Lectures 10 – 11 : Gaussian Ensembles

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Recall that:
1)the density of the Gaussian Ensembles can be represented by

$$\frac{1}{Z_{n,\beta}} e^{-\frac{\beta}{4} trH^2} dH$$

where $\beta = 1, 2, 4$ denote GOE, GUE and GSE respectively.
2)the joint eigenvalue density is given by

$$P(\lambda_1, \cdots, \lambda_n) = \frac{1}{\tilde{Z}_{n,\beta}} \prod_{i>j} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta}{4} \sum_{i=1}^n \lambda_i^2} \tag{1}$$

where $\lambda_1 < \lambda_2 < \cdots < \lambda_n$.
Now we consider the $n \times n$ symmetric matrix M_n to a tridiagonal symmetric matrix

$$\begin{bmatrix} a_n & b_{n-1} & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & \\ & b_{n-2} & \ddots & \ddots & \\ & & \ddots & \ddots & b_1 \\ & & & b_1 & a_1 \end{bmatrix}_{n\times n}$$

where $b_i \geq 0$.
Assume the eigenvector of M is $\vec{u} = (u_1, u_2, ...u_n)^T$, and then we have the equation $M \cdot \vec{u} = \lambda \cdot \vec{u}$, which in other form: $b_{k-1}u_{k-1} + (a_k - \lambda)u_k + b_k u_{k+1} = 0$.
Now we consider the normalized eigenvalues: $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$

- $\langle \vec{v_i}, \vec{v_j} \rangle = \delta_{ij}$
- Assume that $v_{i1} > 0$, and $\displaystyle\sum_{i=1}^n v_{i1}^2 = 1$

Now we want to show that there is a map between $(\lambda_1, \lambda_2, ...\lambda_n, v_{11}, v_{21}, ...v_{n-11})$ and $(a_1, a_2, ...a_n, b_1, ...b_{n-1})$. That is , we have to show that: if we have $\lambda_1 < \lambda_2 < ... < \lambda_n$ and $v_{11}, v_{21}, ...v_{(n-1)1} > 0$ with$\displaystyle\sum_{i=1}^n v_{i1}^2 < 1$,then there is a unique vector $(a_1, a_2, ...a_n, b_1, ...b_{n-1})$ for which the corresponding diagonal matrix has the appropriate eigenvalues and eigenvectors.
Now consider

$V = [\vec{v_1}, \vec{v_2}, ...\vec{v_n}]$
$V^T = [\vec{q_1}, \vec{q_2}, ...\vec{q_n}]$,
so we have $VV^T = I,\ VM = DV,\ D = diag(\lambda_1, ...\lambda_n)$
We would like to show that given $\vec{q_1}$ and D, we can compute $\vec{q_2}, ..., \vec{q_n}$.
Claim: From the eigenvalue equation we get $b_{k-1}\vec{q}_{(k-1)} + a_k\vec{q_k} + b_k\vec{q}_{k+1} = D\vec{q_k}$,if we look at row i, we have $b_{k-1}u_{k-1}^{(i)} + a_k u_k^{(i)} + b_k u_{k+1}^{(i)} = \lambda_i u_k^{(i)}$, $v_i = [u_1^{(i)}, u_2^{(i)}, ..., u_n^{(i)}]$.
Assume $b_0 = 0$,

$$a_1\vec{q_1} + b_1\vec{q_2} = D\vec{q_1} \tag{2}$$
$$b_1\vec{q_1} + a_2\vec{q_1} + b_2\vec{q_2} = D\vec{q_2} \tag{3}$$

