Lecture 14 – 15 : Bulk scaling limit of GUE near 0

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Now we continue working with the Bulk scaling limiting, i.e., we are looking for a_n and b_n such that $b_n(\Lambda_n - a_n)$ converges to a "nice" limit as a point process. In the previous lecture, we have already found the Bulk scaling. From now on, we are trying to identify the limit.

Recall that:

- For $|c| < 2$, we will use the Bulk scaling $\sigma(c)\sqrt{n}(\Lambda_n - \sigma(c)\sqrt{n})$, where $\sigma(x) = \frac{1}{2\pi}\sqrt{4-x^2}1_{|x|<2}$ and $\Lambda = \{\lambda_1, \cdots, \lambda_n\}$.
- For $\beta = 2$ case (GUE), the Bulk scaling at 0 is $\frac{1}{\pi}\sqrt{n}\Lambda_n$, and the scaled joint intensities will converges to something nice. (Here $c = 0, \sigma(c) = \frac{1}{\pi}$).
- The joint eigenvalue density of GUE is

  $$P_n(\lambda_1, \cdots, \lambda_n) = \frac{1}{Z_n}\triangle(\lambda)^2 e^{-\frac{1}{2}\sum \lambda_i^2},$$

  where $\triangle(\lambda) = \prod(\lambda_i - \lambda_j)$. This is unordered n-tuples live on $\mathbb{R}^n$

Definition 1. $\zeta_{k,n}(\nu_1, \cdots, \nu_k) = \int P_n(\nu_1, \cdots, \nu_n) d\nu_{k+1} \cdots d\nu_n$, where k is fixed, and $n \to \infty$.

Hermite Polynomials

Definition 2 (n^{th} Hermite Polynomial)**.** The n^{th} Hermite Polynomial is defined as

$$h_n = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}})$$

This is a polynomial of degree n with the main term x^n, and $h_0(x) = 1$.

Proposition 3. *One property of Hermite Polynomials: $h_0, h_1, h_2, \cdots$ gives an orthogonal basis w.r.t. the measure $e^{-\frac{x^2}{2}}dx$, i.e.,*

$$\int h_k(x)h_n(x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx = \delta_{kn} \cdot n!$$

Proof. For $k \geq n$,

$$\begin{aligned}
&\int h_n(x)h_k(x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\\
=&\frac{1}{\sqrt{2\pi}}\int h_n(x)(-1)^k\frac{d^k}{dx^k}(e^{-\frac{x^2}{2}})dx\\
=&\frac{1}{\sqrt{2\pi}}\int (h_n(x))^{(k)}e^{-\frac{x^2}{2}}dx \text{ (integration by parts)}\\
=&\begin{cases} 0, & k>n \\ n!\int\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx, & k=n \end{cases} = \begin{cases} 0, & k>n \\ n!, & k=n \end{cases}
\end{aligned}$$

□

The n^{th} normalized oscillator wave function is

$$\psi_n(x) = h_n(x)\frac{e^{-\frac{x^2}{4}}}{\sqrt{\sqrt{2n}n!}}$$

And $\psi_0,\ \psi_1, \cdots$ defines an orthonormal basis w.r.t. Lebesgue measure.

Let's go back to GUE

Now let's consider $\zeta_{k,n}(\nu_1,\cdots,\nu_k) = \int P_n(\nu_1,\cdots,\nu_n)d\nu_{k+1}\cdots d\nu_n$.

Lemma 4.

$$\zeta_{k,n}(\nu_1,\cdots,\nu_k) = \frac{(n-k)!}{n!}\det[K^{(n)}(\nu_i,\nu_j)]_{i,j=1}^k,$$

where $K^{(n)}(x,y) = \sum_{k=0}^{n-1}\psi_k(x)\psi_k(y)$.

Proof. First consider $k=n$ case. We know that when $k=n$,

$$\zeta_{k,n}(\nu_1,\cdots,\nu_k) = P_n(\underline{\nu}) = \frac{1}{Z_n}[\triangle(\underline{\nu})]^2e^{-\sum\lambda_i^2/2},$$

where $\triangle(\underline{X}) = \prod(X_i - X_j) = \det(X_i^{j-1})_{i,j=1}^n$. Note that the first (highest order) term of $h_k(x)$ is x_k, we have,

$$\det\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} = \det[h_{j-1}(x_i)]_{i,j=1}^n.$$

Therefore,

$$\zeta_{k,n}(\nu_1,\cdots,\nu_k) = \frac{1}{Z_n}[\det[h_{j-1}(\nu_i)]_{i,j=1}^n]^2e^{-\sum\lambda_i^2/2} = \frac{1}{\widetilde{Z_n}}[\det\psi_{j-1}(\nu_i)]^2 = \frac{1}{\widetilde{Z_n}}\det[K^{(n)}(\nu_i,\nu_j)]$$

Hence, we only need to show that $\widetilde{Z_n} = n!$. (That's because here $k = n$, thus $(n-k)! = 1$.) To prove this, we need a lemma.

