Math/Stat 832 – Spring 2011

Homework 2

Due: February 24, 2011

1. Let $\xi_0, \xi_1, \dots$ be i.i.d. Bernoulli(1/2) random variables. Show that $X_n = (\xi_n, \xi_{n+1})$ is a Markov chain and compute its transition probability p. What is $p^{(2)}$?

2. Let $\xi_0, \xi_1, \dots$ be i.i.d. uniform on the set $\{1, 2, \dots, N\}$. Show that $X_n = |\{\xi_1, \dots, \xi_n\}|$ is a Markov chain and compute its transition probability. Find ET_n where $T_n = \inf\{n : X_n = N\}$.
   Hint: Where did we see this process in the first semester?

3. Let X_n be a time-homogeneous discrete Markov chain on S and $f : S \to \mathbb{R}$ a bounded function. Construct a function $g : S \to \mathbb{R}$ such that $M_n = \sum_{k=1}^{n} (f(X_k) - g(X_{k-1}))$ is a martingale.

4. Show that if the transition probability of a Markov chain is symmetric (i.e. $p(x, y) = p(y, x)$) then it has a stationary measure.

5. Show that a Markov chain on a finite state space Ω always has a stationary distribution using the following outline.
   Denote the transition probability matrix by P, let μ be a distribution on Ω and define
   $$\nu_n = \frac{1}{n}(\mu + \mu P + \mu P^2 + \cdots + \mu P^{n-1}).$$
   (a) Give an estimate on $\sup_{x\in\Omega} |\nu_n P(x) - \nu_n(x)|$ in terms of n.
   (b) Show that there is a subsequence n_k along which $\nu(x) = \lim_{k\to\infty} \nu_{n_k}(x)$ exists for all x.
   (c) Show that ν is a stationary distribution.

Bonus problem: We roll a fair die infinitely many times and record the partial sums. Let q_n be the probability that n is equal to one of these sums, find the limit of q_n as $n \to \infty$.