...
Recall: $\langle\vec{q_i}, \vec{q_j}\rangle = \delta_{ij}$,and $a_k\langle\vec{q_k}, \vec{q_k}\rangle = \vec{q_k}^T D\vec{q_k}$.
We have D, $\vec{q_1}$,then:

$$(1) \Rightarrow a_1 = \vec{q_1}^T D\vec{q_1} \tag{4}$$
$$(2) \Rightarrow b_1\vec{q_2} = D\vec{q_1} - a\vec{q_1} \Rightarrow b_1, \vec{q_2} \tag{5}$$

so we have

$$b_1 = \parallel D\vec{q_1} - a_1\vec{q_1} \parallel_2$$

$$\vec{q_2} = \frac{D\vec{q_1} - a_1\vec{q_1}}{b_1}$$

Thus, following this process, we have $D, \vec{q_1} \Rightarrow a_1 \Rightarrow b_1, q_2 \Rightarrow a_2 \Rightarrow b_2, \vec{q_3}, ...$
So we can say:

$$a_k = \vec{q_k}^T D\vec{q} \tag{6}$$
$$b_k\vec{q}_{k+1} = D\vec{q_k} - a_k\vec{q_k} - b_{k-1}\vec{q}_{k-1} \tag{7}$$

Now we would like to show that for any $\lambda_1 < \lambda_2 < ... < \lambda_n, v_{11}, v_{21}, ...v_{(n-1)1} > 0, \sum_{i=1}^{n-1} v_{i1}^2 < 1$, there exists a tridiagonal matrix. For this,we use the same recursion. We need (7) cannot be zero,for $k \leq n-1$
Let $P_k(\lambda_i) = \frac{q_{k,i}}{q_{1,i}}$, then we can consider P_k as a map from $(\lambda_1, \lambda_2, ...\lambda_n)$ to R.
Thus:

$$P_1(x) = 1$$

$$P_1 \to P_2 \to P_3...$$

So we say P_k is a polynomial of degree at most $k-1$
If $b_k\vec{q}_{k+1}=0$,then $b_kP_{k+1}(\lambda)=0$,which cannot happen if $k<n$.(Because we would get a polynomial of degree $\leq n-1$ vanishing at $\lambda_1,...\lambda_n$).
Note:$P_1,P_2,...P_n$ is the system of orthonormal polynomials on the measure $\mu(\lambda_i)=q_{1i}^2$.Here by orthonormal polynomials we mean $\langle P_i,P_j\rangle=\int P_i(x)P_j(x)d\mu(x)=\Sigma P_i(\lambda_k)P_j(\lambda_k)\mu(\lambda_k)=\langle\vec{q_i},\vec{q_j}\rangle=\delta_{ij}$.
Now we go back to consider the map:

$$\varphi:(\tilde{a},\tilde{b})\to(\tilde{\lambda},\tilde{q}_1)$$

here $\tilde{q}_1=(v_{11},v_{21},...v_{(n-1)1})$.

Theorem 1. *1)*$Jac(\varphi)=\frac{\Delta(\tilde{\lambda})}{\Pi_{i=1}^{n-1}b_i^{i-1}}\cdot\frac{1}{q_{1n}}$,*where* $\Delta(\tilde{\lambda})=\Pi_{i>j}(\lambda_i-\lambda_j)$
2)$\Delta(\tilde{\lambda})=\frac{\Pi_{i=1}^{n-1}b_i^i}{\Pi_{i=1}^{n}q_i}$

Corollary 2. $Jac(\varphi)=\frac{\Pi_{i=1}^{n-1}b_i}{\Pi_{i=1}^{n}q_i}\cdot\frac{1}{q_n}$

BTW,from now on,

$$M=\begin{bmatrix} a_n & b_{n-1} & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & \\ & b_{n-2} & \ddots & \ddots & \\ & & \ddots & \ddots & b_1 \\ & & & b_1 & a_1 \end{bmatrix}_{n\times n}$$

Now first we proof the existence of the map.

Proof. 1) If we got $(\tilde{\lambda},\vec{q})$ from a certain $(\tilde{a},\tilde{b})$, then we can recover $(\tilde{a},\tilde{b})$ (and also $\vec{q}_2,\vec{q}_3,...\vec{q}_n$)
$V=(v_1,v_2,...v_n)=\begin{bmatrix}\tilde{q}_1^T\\ \vdots\\ \tilde{q}_n^T\end{bmatrix}$
We have the recursion equation:

$$b_{k-1}\vec{q}_{k-1}+a_k\vec{q}_k+b_k\vec{q}_{k+1}=Dq_k$$

$$a_k=\vec{q}_k^TD\vec{q}_k$$

$$b_k\vec{q}_{k+1}=D\vec{q}_k-a\vec{q}_k-b_{k-1}\vec{q}_{k-1}$$

$$\therefore D_1\vec{q}_1 \Rightarrow a_1 \Rightarrow b_1, \vec{q}_2 \Rightarrow ...$$

