

# Lectures 12 – 13 : Gaussian Ensembles and Local Limits

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## 1 Gaussian Ensembles

Recall the joint eigenvalue density of gaussian ensembles, where  $\beta = 1, 2, 4$  denotes GOE, GUE and GSE, respectively, is given by

$$P(\lambda_1, \dots, \lambda_n) = \frac{1}{\tilde{Z}_{n,\beta}} \prod_{i>j} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta}{4} \sum_{i=1}^n \lambda_i^2}. \quad (1)$$

Consider the  $n \times n$  tridiagonal symmetric matrix  $M_n$ . Last time we showed  $M_n$  forms a one-to-one mapping with  $((\lambda, \mathbf{q}))$ :

$$\begin{bmatrix} a_n & b_{n-1} & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & \\ & b_{n-2} & \ddots & \ddots & \\ & & \ddots & \ddots & b_1 \\ & & & b_1 & a_1 \end{bmatrix}_{n \times n} \longleftrightarrow (\lambda, \mathbf{q}) \quad (2)$$

where

$$\mathbf{q} = (q_1, q_2, \dots, q_n) \quad \sum q_i^2 = 1. \quad (3)$$

Last time we showed

$$\Delta(\lambda) \triangleq \prod_{i<j} |\lambda_i - \lambda_j| = \frac{\prod_{i=1}^{n-1} b_i^i}{\prod_{i=1}^n q_i}. \quad (4)$$

We would like to show the Jacobian of the mapping  $\psi : (\mathbf{a}, \mathbf{b}) \rightarrow (\lambda, \mathbf{q})$  is

$$\text{Jac}(\psi) = \frac{\Delta(\lambda)}{\prod_{i=1}^{n-1} b_i^{i-1} q_n}. \quad (5)$$

Let us introduce another description of the tridiagonal matrix  $M_n$ . Consider  $m_1, m_2, \dots, m_{2n-1}$  where

$$m_k = \sum_{i=1}^n \lambda_i^k q_i^2 = \int \lambda^k du(\lambda) \quad (6)$$

How do we express  $m$  using  $a, b$ ? First, notice

$$m_k = [M^k]_{1,1} = e_1^T M^k e_1 \quad (7)$$

From above, we can writing out  $m$  as

$$m_1 = a_n \quad (8)$$

$$m_2 = b_{n-1}^2 + a_n^2 \quad (9)$$

$$\vdots \quad (10)$$

$$m_k = \begin{cases} (b_{n-1}b_{n-2}\dots b_{n-l})^2 + f(m_{k-1}) & k = 2l \\ (b_{n-1}b_{n-2}\dots b_{n-l})^2 b_{n-l} + f(m_{k-1}) & k = 2l + 1 \end{cases} \quad (11)$$

where  $f(m_{k-1})$  is some function of the previous term,  $m_{k-1}$ . We can compute  $m_k$  sequentially. In particular, we can recursively compute  $(m_1, \dots, m_{2n-1}) \rightarrow (\mathbf{a}, \mathbf{b})$  by solving for  $a_n, b_{n-1}, a_{n-1}, \dots$ . Hence

$$\begin{bmatrix} a_n & b_{n-1} & & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & & \\ & b_{n-2} & \ddots & \ddots & & \\ & & \ddots & \ddots & b_1 & \\ & & & b_1 & a_1 & \end{bmatrix}_{n \times n} \longleftrightarrow (\lambda, \mathbf{q}) \longleftrightarrow (m_1, \dots, m_{2n-1}) \quad (12)$$

How do we get the Jacobian? Write

$$J = \det \begin{bmatrix} \frac{dm_1}{da_n} & \frac{dm_2}{da_n} & \dots & \frac{dm_{2n-1}}{da_n} \\ \frac{dm_1}{db_{n-1}} & \frac{dm_2}{db_{n-1}} & & \frac{dm_{2n-1}}{db_{n-1}} \\ \frac{dm_1}{da_{n-1}} & \frac{dm_2}{da_{n-1}} & & \frac{dm_{2n-1}}{da_{n-1}} \\ \vdots & & \ddots & \vdots \\ \frac{dm_1}{da_1} & \frac{dm_2}{da_1} & \dots & \frac{dm_{2n-1}}{da_1} \end{bmatrix} = \det \begin{bmatrix} \frac{dm_1}{da_n} & \frac{dm_2}{da_n} & \dots & \frac{dm_{2n-1}}{da_n} \\ 0 & \frac{dm_2}{db_{n-1}} & & \frac{dm_{2n-1}}{db_{n-1}} \\ \vdots & \ddots & \ddots & \frac{dm_{2n-1}}{da_{n-1}} \\ 0 & \dots & 0 & \frac{dm_{2n-1}}{da_1} \end{bmatrix} = 2^{n-1} \prod_{i=1}^{n-1} b_{n-i}^{4(n-1)+3} \quad (13)$$

