Math/Stat 832 – Spring 2011

## Homework 5

**Due: May 3, 2011**

1. $\mathbf{B}(t) = (B_1(t), B_2(t))$ is a two-dimensional (standard) Brownian motion if $B_1, B_2$ are independent standard Brownian motions.
   (a) Show that if $\mathbf{B}(t)$ is a 2D BM (starting from $(0,0)$) and $T$ is a rotation around $(0,0)$ by a fixed angle then $T(\mathbf{B}(t))$ is also a 2D BM.
   (b) Show that $\mathbf{B}(t)$ will hit any given line on the plane with probability one.

2. Show that for any $t \geq 0$

$$P(\text{the Brownian motion has a local maximum at } t) = 0$$

3. Find the density of $R = \inf\{t > 1 : B_t = 0\}$.
   (Hint: condition on $B_1$ and use the density of the hitting times.)

4. Let $B(t)$ be a standard BM, and $f(t)$ a continuous function on $[0,1]$.
   (a) Show that $X = \int_0^1 f(t)B(t)dt$ is an a.s. finite random variable.
   (b) Show that $X$ is normal and find its expectation and variance.
   (c) Let $Y = \int_0^1 g(t)B(t)dt$ where $g$ is continuous on $[0,1]$. Find the covariance of $X$ and $Y$.

   Note: for the variance it might be useful to use $(\int_0^1 h(s)ds)^2 = \int_0^1 \int_0^1 h(s)h(t)dsdt$ which holds for bounded integrable functions.

5. (a) Let $X_t$ be a continuous non-negative martingale with $X_0 = 1$ and $X_t \to 0$ a.s. Show that for any $x \geq 1$

$$P(\sup_{t\geq 0} X_t \geq x) = 1/x.$$

   Hint: consider the hitting time $\tau_x$ of $x$ and the stopped martingale $X_{t\wedge\tau_x}$.
   (b) Use the previous result to show that if $b > 0$ then $Y = \sup_{t\geq 0}(B_t - bt)$ is an exponential random variable with parameter $2b$. (I.e. $P(Y \leq x) = 1 - e^{-2bx}$ if $x \geq 0$.)

**Bonus problem:** Use the reflection principle (a bunch of times) to get a formula for the distribution of $\max_{0\leq t\leq 1} |B(t)|$.