2) For each $(\lambda, \vec{q})$, there is an $(\tilde{a}, \tilde{b})$.
Set $\mu(\lambda_i) = q_{1i}^2$,here μ is a probability measure on $\lambda_1, ...\lambda_n$.
$P_1(\lambda_1) = 1$, $P_1, P_2, ...P_{n-1}$is the system of orthonormal polynomial w.r.t. μ.
$\langle P_i, P_j \rangle_\mu = \delta_{ij} = \Sigma P_i(\lambda_k)P_j(\lambda_k)q_{1k}^2$
Set

$$a_k = \langle P_k, \lambda P_k \rangle, k = 1, 2, ..., n$$

$$b_k = \langle P_{k+1}, \lambda P_k \rangle, k = 1, 2, ...n-1$$

With this choice of $\tilde{a}, \tilde{b}$,$b_k\vec{q}_{k+1} = D\vec{q}_k - a\vec{q}_k - b_{k-1}\vec{q}_{k-1}$,which is a polynomial of degree k.

$$\because \lambda P_k - a_k P_k - b_{k-1}P_{k-1} = \sum_{i=1}^{k+1} C_i P_i$$

$$\therefore C_i = \langle P_i, \lambda P_k - a_k P_k - b_{k-1}P_{k-1} \rangle$$

So we have:

$$i = 1, C_{k+1} = \langle P_{k+1}, \lambda P_k \rangle = b_k$$

$$i < k-1, C_i = \langle P_i, \lambda P_{k+1} \rangle = 0$$

$$i = k, C_k = \langle P_k, \lambda P_k \rangle = \lambda$$

Now set $q_{ki} = P_k(\lambda_i)q_{1i}$,then q_k will satisfy $b_{k-1}\vec{q}_{k-1} + a_k\vec{q}_k + b_k\vec{q}_{k+1} = D\vec{q}_k$. □

Now we go back to Gaussian O/U/S ensemble.

$$M = \begin{bmatrix} a_n & b_{n-1} & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & \\ & b_{n-2} & \ddots & \ddots & \\ & & \ddots & \ddots & b_1 \\ & & & b_1 & a_1 \end{bmatrix}_{n \times n}$$

where $a_i \sim \frac{1}{\sqrt{\beta}} N(0,\sqrt{2})$,$b_i \sim \frac{1}{\beta}\chi_{i\beta}$.
Joint density of $(\tilde{a},\tilde{b})$:

$$f(\tilde{a},\tilde{b}) = \frac{1}{Z_{n\beta}} e^{-\frac{\beta}{4}\sum_{i=1}^{n} a_i^2 - \frac{\beta}{2}\sum_{i=1}^{n-1} b_i^2} \prod_{i=1}^{n-1} b_i^{i\beta-1}$$

Joint density of $(\tilde{\lambda,\vec{q}})$: $f(\tilde{\lambda},\vec{q}) = |Jac|\cdot$ same expression with $(\tilde{\lambda},\vec{q})$.

$$\because \sum_i a_i^2 + 2_i b_i^2 = TrM^2 = \sum_{i=1}^{n} \lambda_i^2$$

and

$$Jac(\varphi) = \frac{\Delta(\tilde{\lambda})}{\Pi_{i=1}^{n-1} b_i^{i-1}} \cdot \frac{1}{q_{1n}}$$

$$\Delta(\tilde{\lambda}) = \frac{\Pi_{i=1}^{n-1} b_i^{i}}{\Pi_{i=1}^{n} q_i}$$

$$\therefore |Jac| \prod_{i=1}^{n-1} b_i^{i\beta-1} = \frac{1}{q_n} \triangle(\lambda)^{\beta} \prod_{i=1}^{n-1} q_i^{\beta-1}$$

$$\therefore f(\lambda,\vec{q}) = \frac{1}{Z_{n,\beta}} e^{-\frac{\beta}{4}\sum_{i=1}^{n}\lambda_i^2} \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} \prod_{i=1}^{n} q_i^{\beta-1} \frac{1}{q_n}$$

