Lecture 1 : Basic random matrix models

Notes prepared by: B. Valkó

Our aim in this course to study the asymptotic behavior of the spectrum of certain random matrices.

Wigner Matrices

Definition 1 (real Wigner matrices)**.** For $1 \le i < j < \infty$ let $X_{i,j}$ be i.i.d. (real) random variables with mean 0 and variance 1 and set $X_{j,i} = X_{i,j}$. Let $X_{i,i}$ be i.i.d. (real) random variables (with possibly a different distribution) with mean 0 and variance 1. Then $M_n = [X_{i,j}]_{i,j=1}^n$ will be a random $n \times n$ symmetric matrix.

Definition 2 (complex Wigner matrices)**.** For $1 \le i < j < \infty$ let $X_{i,j}$ be i.i.d. (complex) random variables with mean 0, $\mathbf{E}|X_{i,j}|^2 = 1$ and set $X_{j,i} = \overline{X_{i,j}}$. Let $X_{i,i}$ be i.i.d. (real) random variables with mean 0 and variance 1. Then $M_n = [X_{i,j}]_{i,j=1}^n$ will be a random $n \times n$ hermitian matrix.

In both cases there are n random eigenvalues which we will denote by

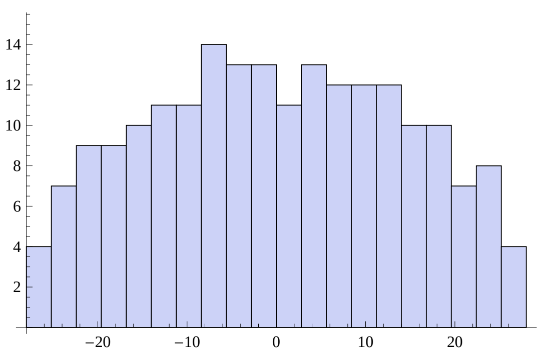
$$\lambda_1 \le \lambda_2 \le \dots \lambda_n.$$

(We will denote the dependence on n). Fact (which we will prove later): these are continuous functions of M_n hence they are random variables themselves.

We would like to study the scaling limit of the empirical spectral measure

$$\nu_n^* = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}.$$

This is a random discrete probability measure which puts n^{-1} mass to each (random) eigenvalue. The following picture shows the histogram of eigenvalues for a certain 200×200 Wigner matrix.



The picture suggests that there is a nice deterministic limiting behavior. In order to figure out the right scaling, we first compute the order of the empirical mean and second moment of the eigenvalues.

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\lambda_i &= \frac{1}{n}\mathbf{T}rM_n = \frac{1}{n}\sum_{i=1}^{n}X_{i,i}\\ \frac{1}{n}\sum_{i=1}^{n}\lambda_i &= \frac{1}{n}\mathbf{T}rM_n^2 = \frac{1}{n}\sum_{i,j=1}^{n}X_{i,j}^2\end{aligned}$$

The first moment converges to 0 by the strong law of large numbers. However the second moment is of $O(n)$ as we have about $n^2/2$ independent terms in the sum with a normalization of $\frac{1}{n}$ instead of $\frac{1}{n^2}$. This suggests that in order to see a meaningful limit, we need to scale the eigenvalues (or the matrix) by $\frac{1}{\sqrt{n}}$.

The following theorem states that in case we indeed have a deterministic limit.

Theorem 3 (Wigner's semicircle law)**.** *Let*

$$\nu_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{\frac{\lambda_i}{\sqrt{n}}}.$$

be the normalized empirical spectral measure. Then as $n \to \infty$ we have

$$\nu_n \Rightarrow \nu \qquad \text{a.s.}$$

where ν has density

$$\frac{d\nu}{dx} = \frac{1}{2\pi}\sqrt{4-x^2}\,\mathbf{1}\{|x| \le 2\}.$$

(There willl be some assumptions on the distribution of the random entries of M_n...)

Gaussian Ensembles

We also discussed some special Wigner matrix models.

