Math/Stat 832 – Spring 2011

Homework 3

Due: March 10, 2011

1. (**6.5.8.**) Compute the expected number of moves it takes a knight to return to its initial position if it starts in a corner of the chessboard, assuming there are no other pieces on the board, and each time it chooses a move at random from its legal moves. (Note: A chessboard is $\{0, 1, ..., 7\}^2$. A knights move is L-shaped; two steps in one direction followed by one step in a perpendicular direction.)

2. Let P be the transition matrix of a Markov chain with a state space of size n. Let I be the $n \times n$ identity matrix, U the $n \times n$ matrix filled with 1's and $\mathbf{e}$ the n-dimensional row-vector filled with 1's. Let π be a non-negative vector with $\sum_i \pi_i = 1$. Prove that $\pi P = \pi$ if and only if $\pi(I - P + U) = \mathbf{e}$. Show that if P is irreducible (i.e. the corresponding Markov chain is irreducible) then $\pi = \mathbf{e}(I - P + U)^{-1}$.
(You also have to show that the right hand side makes sense!)

3. We have a symmetric random walker on $\mathbb{Z}$ (he goes left or right one unit with probability 1/2–1/2) which is trapped whenever it reaches 100 or -100. (I.e. it will stay there forever.) What is the expected number of times it visits 50 if it starts at the origin?

4. **(The exclusion process)** Consider the following interacting particle system. We have a ring with N sites (you can label the sites with the elements of $\mathbb{Z}_N$) and at each site we either have a particle or not. (There can only be at most one particle at any given site.) We can describe a given state of the system by an element of $\{0, 1\}^{\mathbb{Z}_N}$, where 1 denotes a particle and 0 denotes an empty site. The particles will try to move clockwise on the ring the following way: in each step we choose one of the sites uniformly, and if there is a particle at this site which can move then we move it one step (clockwise). (A particle at site i can move if there is no particle at site $i + 1 \mod N$.)

   (a) Show that this defines a Markov process. Prove that it is not irreducible and identify the irreducible components.

   (b) Show that for any $0 \leq p \leq 1$ the Bernoulli(p) product distribution is invariant. (I.e. we flip a biased coin for each site and put a particle there depending on the outcome.) Is it reversible?

   (c) Suppose that $N = 20$ and initially we have particles at sites 2,4,5,9,14,19. Describe the limiting distribution of the process.

Bonus problem: Let P be the transition probability matrix of a finite irreducible Markov chain with period d and let $\zeta = \exp(i2\pi/d)$. Show that $1, \zeta, \zeta^2, \ldots, \zeta^{d-1}$ are eigenvalues of P and all other eigenvalues are less than 1 in absolute value.
(This is basically a linear algebra problem, but the result can be used to describe the asymptotic behavior of P^n.)
Hint: you might want to try a simpler version first: e.g. $d = 1$ and $p(x, y) > 0$ for all x, y.