Math/Stat 832 – Spring 2011

Homework 4

Due: April 7, 2011

1. The baker's map is a $[0,1)^2 \to [0,1)^2$ function given by

$$\phi(u,v) = \begin{cases} (\{2u\}, v/2) & \text{if} \quad u < 1/2 \\ (\{2u\}, v/2 + 1/2) & \text{if} \quad u \geq 1/2 \end{cases}$$

   where $\{x\}$ denotes the fractional part.

   Show that this map is invariant with respect to the Lebesgue measure λ on $[0,1)^2$ and for any $f : [0,1)^2 \to \mathbb{R}$ the sequence $(f(\omega), f(\phi(\omega)), \dots, f(\phi^k\omega), \dots)$ is stationary and ergodic on the probability space $([0,1)^2, \mathcal{B}^2, \lambda)$.

2. Give an example of an ergodic measure preserving transformation $T : \Omega \to \Omega$ and probability space $(\Omega, \mathcal{F}, P)$ for which T^2 is not ergodic.

3. Let $\varphi(x) = \{\frac{1}{x}\}$ for $x \in (0,1)$. Show that φ preserves the measure

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{dx}{1+x}, \quad A \subset (0,1).$$

4. Let $X_0, X_1, \dots$ be a stationary sequence and denote the distribution of X_0 by μ. Show that the sequence is ergodic if and only if

$$\text{for every } f, g \in L^2(\mu) \qquad \lim_{t\to\infty} \frac{1}{n} E \sum_{k=0}^{n-1} f(X_k) g(X_0) = Ef(X_0)\, Eg(X_0)$$

Bonus problem: Let θ be irrational. Show that the map $T : (x,y) \to (x+\theta, x+y)$ is invariant for the Lebesgue measure on the unit square (the addition is meant mod 1 here) and that it is ergodic.