Lemma 5 (Jacobi Identity)**.** *Suppose $f_1,\ f_2, \cdots,\ f_n,\ g_1,\ g_2, \cdots,\ g_n$ are square integrable functions, then*

$$\begin{aligned}
&\frac{1}{n!}\int\cdots\int \det[\sum_{k=1}^{n} f_k(x_i)g_k(x_j)]dx_1\cdots dx_n\\
=&\frac{1}{n!}\int\cdots\int \det f_i(x_j)\det g_j(x_i)]d\underline{x}\\
=&\det[\int f_i(x)g_j(x)dx]_{i,j=1}^{n}
\end{aligned}$$

Proof. Part I: Note that $\det AB = (\det A)(\det B)$. Here let $A = [f_k(x_j)]$ and $B = [g_i(x_k)]$, then we have

$$AB = [f_k(x_j)][g_i(x_k)] = [\sum f_k(x_i)g_k(x_j)].$$

Therefore, the first equation holds.

Part II: Define index mapping $\nu = \tau \circ \sigma^{-1}$, then

$$\begin{aligned}
&\frac{1}{n!}\int\cdots\int \det f_i(x_j)\det g_j(x_i)]d\underline{x}\\
=&\sum_{\sigma,\tau\in S_n}\epsilon(\sigma)\epsilon(\tau)\int\cdots\int\prod f_{\sigma(i)}(x_i)g_{\tau(i)}(x_i)(\prod dx_i)\\
=&\sum_{\sigma,\nu}\epsilon(\nu)\int\cdots\int\prod f_{\sigma(i)}(x_i)g_{\nu(\sigma(i))}(x_i)(\prod dx_i)\\
=&\sum_{\sigma,\nu}\epsilon(\nu)\int\cdots\int\prod f_k(x_{\sigma(k)^{-1}})g_{\nu(k)}(x_{\sigma(k)^{-1}})(\prod dx_k)\\
=&\sum_{\sigma}\det[\int f_i(x)g_j(x)dx]\\
=&n!\det[\int f_i(x)g_j(x)dx]
\end{aligned}$$

□

Now back to Lemma 4,

$$\int\cdots\int \det[K^{(n)}(\nu_i,\nu_j)]d\underline{\nu} = n!\det\int\psi_{i-1}(x)\psi_{j-1}(x)dx = n!$$

This implies $\widetilde{Z_n} = n!$.

Now let's consider $k < n$ case. If $k < n$, define X_i as $Xi = \nu_i,\ i \le k$ and $X_i = \xi_i,\ i > k$, then we have

$$
\begin{aligned}
\zeta_{k,n}(\nu_1,\cdots,\nu_k) &= C_{k,n}\int(\det\psi_{j-1}(x_i))^2\prod_{i=k+1}^{n}d\xi_i \\
&= C_{k,n}\sum_{\sigma,\tau}\epsilon(\sigma)\epsilon(\tau)\underbrace{\int\cdots\int}_{n-k}\prod\psi_{\sigma(j)-1}(X_j)\psi_{\tau(j)-1}(X_j)\prod_{i=k+1}^{n}d\xi_i \\
&\quad \text{if } \sigma_j\neq\tau_j \text{ for } j>k\Rightarrow \text{ the appropriate term is 0 (orthogonal)} \\
&= C_{k,n}\sum_{1\leq\nu_1<\cdots<\nu_k\leq n}\sum_{\sigma,\tau}\epsilon(\sigma)\epsilon(\tau)\prod_{i=1}^{k}\psi_{\sigma(i)-1}(\nu_i)\psi_{\tau(i)-1}(\nu_i) \\
&\quad \text{where } \sigma,\tau:\{\nu_1,\nu_2,\cdots,\nu_k\}=\{\sigma(1),\sigma(2),\cdots,\sigma(k)\}=\{\tau(1),\tau(2),\cdots,\tau(k)\} \\
&= C_{k,n}\det K^{(n)}(\nu_i,\nu_j)\ \text{(by Cauchy-Binet)}
\end{aligned}
$$

