Math 831 – Fall 2010

Homework 1

Due: September 16, 2010, beginning of the class. Late homework will **not** be accepted.

1. In each of the following cases construct a probability space to model the corresponding random experiment. Be sure to describe each component in $(\Omega, \mathcal{F}, \mathbf{P})$ carefully. (There might be several possible correct solutions.)
   (a) We flip three fair coins and throw two dice.
   (b) We throw a fair die and if it shows the number n then we flip a coin n times.

2. Let $\{\mathcal{F}_\alpha, \alpha \in A\}$ be a (not necessarily countable) collection of σ-fields on the sample space Ω.
   (a) Show that $\cap_{\alpha \in A} F_\alpha$ is a σ-field.
   (b) Show that for any collection of subsets G in Ω there is a smallest σ-field containing G. (This is the σ-field generated by G: $\sigma(G)$.)

3. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Show that for every $A, B, C \in \mathcal{F}$
$$\mathbf{P}(A \circ B) \leq \mathbf{P}(B \circ C) + \mathbf{P}(A \circ C)$$
   $A \circ B$ denotes the symmetric difference: $(A^c \cap B) \cup (A \cap B^c)$.

4. We have a large empty urn and infinitely many balls numbered with the positive integers. At $t = 0$ we add the balls numbered with $1, 2, \ldots, 10$ into the urn then choose one randomly and throw it away. At $t = 1/2$ we add the balls $11, 12, \ldots, 20$ into the urn then choose one randomly (out of the 19) and throw it away. We repeat this infinitely many times: at time $t = 1 - 1/2^n$ we add the balls $10n + 1, 10n + 2, ..., 10(n + 1)$ into the urn, choose one randomly and we throw it away. Show that with probability one at time $t = 1$ the urn will be empty.
   Hint: Show that for any $k \in \mathbb{Z}_+$ the probability that the ball k will be in the urn at time $t = 1$ is zero.

5. Let X be a uniform random variable on $[0, 1]$ and consider $Y = X^2$. Describe
   (a) the distribution of Y,
   (b) the distribution function of Y.

Bonus problem. The following statement shows that if you want to prove an identity or inequality relating probabilities of certain events – like in problem 2 – then it is enough to check it on the trivial probability space (i.e. where $\mathcal{F} = \{\emptyset, \Omega\}$).

Suppose that $A_1, A_2, \ldots A_n \in \mathcal{F}$ and $B_1, B_2, \ldots, B_k \in \sigma(A_i : i = 1 \ldots n)$. (This means that each B_j may be expressed from the A_i's using the usual set operations.) Let $c_1, c_2, \ldots, c_k$ be real numbers. Then
$$\sum_{j=1}^{k} c_j \mathbf{P}(B_j) \geq 0$$
holds for all probability spaces if and only if it holds for the trivial probability space. (The same statement holds with $=$ instead of $\geq$.)
Hint: Try to find 'building blocks' for the σ-field generated by the A_i's.