since the lower left terms are identically zero - and the upper right terms are given by

$$\frac{dm_{2l}}{db_{n-l}} = 2b_{n-l}(b_{n-1}\dots b_{n-l+1})^2 \quad (14)$$

$$\frac{dm_{2l+1}}{da_{n-l}} = (b_{n-1}\dots b_{n-l})^2. \quad (15)$$

Thus,  $J$  is upper triangular, and the determinant is given by the product of the diagonal entries. We'd like to be able to compute

$$\frac{dm_j}{d\lambda_k} = j\lambda_k^{j-1}q_k^2 \quad (16)$$

$$\frac{dm_j}{dq_k} = 2(\lambda_k^j - \lambda_n^j)q_k \quad (17)$$

$$(18)$$

which can again be computed recursively. Writing out the Jacobian

$$J = \det \left[ \begin{array}{ccc|ccc} \frac{dm_1}{dq_1} & \dots & \frac{dm_1}{dq_{n-1}} & \frac{dm_1}{d\lambda_1} & \dots & \frac{dm_1}{d\lambda_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{dm_{2n-1}}{dq_1} & \dots & \frac{dm_{2n-1}}{dq_{n-1}} & \frac{dm_{2n-1}}{d\lambda_1} & \dots & \frac{dm_{2n-1}}{d\lambda_n} \end{array} \right] \quad (19)$$

$$= \det \left[ \begin{array}{c|c} 2q_i(\lambda_i^j - \lambda_n^j) & \\ \hline & j\lambda_i^{j-1}q_i^2 \end{array} \right] \quad (20)$$

$$= \frac{2^{n-1}}{q_n} \prod_{i=1}^n q_i^3 \det \left[ \begin{array}{c|c} \lambda_i^j - \lambda_n^j & \\ \hline & j\lambda_i^{j-1} \end{array} \right] \quad (21)$$

$$= \frac{2^{n-1}}{q_n} \prod_{i=1}^n q_i^3 \Delta(\lambda)^4 \quad (22)$$

where the last step comes from the claim that the determinant in (21) is a homogeneous polynomial of  $\lambda_1, \lambda_2, \dots, \lambda_n$  of degree 4 times the degree of  $\Delta(\lambda)$ . **Claim:** We can extend the matrix in (21) as follows

$$\det \left[ \begin{array}{ccc|c} 1 & f(\lambda_i) & \dots & \\ 0 & & & \\ \vdots & \lambda_i^j - \lambda_n^j & & j\lambda_i^{j-1} \\ 0 & & & \end{array} \right] = \det \left[ \begin{array}{c|c} \lambda_i^j & \\ \hline & j\lambda_i^{j-1} \end{array} \right] \quad (23)$$

where the first row consists of the  $n$ th column subtracted from  $\{1, 2, \dots, n-1\}$ . Then, replace  $\lambda$  in the second block of the matrix with a new variable, and since  $\frac{d}{d\nu_i}(\nu_i^j) = j\nu_i^{j-1}$ :

$$f(\lambda, \nu) = \det \left[ \begin{array}{c|c} \lambda_i^j & \\ \hline & j\nu_i^{j-1} \end{array} \right] \quad (24)$$

$$= \frac{d^n}{d\nu_1 \dots d\nu_n} \det \left[ \begin{array}{c|c} \lambda_i^j & \\ \hline & \nu_i^j \end{array} \right] \quad (25)$$

$$= \frac{d^n}{d\nu_1 \dots d\nu_n} \Delta(\lambda, \nu) \quad (26)$$

$$= \frac{d^n}{d\nu_1 \dots d\nu_n} \prod_{i,j} (\lambda_i - \lambda_j) \prod_{i < j} (\nu_i - \nu_j) \prod_{i < j} (\lambda_i - \nu_j) \quad (27)$$

$$= \frac{d^n}{d\nu_1 \dots d\nu_n} g(\lambda, \nu) \Big|_{\nu_i = \lambda_i} \quad (28)$$

