\frac{1}{6}}}{\sqrt{2\beta}}\begin{bmatrix} 2\rho_0 & \zeta_1 & & & \\ \zeta_1 & 2\rho_1 & \zeta_2 & & \\ & \zeta_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \zeta_{n-1} \\ & & & \zeta_{n-1} & 2\rho_{n-1} \end{bmatrix}.$$

where ζ_i, ρ_i are standard normals. These matrices act on vectors $v = [v_1, v_2, ..., v_n]^T$ where $v_k = f(kn^{-1/3})$ in other words v is the discretization of a function f with mesh size $n^{-1/3}$. In this setting the first matrix corresponds roughly to twice differentiating, the second multiplying by x, and the last to the white noise term.

Now that we understand the general idea behind the conjecture we wish to make the proof more rigorous. The proof is due to Ramirez, Rider, and Virag.

Take

$$\mathcal{H}_\beta = -\partial_{xx} + x + \frac{2}{\sqrt{\beta}}B'(x).$$

Assume that $f \in \mathcal{H}^1_{loc}$, $f : \mathbb{R}^+ \to \mathbb{R}$ and $f'1_I \in L^2(\mathbb{R}^+)$ for every compact interval I. Let $\mathcal{D}$ be the space of distributions, then $\mathcal{H}_\beta$ is a random linear operator $H^1_{loc} \to D$, defined by

$$\mathcal{H}_\beta f = -f''(x) + xf(x) + \frac{2}{\sqrt{\beta}}f(x)B'(x).$$

For the last term we have that

$$\int_0^y f(x)B'(x)dx = -\int_0^y B(x)f'(x)dx + f(y)B(y)$$

and this is a continuous function in y. We define $f(x)B'(x)$ as the generalized derivative of

$$\begin{aligned}\langle \varphi, \mathcal{H}_\beta f\rangle &= -\int_0^\infty \varphi''(x)f(x)dx + \int_0^\infty xf(x)\varphi(x)dx + \frac{2}{\sqrt{\beta}}\int_0^\infty \varphi(x)f(x)B'(x)dx \\ &= -\int_0^\infty \varphi''(x)f(x)dx + \int_0^\infty xf(x)\varphi(x)dx - \frac{2}{\sqrt{\beta}}\int_0^\infty \varphi'(x)f(x)B(x) + \varphi(x)f'(x)B(x)dx\end{aligned}$$

2.1 Finding the Eigenvalues

Now that the operator is defined how do we find its eigenvalues? We say that (λ, f) is an eigenvalue-eigenfunction pair for $\mathcal{H}_\beta$ if for all $\varphi \in C_0^\infty$ we have that

$$\begin{aligned}-\lambda \int_0^\infty f(x)\varphi(x)dx &= \int_0^\infty [\varphi''(x)f(x) - x\varphi(x)f(x)]dx - \frac{2}{\sqrt{\beta}}\int_0^\infty \varphi'(x)f(x)B(x) + \varphi(x)f'(x)B(x)dx \\ &\quad -f'(0)\varphi(0)\end{aligned}$$

where f has the boundary conditions $f(0) = 0$, $\int f^2(x)dx = 1$ and $f \in \mathcal{H}^1_{loc} \cap L^2$.

To find the eigenvalue-eigenfunction pairs we use a technique from linear algebra. For A a symmetric matrix we have that the minimal eigenvalue is given by $\lambda_{\min} = \min_{|x|=1}\langle x, Ax\rangle$. We consider

$$\langle f, \mathcal{H}_\beta f\rangle = \int_0^\infty f'(x)f'(x)dx + \int_0^\infty xf^2(x)dx + \frac{2}{\sqrt{\beta}}\int_0^\infty f^2(x)dB(x).$$

Define L^* to be the space of functions with the following conditions,

$$L^* = \{f(x) : f(0) = 0, \int_0^\infty (f'(x))^2 + (x+1)f^2(x)dx < \infty\}$$

with the norm on L^* defined to be

$$\|f\|^2 = \int (f'(x))^2 + (1+x)f^2(x)dx$$

Note that the $(x+1)$ term gives us that $\int f^2(x)dx < \infty$ and so $f \in L^2$. We would like to show that

$$\widetilde{\Lambda}_0 = \inf_{f \in L^*, \|f\|_2 = 1} \langle f, \mathcal{H}_\beta f\rangle \text{ is finite a.s..}$$

That is, there exists a minimal eigenvalue, and its corresponding eigenfunction is in L^*. We would like to show that the quadratic form $\langle f, \mathcal{H}f\rangle$ is nice enough by bounding $\int f^2(x)dB(x)$ with $\int (f'(x))^2 + xf^2(x)dx$.

