A Martingale Central Limit Theorem

We will prove the following version of the martingale central limit theorem:

Theorem 1. *Let $X_{n,k}, 1 \le k \le m_n$ be a martingale difference array with respect to $\mathcal{F}_{n,k}$ and let $S_{n,k} = \sum_{i=1}^{k} X_{n,i}$. If $\mathrm{E} \max_{j \le m_n} |X_{n,j}| \to 0$ and $\sum_{j=1}^{m_n} X_{j,n}^2 \xrightarrow{P} \sigma^2$ then $S_{n,m_n} \Rightarrow N(0, \sigma^2)$.*

We give a proof due to McLeish based on Sunder Sethuraman's notes. We start with a lemma:

Lemma 2. *For $n \ge 1$ let U_n, T_n be random variables satisfying the following conditions:*

1. *$U_n \xrightarrow{P} a$*
2. *$\{T_n\}$ and $\{T_n U_n\}$ are uniformly integrable sequences.*
3. *$\mathrm{E}T_n \to 1$.*

Then $\mathrm{E}T_n U_n \to a$.

Proof. Using (1) it is enough to show that $\mathrm{E}\left[T_n(U_n - a)\right] \to 0$. Since sum of u.i. sequences is also u.i. by (2) we get that $T_n(U_n - a)$ is u.i. We will show that $T_n(U_n - a) \xrightarrow{P} 0$, from this we will also get the L^1 convergence (by the u.i.) and thus the convergence of the expectation. This follows from

$$\mathbf{P}(|T_n(U_n - a)| > \varepsilon) \quad \le \quad \mathbf{P}(|U_n - a| > \varepsilon/K) + \mathbf{P}(|T_n| > K).$$

Indeed: by first choosing K big enough (by the uniform integrability of T_n) and then n large enough (by the convergence in probability) the left hand side can be made as small as we wish. $\square$

Proof of Theorem 1. We will prove the theorem with $\sigma = 1$, the general case is the same. We start with a cutoff: we define $Z_{n,1} = X_{n,1}$ and $Z_{n,j} = X_{n,j} \mathbb{1}(\sum_{r \le j-1} X_{n.r}^2 \le 2)$ for $2 \le j \le n$. Then $Z_{n,j}$ is also a martingale difference array (since the indicator is measurable wrt $\mathcal{F}_{n,j-1}$). Let $J = \inf\{j : \sum_{1 \le r \le j} X_{n,r}^2 > 2\} \wedge m_n$. Then

$$P(X_{n,r} \ne Z_{n,r} \text{ for some } r \le m_n) = \mathbf{P}(J \le m_n - 1) \le \mathbf{P}(\sum_{r \le m_n} X_{n,r}^2 > 2) \to 0 \tag{1}$$

where the last limit follows from the assumed convergence in probability of $\sum_{r \le m_n} X_{n,r}^2$. Let $\tilde{S}_{n,k} = \sum_{r \le k} Z_{n,r}$, we just proved that $S_{n,m_n} - \tilde{S}_{n,m_n} \xrightarrow{P} 0$ which means that it is enough to prove the weak convergence of $\tilde{S}_n := \tilde{S}_{n,m_n}$. We will do this by proving that $\mathrm{E}\exp(it\tilde{S}_n) \to e^{-it^2/2}$.

By Taylor-expansion we have

$$\exp(ix) = (1 + ix)\exp(-x^2/2 + r(x))$$

where the error term satisfies $|r(x)| \le |x|^3$ for all x.

Thus

$$\exp(it\tilde{S}_n) = \left(\prod_{j \le m_n} (1 + itZ_{n,j})\right) \exp\left(-\frac{t^2}{2}\sum_{j \le m_n} Z_{n,j}^2 + \sum_{j \le m_n} r(tZ_{n,j})\right)$$

Let $T_n = \prod_{j \le m_n}(1 + itZ_{n,j})$ and $U_n = \exp\left(-\frac{t^2}{2}\sum_{j \le m_n} Z_{n,j}^2 + \sum_{j \le m_n} r(tZ_{n,j})\right)$. We will show that the conditions of Lemma 2 are satisfied.

For a fixed t

$$\left| \sum_{j \le m_n} r(tZ_{n,j}) \right| \le |t|^3 \sum_j |Z_{n,j}|^3 \le |t|^3 \sum_j |X_{n,j}|^3 \le |t|^3 \max_j |X_{n,j}| \sum_j X_{n,j}^2 \xrightarrow{P} 0$$

(In the last step we used the conditions of the theorem.) By (1) and the conditions of the theorem we have $\sum_{j \le m_n} Z_{n,j}^2 \xrightarrow{P} 1$ which leads to $U_n \xrightarrow{P} e^{-t^2/2}$.

Since $Z_{n,j}$ is a martingale difference array, we get

$$\mathrm{E}T_n = \mathrm{E}\left[\mathrm{E}[(1 + itZ_{n,m_n}) | \mathcal{F}_{n,m_n-1}] \prod_{j \le m_n - 1} (1 + itZ_{n,j}) \right] = \mathrm{E} \prod_{j \le m_n - 1} (1 + itZ_{n,j})$$

and continuing this we get $\mathrm{E}T_n = 1$. Also $|T_n U_n| = 1$ by definition so the sequence $T_n U_n$ is u.i.

Thus the only thing left to prove is the uniform integrability of T_n. Using the definition of $Z_{n,j}$, J and the inequality $|1 + iz|^2 \le \exp(z^2)$ we get

$$\begin{aligned} |T_n| &= \prod_{r \le J-1} |1 + itX_{n,j}| \, (1 + |tX_{n,J}|) \le \exp\left(\frac{t^2}{2} \sum_{r \le J-1} X_{n,r}^2 \right) (1 + |tX_{n,J})\\ &\le \exp(t^2)(1 + |t| \max_j |X_{n,j}|). \end{aligned}$$

Since $\{\max_{j \le m_n} |X_{n,j}|\}$ is u.i. (since it converges to 0 in L^1) this will also hold for $\{T_n\}$.

This shows that all the conditions of Lemma 1 are satisfied and this means that $\exp(i\tilde{S}_n) = T_n U_n$ converges in expectation to $e^{-t^2/2}$ which finishes the proof. $\square$