$$= \Delta(\lambda)^4. \quad (29)$$

Collecting all results we have

$$(\lambda, q) \longrightarrow m \quad (30)$$

$$\text{Jac} \longrightarrow \frac{2^{n-1}}{q_n} \prod_{i=1}^n q_i^3 \Delta(\lambda)^4 \quad (31)$$

$$(a, b) \longrightarrow (\lambda, q). \quad (32)$$

Finally

$$\text{Jac}(\psi) = \frac{\frac{2^{n-1}}{q_n} \prod_{i=1}^n q_i^3 \Delta(\lambda)^4}{2^{n-1} \prod_{i=1}^{n-1} b_i^{4(i-1)+3}} \quad (33)$$

$$= \frac{1}{q_n} \prod_{i=1}^n q_i^3 \frac{\prod_{i=1}^n b_i^{4i}}{\prod_{i=1}^n q_i^4 \prod_{i=1}^n b_i^{4i-1}} \quad (34)$$

$$= \frac{1}{q_n} \frac{\prod_{i=1}^n b_i}{\prod_{i=1}^n q_i}. \quad (35)$$

## 2 Local Limits

In this section, we only consider classical ensembles. Let  $\Lambda_n$  be a length  $n$  vector drawn from the joint eigenvalue density:

$$P(\lambda_1, \dots, \lambda_n) = \frac{1}{\tilde{Z}_{n,\beta}} \prod_{i>j} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta}{4} \sum_{i=1}^n \lambda_i^2} \quad (36)$$

We would like to show that  $\nu_n = \frac{1}{n} \sum_{i=1}^n \frac{\delta_{\lambda_i}}{\sqrt{n}}$  converges almost surely to the Wiegner semi-circle law. How can we understand the microscopic picture? Our aim is to prove a scaling property of  $\Lambda_n$  by choosing appropriate  $\{a_n, b_n\}$  so that

$$b_n(\Lambda_n - a_n) \quad (37)$$

is a limiting point process. Beginning with a toy model, consider the joint eigenvalue density for  $\beta = 0$ .

$$\Lambda_n = \{\lambda_1, \lambda_2, \dots, \lambda_n\} \quad (38)$$

$$\lambda_i \sim \mathcal{N}(0, 1) \quad (39)$$

$$\nu_n = \frac{1}{n} \sum_{i=1}^n \frac{\delta_{\lambda_i}}{\sqrt{n}} \implies \mathcal{N}(0, 1) \quad \text{a.s.} \quad (40)$$

How should we scale  $\Lambda_n$  to get a limiting point process? Suppose we want to scale  $\Lambda_n$  near zero ( $a_n = 0$ ). The scaled process should have a density  $O(1)$ . Choosing  $b_n = n$  is a logical choice.

**Definition 1.** Let  $N_n([a, b])$  be the number of points of  $n\Lambda_n$  in the interval  $[a, b]$ . Write

$$N_n([a, b]) \triangleq \sum_{i=1}^n \mathbf{1}(\lambda_i \in [a/n, b/n]) \quad (41)$$

$N_n$  will be a sum of iid Bernoulli random variables with probability  $p_n$ , where

$$p_n = \int_{\frac{a}{n}}^{\frac{b}{n}} \Psi(x) dx \approx \Psi(0) \frac{b-a}{n} \quad (42)$$

and,

$$np_n \longrightarrow (a-b)\Psi(0). \quad (43)$$

By the Poisson approximation

$$N_n([a, b]) \implies \text{Poisson}((a-b)\Psi(0)). \quad (44)$$

Thus,  $n\Lambda_n$  converges to a Poisson point process with density  $\Psi(0)$ .

Suppose we wanted to look at the microscopic picture near  $c$  (set  $a_n = c$ ). In this case,  $n(\Lambda_n - c)$  converges to a Poisson point process with density  $\Psi(c)$ .

Returning to  $\beta$ -ensembles, recall

$$P(\lambda_1, \dots, \lambda_n) = \frac{1}{\tilde{Z}_{n,\beta}} \Delta(\lambda)^\beta e^{-\frac{\beta}{4} \sum_{i=1}^n \lambda_i^2} \quad (45)$$

How should we choose  $\{a_n, b_n\}$  for  $\beta = 1, 2, 4$ ? The semicircle law tells us there are approximately  $n \int_a^b \frac{1}{2\pi} \sqrt{4-x^2} dx$  points in the interval  $[a\sqrt{n}, b\sqrt{n}]$ .