Definition 4 (GOE)**.** Consider a real Wigner matrix where $X_{i,j} \sim N(0,1)$ and $X_{i,i} \sim \sqrt{2}N(0,1)$. The resulting random matrix model is called *Gaussian Orthogonal Ensemble* (or GOE).
Another construction: let $a_{i,j}, i,j \in \mathbb{Z}$ be i.i.d. standard normals and $A_n = [a_{i,j}]_{i,j=1}^n$. (Note that this is not a symmetric matrix!). Then the distribution of $M_n = \frac{A_n + A_n^T}{\sqrt{2}}$ is GOE.

It is easy to check the following useful fact: if $C \in \mathbb{R}^{n\times n}$ is orthogonal (i.e. $CC^T = I$) the $C^T M_n C$ has the same distribution as M_n. (The GOE is invariant to orthogonal conjugation.) It is a bit harder (we will prove it later) that one can actually compute the joint eigenvalue density which is given by

$$f(\lambda_1,\dots,\lambda_n) = \frac{1}{Z_1}\prod_{i<j}|\lambda_j - \lambda_i|e^{-\frac{1}{4}\sum_{i=1}^n \lambda_i^2}.$$

Here Z_1 is an explicitly computable normalizing constant (which also depends on n).

Definition 5 (GUE)**.** Consider a complex Wigner matrix where $X_{i,j}$ is standard complex Gaussian (i.e. $X_{i,j} \sim N(0, \frac{1}{2}) + iN(0, \frac{1}{2})$) and $X_{i,i} \sim N(0, 1)$ (real). The resulting random hermitian matrix model is called *Gaussian Unitary Ensemble* (or GUE).
Another construction: let $a_{i,j}, i, j \in \mathbb{Z}$ be i.i.d. standard complex Gaussians and $A_n = [a_{i,j}]_{i,j=1}^n$. (Note that this is not a symmetric matrix!). Then the distribution of $M_n = \frac{A_n + A_n^*}{\sqrt{2}}$ is GUE.

As the name suggests, GUE is invariant under unitary conjugation. If $C \in \mathbb{C}^{n\times n}$ is unitary (i.e. $CC^* = I$) the $C^T M_n C$ has the same distribution as M_n. (We will later show that the joint eigenvalue density is given by

$$f(\lambda_1, \dots, \lambda_n) = \frac{1}{Z_2} \prod_{i<j} |\lambda_j - \lambda_i|^2 e^{-\frac{1}{2}\sum_{i=1}^n \lambda_i^2}.$$

Here Z_2 is an explicitly computable normalizing constant (which also depends on n).

One can see the similarity between the two densities: they are contained in the following one-parameter family of densities:

$$f(\lambda_1, \dots, \lambda_n) = \frac{1}{Z_\beta} \prod_{i<j} |\lambda_j - \lambda_i|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^n \lambda_i^2}. \tag{1}$$

For a given $\beta > 0$ the resulting distribution (on ordered n-tuples in $\mathbb{R}$) is called *Dyson's β-ensemble.* For $\beta = 1$ one gets the eigenvalue density of GOE, for $\beta = 2$ we get the GUE. The $\beta = 4$ case is also special: it is related another classical random matrix model, the Gaussian Symplectic Ensemble (GSE), which can be defined using quaternions.

For other values of β there are no 'nice' random matrices in the background. (We will see that one can still build random matrices from which we get the general β-ensemble, but they won't have such nice symmetry properties.)

Later in the semester we will show that if one scales the β ensembles properly ('zooming in' to see the individual eigenvalues near a point) then one gets a point process limit. The limiting point process is especially nice in the $\beta = 2$ case (GUE). It is conjectured that its distribution appears among the critical line zeros of the Riemann-ζ function.

Another symmetric random matrix model

Another way of constructing a symmetric random matrix is the following. Let $a_{i,j}$ be i.i.d random real random variables with mean 0 and variance 1. Let $A = [a_{i,j}]_{i,j=1}^{n,m} \in \mathbb{R}^{n\times m}$ be a random matrix with n rows and m columns (with $n \le m$). Then $M_n = AA^T$ is a (positive) symmetric random matrix. We will show that the appropriately normalized empirical spectral measure will converge to a deterministic limit. (This is the Marchenko-Pastur law). A similar statement holds if we construct our matrix from i.i.d. complex random variables.