Math/Stat 832 – Spring 2010

Homework 1

Due: February 4, 2011

1. Suppose that X_n and Y_n are both uniformly integrable sequences and all the random variables are in L^2.
   (a) Show that if X_n, Y_n are independent then the sequence $X_n Y_n$ is u.i.
   (b) What happens if we do not assume independence?
2. Let X_n be a martingale, τ a stopping time and assume that $X_{n\wedge\tau}$ is uniformly integrable. Show that $EX_\tau = EX_0$.
3. **(5.5.8)** Show that if $\mathcal{F}_n \uparrow \mathcal{F}$ and $Y_n \to Y$ in L^1 then $E[Y_n|\mathcal{F}_n] \to E[Y|\mathcal{F}]$ in L^1.
4. **(5.6.4)** Show that if $X_1, X_2, \dots$ are exchangeable real random variables and $EX_i^2 < \infty$ then $EX_1X_2 \geq 0$.
5. Let $X_1, \dots$ be i.i.d. random variables with mean μ and variance σ^2. Show that the empirical variance $\frac{1}{n-1}\sum_{i=1}^n (X_i - S_n/n)^2$ converges a.s. to σ^2. (Here $S_n = X_1 + \cdots + X_n$.) Hint: Check out **5.6.5**.

Bonus problem: Consider the Pólya urn model (Section 5.3.2). Let X_n be the fraction of green balls after the n^{th} and let Y_n be the indicator of drawing a green ball for the n^{th} draw. Show that X_n converges in distribution to a beta distribution with parameters g/c and r/c (check the end of 5.3.2 for the definition).
Hints: show that $Z_n = \sum_{k=1}^n Y_k/n$ converges to the same limit as X_n. Then prove that $Y_1, Y_2, \dots$ is an exchangeable sequence and compute the limits of the moments of Z_n.