Lemma 6 (Cauchy-Binet Formula)**.** *A is $m\times k$ matrix, B is $k\times n$ matrix, and $C=AB$. $r\leq\min(m,k,n)$ and $|I|=|J|=r$, then $\det C_{I,J}=\sum_{|K|=r}\det A_{I\times K}\det B_{K,J}$, where $K\subset\{1,\cdots,k\}$*

Proof. Use linear algebra or Jacobi identity. □

Then we will prove that $C_{k,n}=\frac{(n-k)!}{n!}$. In fact, we have:

$$
\begin{aligned}
1 &= C_{k,n}\int\cdots\int\det K^{(n)}(\nu_i,\nu_j)d\underline{\nu} \\
&= C_{k,n}\sum_{1\leq\nu_1<\cdots<\nu_k\leq n}\int(\det\psi_{\nu_j-1(\nu_i)})^2 d\nu \\
&= C_{k,n}\sum_{1\leq\nu_1<\cdots<\nu_k\leq n}k!\ \text{by Jacobi identity, part II} \\
&= C_{k,n}\binom{n}{k}k! = C_{k,n}\frac{n!}{(n-k)!}
\end{aligned}
$$

This implies $C_{k,n}=\frac{(n-k)!}{n!}$

Therefore, we finished the proof of Lemma 4. □

Recall that we want to show the joint intensities will converge to something nice. First let's define what the joint intensity is.

Definition 7 (Joint Intensity of order k)**.** Suppose $D_1,D_2,\cdots,D_k\subset\mathbb{R}$ are disjoint, and $\chi(D_i)$ is the number of points in D_i, then the expectation of $\prod_{i=1}^{k}\chi(D_i)$ can be presented as

$$
E[\prod_{i=1}^{k}\chi(D_i)]=\int_{D_1}\int_{D_2}\cdots\int_{D_k}R_k^{(n)}(\nu_1,\cdots,\nu_k)d\nu_1\cdots d\nu_k
$$

Then $R_k^{(n)}(\nu_1,\cdots,\nu_k)$ is called the joint intensity of order k.

Note that $\prod_{i=1}^k \chi(D_i) = \sum_{\lambda_{i_1},\cdots,\lambda_{i_k}} 1(\lambda_{i_1} \in D_1)1(\lambda_{i_2} \in D_2)\cdots 1(\lambda_{i_k} \in D_k)$. Therefore, we have $E[\prod_{i=1}^k \chi(D_i)]$ equals to the marginal of first k variables. Hence

$$\begin{aligned} E[\prod_{i=1}^k \chi(D_i)] &= \int P_n(\nu_1,\cdots,\nu_n)d\nu_k + 1\cdots\nu_n \\ &= \frac{n!}{(n-k)!}\int_{D_1}\cdots\int_{D_k} \zeta_{k,n}(\nu_1,\cdots,\nu_k)d\nu_1\cdots d\nu_k \\ &= \int_{D_1}\cdots\int_{D_k} \det_{i,j=1}^k [K^{(n)}(\nu_i,\nu_j)]d\nu_1\cdots d\nu_k \end{aligned}$$

Therefore, $R_k^{(n)}(\nu_1,\cdots,\nu_k) = \det_{i,j=1}^k[K^{(n)}(\nu_i,\nu_j)]$, where $K^{(n)}(\nu_i,\nu_j) = \sum_{j=0}^{n-1}\psi_i(x)\psi_j(y)$. When we rescale it, we have

$$(\frac{\pi}{\sqrt{n}})^k R_k^{(n)}(\frac{\pi}{\sqrt{n}}t_1,\cdots,\frac{\pi}{\sqrt{n}}t_k) = \det_{i,j=1}^k[\frac{\pi}{\sqrt{n}}K^{(n)}(\frac{\pi}{\sqrt{n}}t_i,\frac{\pi}{\sqrt{n}}t_j)]$$

And we will show that

$$\frac{\pi}{\sqrt{n}}t_1K^{(n)}(\frac{\pi}{\sqrt{n}},\frac{\pi}{\sqrt{n}}t_j) \to \frac{\sin(\pi(x-y))}{\pi(x-y)} \text{ uniformly on compacts.}$$

For the convenience, let's define $S^{(n)}(x,y) = \frac{1}{\sqrt{n}}K^{(n)}(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}})$, then what we need to prove is $S^{(n)}(x,y) \to \frac{1}{\pi}\frac{\sin(x-y)}{(x-y)}$.