One possible attempt would be the following: we have that

$$\begin{aligned}\int_0^\infty f^2(x)dB(x) &= -2\int_0^\infty f(x)f'(x)B(x)dx \qquad \text{and} \\ \left|\int_0^\infty f(x)f'(x)B(x)dx\right|^2 &\le \int_0^\infty (f'(x))^2 + B^2(x)f^2(x)dx.\end{aligned}$$

But we know that $B(x)/\sqrt{x} =^d B(1)$ so this bound is not good enough.

Instead note that if we have $\overline{B}(x) = \int_x^{x+1} B(y)dy$, then

$$\begin{aligned}B(x) &= \overline{B}(x) + B(x) - \overline{B}(x) \\ &= \int_x^{x+1} B(y)dy - \int_x^{x+1} (B(y) - B(x))dy\end{aligned}$$

In order to bound these terms we want to consider the following

Claim:

$$\sup_x \sup_{0\le y\le 1} \frac{|B(x+y)-B(x)|}{\log(2+x)} < \infty \tag{2}$$

This is a random variable which is almost surely finite. By considering this random variable we can bound the local fluctuations instead of the global ones.

Proof. (of (2)). Let

$$X_n = \sup_{0\le y\le 1} |B(n+y)-B(n)|\,,$$

then it is enough to show that

$$\sup_n \frac{X_n}{\sqrt{\log(n+1)}} < \infty \qquad a.s.$$

We can show that this is sufficient by considering finitely many cases. For example, one case is $n < x < x+y < n+1$. For this we use the triangle inequality to get

$$|B(x+y)-B(x)| \le |B(x)-B(n)| + |B(x+y)-B(n)|$$

both of which can be written in the form $B(n+y)-B(n)$. Similar equalities can be found for all possible cases of x, y.

Now we show that

$$\sup_n \frac{X_n}{\sqrt{\log(n+1)}} < \infty \qquad a.s$$

First note that the X_n are i.i.d. random variables, so if we can fine C with

$$\sum_{n=2}^{\infty} P(X_n > C\sqrt{\log(n)}) < \infty$$

then the Borel-Cantelli lemma would imply that there are finitely many $X_n > C\sqrt{\log(n)}$. Therefore to find $\sup_n X_n/\sqrt{\log(n+1)}$ we need only take the supremum over finitely many terms which will be finite.

To find C note that we find that

$$P(\sup_{0\le x\le 1} |B(x)| < a) \le 2P(|B(1)| > a$$

by using the reflection principle. Some computation gives us that

$$\begin{aligned} P(\sup_{0\le x\le 1} |B(x)| < a) &\le \frac{2}{a} e^{-\frac{a^2}{2}} \qquad \text{and} \\ P(X_n \ge C\sqrt{\log(n)}) &\le \frac{1}{2C\sqrt{\log n}} e^{-\frac{C^2 \log n}{2}} \\ &= \frac{1}{2C\sqrt{\log n}} n^{-\frac{C^2}{2}} \end{aligned}$$

So if we take $C = 2$ then we have that $\sum_{n=2}^{\infty} P(X_n > 2\sqrt{\log n}) = \sum_{n=2}^{\infty} \frac{1}{4\sqrt{\log n}} n^{-2} < \infty$. Therefore the Borel-Cantelli lemma implies that $X_n \le 2\sqrt{\log n}$ for all except finitely many n, and so $\sup_n X_n/\sqrt{\log(n+1)} < \infty$ a.s.

□

Note that since $\overline{B}'(x) = B(x+1) - B(x)$ and $|B(x) - \overline{B}(x)| \le \sup_{y \in [0,1]} |B(x+y) - B(x)|$ the previous claim gives us that

$$\sup_x \frac{\max(|\overline{B}'(x)|, |B(x) - \overline{B}(x)|)}{\sqrt{\log(2+x)}} < C(B) < \infty \tag{3}$$

Now to bound the $\int f^2 dB$ term we write it as

$$\int_0^\infty f^2(x) dB(x) \quad = \quad \int_0^\infty f^2 \overline{B}'(x) dx + 2 \int_0^\infty f(x) f'(x) (\overline{B}(x) - B(x)) dx \tag{4}$$

by rewriting $B(x)$ and using integration by parts.