**Definition 2.**

$$\sigma(x) \triangleq \frac{1}{2\pi} \sqrt{4-x^2} \mathbf{1}(|x| \leq 2) \quad (46)$$

Roughly speaking, the density near  $c\sqrt{n}$  is approximately  $\sigma(c)\sqrt{n}$ . If we want to 'zoom in' near  $c\sqrt{n}$  then we should scale by  $\sigma(c)\sqrt{n}$ . This is referred to as the **bulk scaling of  $\beta$ -ensembles**.

$$\lim_{n \rightarrow \infty} \sigma(c)\sqrt{n}(\Lambda_n - c\sqrt{n}) \quad (47)$$

Next we investigate how to scale near  $a_n = 2\sqrt{n}$ . Write  $b_n(\Lambda_n - 2\sqrt{n})$ . The number of point in the interval  $[2\sqrt{n} - y, 2\sqrt{n}]$  is given by:

$$N_n([2\sqrt{n} - y, 2\sqrt{n}]) \approx n \int_{2 - \frac{y}{\sqrt{n}}}^2 \sigma(x) dx \approx n \int_0^{\frac{y}{\sqrt{n}}} \frac{1}{\pi} \sqrt{x} dx = cy^{\frac{3}{2}} n^{\frac{1}{4}} \quad (48)$$

If we want the number of points to be  $O(1)$ , we need

$$y^{\frac{3}{2}} n^{\frac{1}{4}} = O(1) \quad (49)$$

$$y \sim n^{-\frac{1}{6}} \quad (50)$$

Thus, we choose  $b_n = n^{\frac{1}{6}}$ . This is referred to as the edge scaling of  $\beta$ -ensembles.

$$\lim_{n \rightarrow \infty} n^{\frac{1}{6}}(\Lambda_n - 2\sqrt{n}) \quad (51)$$

One would expect to have different limiting processes in the two cases. In the bulk case, one would expect at translation invariant process. In the edge case, one would expect the existence of a largest point. How can we identify the limiting point process? Consider  $\beta = 1, 2, 4$ . Let  $\zeta$  be a random point process. Furthermore, let  $X(D)$  be the number of point in  $D$ ,  $D \subset \mathfrak{R}$ . Let  $\rho(x_1, x_2, \dots, x_n)$  be the joint density of order  $n$ . If  $D_1, D_2, \dots, D_n$  are disjoint sets,

$$E \prod X(D_i) = \int_{D_1 \times \dots \times D_n} \rho(x_1, x_2, \dots, x_n) dx_1 \dots dx_n \quad (52)$$

**Example 3.**  $\rho(x_1, x_2)\epsilon^2 \approx P(\text{point near } x_1 \text{ and } x_2)$

Scaling limit results exist in the classical cases. Let  $\rho_m^{(n)}(x_1, x_2, \dots, x_m)$  be the  $m^{\text{th}}$  marginal of  $\rho$ , the joint distribution.

$$\rho_m^{(n)}(x_1, x_2, \dots, x_m) = \int \rho(x_1, x_2, \dots, x_n) dx_{m+1} \dots dx_n \quad (53)$$

**Theorem 4.**

$$\frac{1}{b_n^m} \rho_m^{(n)}\left(a_n + \frac{t_1}{b_n}, a_n + \frac{t_2}{b_n}, \dots, a_n + \frac{t_m}{b_n}\right) \longrightarrow \rho_m(t_1, \dots, t_2) \quad (54)$$

The limiting intensities will have a nice structure, in particular the  $\beta = 2$  case. For the  $\beta = 2$  case, we have:

$$\rho_m(x_1, \dots, x_m) = \det[K(x_i, x_j)]_{i,j=1}^m \quad (55)$$

where  $K(x_i, x_j)$  is:

Bulk case:

$$\begin{aligned} K(x, y) &= \frac{\sin(\pi(x-y))}{\pi(x-y)} \\ K(x, x) &= 1 \end{aligned} \quad (56)$$

Edge case:

$$K(x, y) = \frac{A_i(x)A_i(y) - A_j(x)A_j(y)}{x-y} \quad (57)$$

where

$$A_i(x) = \frac{1}{2\pi i} \int_C e^{\frac{z^2}{3} - zx} \quad (58)$$

In the  $\beta = 1, 4$  cases, the structures of the limiting intensities have more complicated expressions involving Pfaffians instead of determinants.