Before starting proving, let's discuss some properties of h_n and ψ_n:

1. $\int h_k(x)h_n(x)\frac{1}{2\pi}e^{-x^2/2}dx = \delta_{n,k}n!$
2. $h_n(x)$ is even (or odd) if n is even (or odd).
3. $h_{n+1} = xh_n - h_n'$, b/c $h_n' = [(-1)^n e^{x^2/2}(e^{-x^2/2})^{(n)}]' = xh_n - h_{n+1}$ (Integration by parts)
4. $\int xh_n(x)^2\frac{1}{2\pi}e^{-x^2/2}dx = 0$, b/c xh_n^2 is odd.
5. $h_n' = nh_{n-1}$ (It's equivalent to $xh_n = h_{n+1} + nh_{n-1}$). It can be proved by induction, i.e., for $k \le n-1$, (by property 3)

$$\int h_n'h_k\frac{1}{2\pi}e^{-x^2/2}dx = -\int h_{n+1}h_k\frac{1}{2\pi}e^{-x^2/2}dx + \int h_n(xh_k)\frac{1}{2\pi}e^{-x^2/2}dx.$$

In the RHS, the first term is 0, for the degree of h_k is k, less than $n+1$. And for the second term, the degree of xh_k is $k+1$. Hence for $k+1 < n$, then $\int h_n(xh_k)\frac{1}{2\pi}e^{-x^2/2}dx = 0$. This implies that $h_n' = C_nh_{n-1}$. Therefore,

$$\int h_n(xh_{n-1})\frac{1}{2\pi}e^{-x^2/2}dx = \int h_n'h_{n-1}\frac{1}{2\pi}e^{-x^2/2}dx = C_n\int h_{n-1}^2\frac{1}{2\pi}e^{-x^2/2}dx = C_n(n-1)!.$$

On the other hand,

$$\int h_n(xh_{n-1})\frac{1}{2\pi}e^{-x^2/2}dx = \int h_n(h_n + h'_{n-1})\frac{1}{2\pi}e^{-x^2/2}dx = \int h_n^2\frac{1}{2\pi}e^{-x^2/2}dx = n!$$

So we have $C_n = n$, and $h_n = nh_{n-1}$.

6. Christoffel-Darboux identity

$$\sum_{k=0}^{n-1}\frac{h_k(x)h_k(y)}{k!} = \frac{h_n(x)h_{n-1}(y) - h_{n-1}(x)h_n(y)}{(x-y)(n-1)!}$$

This is because

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{h_k(x)h_k(y)(x-y)}{k!} &= \sum_{k=0}^{n-1}\frac{(h_{k+1}(x)+kh_{k-1}(x))h_k(y)}{k!} - \sum_{k=0}^{n-1}\frac{h_k(x)(h_{k+1}(y)+kh_{k-1}(y))}{k!} \\ &= \frac{h_n(x)h_{n-1}(y) - h_{n-1}(x)h_n(y)}{(n-1)!} \text{ (telescopic)}\end{aligned}$$

7. For $\psi(x) = e^{-x^2/4}\frac{1}{\sqrt[4]{2\pi}}\frac{1}{\sqrt{n!}}h_n(x)$ (ψ_i's are orthonomral), $\psi'_n(x) = \frac{x}{2}\psi_n(x) + \sqrt{n}\psi_{n-1}(x)$ (by property 5 + product rule).

8. $K^{(n)}(x,y) = \sum_{k=0}^{n-1}\psi_k(x)\psi_k(y) = \sqrt{n}\frac{\psi_n(x)\psi_{n-1}(y)-\psi_{n-1}(x)\psi_n(y)}{x-y}$

Lemma 8. *Let $\phi_\gamma(t) = n^{\frac{1}{4}}\psi_\gamma(\frac{t}{\sqrt{n}})$, where $\gamma - n$ is a constant. Then we have*

$$\lim_{\nu\to\infty}|\phi_\gamma(t) - \frac{1}{\sqrt{\pi}}\cos(t - \frac{\pi\gamma}{2})| = 0 \text{ uniformly in } t \text{ on compacts.}$$