$\therefore \tilde{\lambda},\vec{q}$ are independent,and the density of λ is:

$$\frac{1}{\tilde{Z}_{n,\beta}} \prod_{i>j} |\lambda_i - \lambda_j|^{\beta} e^{-\frac{\beta}{4}\sum_{i=1}^{n}\lambda_i^2}$$

the density of $\vec{q}$ is:

$$\prod_{i=1}^{n} q_i^{\beta-1} \cdot \frac{1}{\vec{q}_n}$$

Theorem 3. *In G O/U/S E, $(\vec{V}_1,\vec{V}_2,...\vec{V}_n)$ is independent of $(\lambda_1,...\lambda_n)$ each $\vec{V}_i$ is distributed uniformly on $V : \|V\|_2 = 1, \vec{v}_i > 0$.Furthermore,$(\vec{v}_1,...\vec{v}_n)$ is distributed according to the Haur measure of the appropriate group.*

Important Note: We can go backwards in the previous proof.
$(\lambda_1,...\lambda_n)$ is distributed according to $\frac{1}{\tilde{Z}_{n,\beta}} \prod_{i>j} |\lambda_i - \lambda_j|^{\beta} e^{-\frac{\beta}{4}\sum_{i=1}^{n}\lambda_i^2}$,this has the same distribution as

λ from

$$\frac{1}{\sqrt{\beta}}\begin{bmatrix} N(0,\sqrt{2}) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,\sqrt{2}) & \chi_{(n-2)\beta} & & \\ & \chi_{(n-2)\beta} & \ddots & \ddots & \\ & & \ddots & \ddots & \chi_{\beta} \\ & & & \chi_{\beta} & N(0,\sqrt{2}) \end{bmatrix}_{n\times n}$$

This family is called Gaussian β-ensemble (or Dyson's β-ensemble,Jacobi β-ensemble). Now we prove the Theorem 1.

Proof. **Claim:**$\Delta(\tilde{\lambda}) = \frac{\Pi_{i=1}^{n-1} b_i^i}{\Pi_{i=1}^{n} q_i}$

$$M = UDU^T$$

$$D = diag(\lambda_1, ...\lambda_n)$$

$$U^T = [\vec{q}_1, ...\vec{q}_n]$$

$$A = [e_1, Me_1, ..., M^{n-1}e_1]$$

then

$$Det\ A = \prod_{i=1}^{n-1} b_i^i$$

$$Me_1 = \begin{bmatrix} a_n \\ b_{n-1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}, M^2 e_1 = \begin{bmatrix} * \\ * \\ b_{n-1}b_{n-2} \\ \vdots \\ \vdots \end{bmatrix} \dots M^{k-1}e_1 = \begin{bmatrix} * \\ \vdots \\ \prod_{i=1}^{k-1} b_{n-i} \\ * \end{bmatrix}$$

then

$$U^T\vec{e}_1 = \vec{q}_1$$

$$A = [\vec{e}_1, M\vec{e}_1, ...M^{n-1}\vec{e}_1] = [UU^Te_1, UDU^Te_1, ..., UD^{n-1}U^Te_1] = U[\vec{q}_1, D\vec{q}...D^{n-1}\vec{q}_1]$$

$$B = [\vec{q}_1, D\vec{q}...D^{n-1}\vec{q}_1]$$

then:

$$det\ B = \det A$$

Since

$$B = \begin{bmatrix} q_{11} & \lambda_1 q_{11} & \lambda_1^2 q_{11} & \dots & \lambda_1^{n-1} q_{11} \\ q_{12} & \lambda_2 q_{12} & \dots & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{1n} & \lambda_n q_{1n} & \lambda_n^2 q_{1n} & \dots & \lambda_n^{n-1} q_{1n} \end{bmatrix}$$

$$\therefore detB = \prod_{i=1}^{n} q_{1i} \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{bmatrix} = \prod_{i=1}^{n} q_{1i} \prod_{i<j} (\lambda_i - \lambda_j)$$

□