Math/Stat 831 – Fall 2010

Homework 3

Due: October 14, 2010

1. Prove that if $P(A_n) \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty$ then $P(A_n \text{ i. o.}) = 0$.

2. Let $X_1, X_2, X_3, \ldots$ be i.i.d. with a a distribution which is exponential with parameter λ. (This means that they have a density function $\lambda e^{-\lambda x}$ on $[0, \infty)$.)

   Show that almost surely
   $$\limsup_{n\to\infty} \frac{X_n}{\log n} = \frac{1}{\lambda}$$

3. Let $X_1, X_2, \ldots$ be an i.i.d. sequence of Bernoulli random variables with parameter p. (I.e. $P(X_i = 1) = p = 1 - P(X_i = 0)$.) Consider the random variable $Y = \sum_{i=1}^{\infty} \frac{X_i}{2^i}$, a random number in $[0, 1]$ with binary digits given by the X_i's. Show that if $p \neq 1/2$ then the distribution of Y is *singular* to the Lebesgue measure, meaning that there is a set A of Lebesgue measure 0 so that $P(Y \in A) = 1$.
   Hint: Try to find a set like that using the Strong Law of Large Numbers.

4. Let $X_1, X_2, \ldots$ be i.i.d. with $E|X_1| < \infty$. Show that
   $$\frac{\max(X_1, X_2, \ldots, X_n)}{n} \to 0 \qquad \text{a.s.}$$

5. Let $X_1, X_2, \ldots$ be independent so that X_i is Bernoulli with parameter p_i. Show that
   (a) $X_n \to 0$ in probability if and only if $p_n \to 0$,
   (b) $X_n \to 0$ a.s. if and only if $\sum_{n=1}^{\infty} p_n < \infty$.

6. Show that the "identically distributed" condition cannot be dropped from the strong law of large numbers. Find a sequence of independent, non-negative, mean one random variables $X_1, X_2, \ldots$ so that
   $$\limsup_{n\to\infty} \frac{X_1 + \ldots + X_n}{n} = \infty \qquad \text{a.s.}$$

Bonus problem. Let $X_1, X_2, X_3, \ldots$ be independent random variables with

$$P(X_n = n) = P(X_n = -n) = \frac{1}{2n\log(n+1)}, \qquad P(X_n = 0) = 1 - \frac{1}{n\log(n+1)}.$$

Show that $n^{-1}\sum_{i=1}^{n} X_i$ converges to 0 in probability, but it almost surely does not converge to a finite limit.