Using Lemma 8, we can prove that $S^{(n)}(x,y) \to \frac{1}{\pi}\frac{\sin(x-y)}{(x-y)}$. That's because

$$\begin{aligned}S^{(n)}(x,y) &= \sqrt{n}\frac{\psi_n(\frac{x}{\sqrt{n}})\psi_{n-1}(\frac{y}{\sqrt{n}}) - \psi_{n-1}(\frac{x}{\sqrt{n}})\psi_n(\frac{y}{\sqrt{n}})}{x-y} \\ &\triangleq \sqrt{n}\frac{f(x)g(y)-g(x)f(y)}{x-y} \\ &= \sqrt{n}[\frac{f(x)-f(y)}{x-y}g(y) - \frac{g(x)-g(y)}{x-y}f(y)],\end{aligned}$$

where $\frac{f(x)-f(y)}{x-y}$ can be expressed as $\int_0^1 f'(tx+(1-t)y)dt$, and similarly for $\frac{g(x)-g(y)}{x-y}$, i.e., $\frac{g(x)-g(y)}{x-y} = \int_0^1 g'(tx+(1-t)y)dt$. Let $z = t\frac{x}{\sqrt{n}} + (1-t)\frac{y}{\sqrt{n}}$, then we have

$$
\begin{aligned}
S^{(n)}(x,y) &= \psi_{n-1}(\frac{y}{\sqrt{n}})\int_0^1 \sqrt{n}\psi_{n-1}(z) - \frac{z}{2}\psi_n(z)dt - \psi_n(\frac{y}{\sqrt{n}})\int_0^1 \sqrt{n-1}\psi_{n-2}(z) - \frac{z}{2}\psi_{n-1}(z)dt \\
&= n^{\frac{1}{4}}\psi_{n-1}(\frac{y}{\sqrt{n}})\int_0^1 n^{\frac{1}{4}}\psi_{n-1}(z) - n^{-\frac{1}{4}}\frac{z}{2}\psi_n(z)dt \\
&- n^{\frac{1}{4}}\psi_n(\frac{y}{\sqrt{n}})\int_0^1 \frac{\sqrt{n-1}}{n^{\frac{1}{4}}}\psi_{n-2}(z) - n^{-\frac{1}{4}}\frac{z}{2}\psi_{n-1}(z)dt \\
&\to \frac{1}{\pi}\cos(y - \frac{\pi(n-1)}{2})\int_0^1 \cos(tx + (1-t)y - \frac{\pi(n-1)}{2})dt \\
&- \frac{1}{\pi}\cos(y - \frac{\pi n}{2})\int_0^1 \cos(tx + (1-t)y - \frac{\pi(n-2)}{2})dt \\
&= \frac{1}{\pi}\cos(y - \frac{\pi(n-1)}{2})\frac{\int_x^y \cos(z - \frac{\pi(n-1)}{2})dz}{x-y} - \frac{1}{\pi}\cos(y - \frac{\pi n}{2})\frac{\int_x^y \cos(z - \frac{\pi(n-1)}{2})dz}{x-y} \\
&= \frac{1}{\pi}\frac{sin(x-y)}{x-y}
\end{aligned}
$$

Then the only thing left is how to prove Lemma 8. To do this, we need to use the Laplace Method

Lemma 9 (Laplace Method)**.** *Laplace Method is to consider the asymptotics of* $\int f(x)^s g(x)dx$ $(s \to \infty)$*, where* f *has a global max at* $x = a$*, and it's locally quadratic there,* g *is locally Lipschitz near* a*, and* $|g(a)| \subset K$*. Then we have*

$$
\lim_{s\to\infty} \sqrt{s} f(a)^{-s} \int f(x)^s g(x)dx = \sqrt{-\frac{2\pi f(a)}{f''(a)}}g(a)
$$

Proof. A brief proof of Lemma 9 is, near a, we have $f(x) \approx f(a) + \frac{1}{2}f''(a)(x-a)^2$. Then we have

$$
\begin{aligned}
\int (\frac{f(x)}{f(a)})^s g(x)dx &= \int (1 + \frac{1}{2}\frac{f''(a)}{f(a)}(x-a)^2)^s g(x)dx \\
&\approx \int \exp(\frac{1}{2}s\frac{f''(a)}{f(a)}(x-a)^2)g(x)dx \\
&= \frac{1}{\sqrt{s}}\int \exp(\frac{1}{2}s\frac{f''(a)}{f(a)}y^2)g(a + \frac{y}{\sqrt{s}})dy \text{ (let } x = a + \frac{y}{\sqrt{s}}) \\
&\to \frac{1}{\sqrt{s}}\sqrt{-\frac{2\pi f(a)}{f''(a)}}g(a)
\end{aligned}
$$

If you want to prove the lemma more precisely, you need to consider

$$
\int f(x)^s g(x)dx = \int_{|x-a|<\epsilon} f(x)^s g(a)dx + \int_{|x-a|<\epsilon} f(x)^s (g(x) - g(a))dx + \int_{|x-a|>\epsilon} f(x)^s g(x)dx
$$

□