Math/Stat 831 – Fall 2010

Homework 5

Due: November 18, 2010

1. Assume that $\mu_n \Rightarrow \mu$ and $\int x^k d\mu_n \to m_k < \infty$ for all $k \geq 1$. Show that for every $k > 0$ the k^{th} moment of μ is finite and it is given by m_k.
   (We basically proved this in class, but you should provide a detailed proof.)

2. Let $X_1, X_2, \ldots$ be independent random variables and $S_n = X_1 + \cdots + X_n$. Assume that almost surely $|X_i| \leq M$ for all $i \geq 1$ with a given constant $M < \infty$. Show that if $\operatorname{Var} S_n \to \infty$ then $\frac{S_n - ES_n}{\sqrt{\operatorname{Var} S_n}} \Rightarrow N(0,1)$.

3. Let $X_1, X_2, \ldots$ be i.i.d. random variables with distribution function F and a continuous density f. We know that $F_n(x) := \frac{1}{n}\sum_{k=1}^n 1(X_k \leq x)$ converges uniformly to $F(x)$ with probability one. Now we will look at the empirical distribution on a finer scale. Let $c \in \mathbb{R}$ be a number with $f(c) > 0$ and consider $N_n(a,b) = \sum_{k=1}^n 1\left(X_k \in (c + \frac{a}{n}, c + \frac{b}{n})\right)$. Show that $N_n(a,b)$ converges in distribution for any $a < b$ and find the limit.

4. Let $X_1, X_2, \ldots$ be i.i.d. standard normals. Find a deterministic sequence a_n so that

$$Y_n = a_n \frac{X_1}{\sqrt{X_1^2 + X_2^2 + \cdots + X_n^2}}$$

   converges weakly to a non-constant distribution and identify the limit.

Bonus problem

Let X and Y be identically distributed random variables with mean zero and variance one. Assuming that $X - Y$ and $X + Y$ are independent, find the distribution of X.