Lemma 5.

$$\left| \frac{2}{\beta} \int_0^\infty f^2(x) dB(x) \right| \le \frac{1}{2} \left[\int_0^\infty ((f'(x))^2 + x f^2(x) dx \right] + C(B) \int_0^\infty f^2(x) dx.$$

Proof. Using the inequality in (3) we have that for any $\epsilon > 0$ there is a random $C(B, \epsilon) < \infty$ a.s. such that $\max(|\overline{B}'(x)|, |B(x) - \overline{B}(x)|) < \sqrt{\epsilon(C + x)}$. Therefore we have the following bounds on the right hand side terms of (4).

$$\left| \int_0^\infty f^2(x) \overline{B}'(x) dx \right| \le \epsilon \int_0^\infty (C + x) f^2(x) dx \qquad \text{and since } ab \le 1/2(a^2 + b^2)$$

$$\begin{aligned} \left| \int_0^\infty f(x) f'(x) (\overline{B}(x) - B(x)) dx \right| &\le \frac{1}{2C^2} \int_0^\infty (f'(x))^2 dx + \frac{C^2}{2} \int_0^\infty f^2(x) (\overline{B}(x) - B(x))^2 dx \\ &\le \frac{1}{2C^2} \int_0^\infty (f'(x))^2 dx + \frac{C^2 \epsilon}{2} \int_0^\infty f^2(x) (C + x)^2 dx \end{aligned}$$

So,

$$\left| \frac{2}{\beta} \int_0^\infty f^2(x) dB(x) \right| \le \frac{1}{2} \left[\int_0^\infty ((f'(x))^2 + x f^2(x) dx \right] + C(B) \int_0^\infty f^2(x) dx.$$

□

This lemma gives us that

$$\langle f, \mathcal{H} f \rangle = \int_0^\infty f'(x) f'(x) dx + \int_0^\infty x f^2(x) dx + \frac{2}{\sqrt{\beta}} \int_0^\infty f^2(x) dB(x) \le C \|f\|_*^2$$

In total we have that

$$\frac{1}{2}\|f\|_*^2 - c\|f\|_2^2 \leq \langle f, \mathcal{H}f\rangle \leq 2\|f\|_*^2 - C\|f\|_2^2$$

This bound on $\langle f, \mathcal{H}f\rangle$ is helpful, because if we take $\widetilde{\Delta}_0 = \inf_{\|f\|_2=1, f\in L^*}\langle f, \mathcal{H}f\rangle$ this is a.s. finite by the previous inequality, and there exists a sequence $\{f_n\}$ with $\|f_n\|_2 = 1$ and $\|f\|_* < K$ where K is a random upper bound, such that $\langle f_n, \mathcal{H}f_n\rangle \to \widetilde{\Delta}_0$. Note that the upper bound K comes from the above inequality which gives us that $\|f_n\|_*^2 \leq 2\langle f_n, \mathcal{H}f_n\rangle + 2C\|f\|_2^2$.

Lemma 6. *Suppose that $\{f_n\}$ is bounded in L^*, then we can find a subsequence so that,*

1. *$f_n \to f$ in L^2*
2. *$f_n' \to f'$ weakly in L^2*
3. *$f_n \to f$ uniformly on compact subsets.*
4. *$f_n \to f$ weakly in L^*.*

Proof. First notice that (2) and (4) follow as a direct consequence of the Banach-Alaoglu theorem which states that the closed unit ball in the dual space of a normed vector space is compact in the weak* topology. Now to prove (3) first note that if $f \in L^*$ then we have that

$$\begin{aligned} |f(x) - f(y)| &\leq \int_x^y |f'(t)|dt \\ &\leq \sqrt{x-y}\left[\int_x^y \left(f'(t)\right)^2 dt\right]^{1/2} \\ &\leq \sqrt{x-y}\|f\|_* \end{aligned}$$

so if $0 < x < M$ then $|f_n(x)| < \sqrt{M}K$ which means that the f_n are equicontinuous in $[0, M]$. Therefore there exists a subsequence $\{f_{n_k}\}$ along which f_n converges uniformly to f on $[0, M]$. We can then find a subsequence of $\{f_{n_k}\}$ along which f_n converges uniformly on $[0, 2M]$. A diagonalization argument using this process yields a subsequence of f_n which converges uniformly on all compact subsets.

Lastly, to prove (1) notice that $\sup_n \int (x+1)f_n^2(x)dx < K$ for some K which means that f_n^2 is uniformly integrable. Therefore we have that

$$\int_a^\infty f_n^2(x)dx \leq \int_a^\infty \left(\frac{x}{a}\right) f_n^2(x)dx < K/a.$$

Uniform convergence on compact sets together with the bound on the tail of the the integrals implies L^2 convergence.

□